Exponential Convergence of hp-FEM for Spectral Fractional Diffusion in Polygons

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Outline

- Synopsis
- Spectral Fractional Diffusion
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- Fractional Diffusion: Exponential Convergence of Sinc-BK and hp-FEM
- Conclusion and Future Directions
- References

Synopsis

- **Exponential Convergence** of two classes of hp-FEM Discretizations for spectral fractional diffusion in curvilinear polygonal domains $\Omega \subset \mathbb{R}^2$
- **Localize** the nonlocal operator equation by extension or BK functional integral
- **Semi-discretize** by hp-FEM or by sinc-quadrature $\Rightarrow$ collection of **local, second-order, elliptic BVPs** in Ω, exponential consistency
- Leverage **robust exponential convergence** of hp-FEM for **singularly perturbed** reaction-diffusion in Ω
- Exponential bounds on Kolmogorov n-widths of solutions.
- L. Banjai and J.M. Melenk and Ch. Schwab: hp-FEM for reaction-diffusion equations.
  II: Robust exponential convergence for **multiple length scales in corner domains**,
  SAM Report 2020-28, arXiv:2004.10517 (in review)
- L. Banjai and J. Melenk and R. Nochetto and E. Otarola and A. Salgado and Ch. Schwab:
  Tensor FEM for spectral fractional diffusion,
  Journ. Found. Comp. Math. **19**(4) (2019) 901-962.

Curvilinear Polygon $\Omega \subset \mathbb{R}^2$

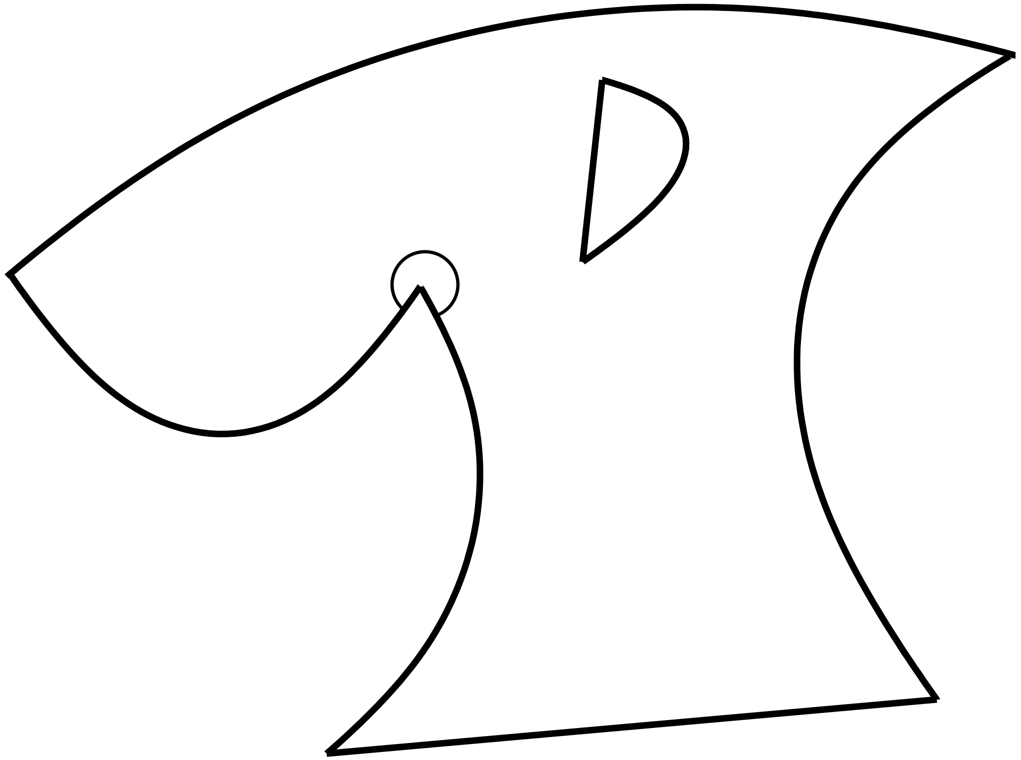

Figure 1: Example of a curvilinear polygon $\Omega \subset \mathbb{R}^2$

- $\Gamma = \partial\Omega$ Lipschitz, *finite union* of analytic arcs Γ_j, $j = 1, ..., J$,
- $\Gamma_j = \{\mathbf{x}_j(\vartheta) = (x_j(\vartheta), y_j(\vartheta)) : [0,1] \to \mathbb{R}^2\}$, with x_j,y_j analytic in $[0,1]$ and

$$\min_{\vartheta\in[0,1]} \left\{ \left|\dot{x}_j^{(i)}(\vartheta)\right|^2 + \left|\dot{y}_j^{(i)}(\vartheta)\right|^2 \right\} > 0 \quad j = 1, \ldots, J.$$

Dirichlet Spectral Fractional Diffusion Problem in Ω

Given $0 < s < 1$, $f \in L^2(\Omega)$, $A \in L^\infty(\Omega, \mathsf{GL}(\mathbb{R}^2))$ symmetric, uniformly positive definite, find

$$u \in \mathbb{H}^s(\Omega): \quad \mathcal{L}^s u = f \quad \text{in} \quad \mathbb{H}^{-s}(\Omega) \ .$$

where

- $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega) : w \mapsto \mathcal{L}w = -\mathbf{div}(A\nabla w)$ (2nd order, linear, elliptic self-adjoint),
- $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H_0^1(\Omega)$ eigenpairs of $\mathcal{L}$, $\{\varphi_k\}_{k\in\mathbb{N}}$ ONB of $L^2(\Omega)$
- for $\sigma \geq 0$, domains of fractional powers of $\mathcal{L}$ are

$$\mathbb{H}^\sigma(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k \varphi_k : \|w\|^2_{\mathbb{H}^\sigma(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^\sigma w_k^2 < \infty \right\} .$$

- $\mathbb{H}^{-\sigma}(\Omega)$ dual of $\mathbb{H}^\sigma(\Omega)$ and $\mathbb{H}^\sigma(\Omega) = H_0^\sigma(\Omega)$, $1/2 < \sigma < 1$.
- $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega) : w \mapsto \sum_{k=1}^\infty w_k \lambda_k^s \varphi_k$, where $w_k := (w, \varphi_k)$.

Proposition : For every $0 < s \leq 1$ and every $f \in \mathbb{H}^{-s}(\Omega)$, ex. unique solution $u = \mathcal{L}^{-s}(f) \in \mathbb{H}^s(\Omega)$.

Spaces $\mathbb{H}^\sigma(\Omega)$

Remark [Implicit Boundary Compatibility Conditions in Scale $\mathbb{H}^\sigma(\Omega)$]

- Mapping property: $\mathcal{L}^s : \mathbb{H}^{s+\sigma}(\Omega) \to \mathbb{H}^{-s+\sigma}(\Omega)$, $\sigma \geq 0$.
- $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ boundedly invertible.
- For smooth A and $\partial\Omega$, $\mathbb{H}^{s+\sigma}(\Omega)$, $\sigma \geq 0$, are proper subspaces of $H^{s+\sigma}(\Omega)$ if $s + \sigma > 1/2$.
- For $\sigma > 1/2$, $\mathbb{H}^{-s+\sigma}(\Omega) \subset H^{-s+\sigma}(\Omega)$ encode boundary conditions on $\partial\Omega$.
  E.g., for $f \in \mathbb{H}^{-s+\sigma}(\Omega)$ with $-s + \sigma \geq 1/2$, $f|_{\partial\Omega} = 0$:

  f must satisfy a *compatibility condition* on $\partial\Omega$ in addition to $H^{-s+\sigma}(\Omega)$ for $u \in H^{s+\sigma}(\Omega)$.
- for $f \in \mathbb{H}^{-s+\sigma}(\Omega)$, $u = \mathcal{L}^{-s} f \in \mathbb{H}^{s+\sigma}(\Omega)$ exhibits *corner singularities* at vertices of Ω
  $\Rightarrow$ local, isotropic corner mesh-refinement for FEM: Optimal (h-FEM) rates in Section 5.4 of
  [Banjai, Melenk, Nochetto, Otarola, Salgado, Schwab: JFoCM **19**(4) (2019) 901-962]
- Major issue: Regularity and FEM error analysis for $u = \mathcal{L}^{-s} f$ with $f \in H^{-s+\sigma}(\Omega) \backslash \mathbb{H}^{-s+\sigma}(\Omega)$.
- First *exponential convergence results without compatibility* of f on $\partial\Omega$ in Section 7 of
  [Banjai, Melenk, Nochetto, Otarola, Salgado, Schwab: JFoCM **19**(4) (2019) 901-962]
  ($d = 1$ and $d = 2$ with **analytic boundary**).

Localization I: CST Extension

CST = Cabré, Caffarelli, Silvestre, Sire, Stinga, Torrea ...
Local $d+1$-*dimensional BVP*

$$\begin{cases} \mathfrak{L}\mathscr{U} := -\mathrm{div}\,(y^\alpha \mathfrak{A} \nabla \mathscr{U}) = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ \mathscr{U} = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times (0,\infty), \\ \partial_{\nu^\alpha} \mathscr{U} = d_s f & \text{on } \Omega \times \{0\}, \end{cases} \tag{1}$$

$$\mathfrak{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in L^\infty(\mathcal{C}, \mathsf{GL}(\mathbb{R}^{d+1})),$$

$$d_s := 2^{1-2s}\Gamma(1-s)/\Gamma(s) > 0, \quad \alpha := 1-2s \in (-1,1).$$

Conormal exterior derivative of $\mathscr{U}$ at $\Omega \times \{0\}$:

$$\partial_{\nu^\alpha} \mathscr{U} := -\lim_{y \downarrow 0} y^\alpha \mathscr{U}_y.$$

Then

$$d_s \mathcal{L}^s u = \partial_{\nu^\alpha} \mathscr{U} \quad \text{in } \Omega\ .$$

Used for numerical approximation by R. Nochetto, E. Otarola & A. Salgado (JFoCM 2015).

Remark: $s = 1/2 \Rightarrow \alpha = 0$, i.e. for $A = I$ is $\mathfrak{L}$ the $d+1$-dimensional Laplacean.
$\Rightarrow \partial\Omega$ is an **edge of the cylinder** $\mathcal{C} \Rightarrow$ $\mathscr{U}$ (and, hence, also u) has **algebraic edge singularity** on $\partial\Omega$

Localization I: CST Extension

Notation: $x = (x', y) \in \mathcal{C}$ with $x' \in \Omega$ and $y > 0$. For $D \subset \mathbb{R}^d \times \mathbb{R}^+$, define

- Lebesgue space w.r.to measure $y^\alpha \, \mathrm{d}x$: $L^2(y^\alpha, D)$
- Weighted Sobolev space: $H^1(y^\alpha, D) := \left\{ w \in L^2(y^\alpha, D) \,:\, |\nabla w| \in L^2(y^\alpha, D) \right\}$.
- Norm: $\|w\|_{H^1(y^\alpha,D)} = \left(\|w\|^2_{L^2(y^\alpha,D)} + \|\nabla w\|^2_{L^2(y^\alpha,D)} \right)^{1/2}.$

Homogeneous Essential BCs: $H^1_\Gamma(y^\alpha, \mathcal{C}) := \left\{ w \in H^1(y^\alpha, \mathcal{C}) \,:\, w = 0 \text{ on } \partial_L \mathcal{C} \right\}$.
Continuous and coercive bilinear form $a_\mathcal{C} : H^1_\Gamma(y^\alpha, \mathcal{C}) \times H^1_\Gamma(y^\alpha, \mathcal{C}) \to \mathbb{R}$:

$$a_\mathcal{C}(v, w) = \int_\mathcal{C} y^\alpha (\mathfrak{A} \nabla v \cdot \nabla w) \, \mathrm{d}x' \, \mathrm{d}y.$$

Energy inner product on $H^1_\Gamma(y^\alpha, \mathcal{C})$ and **energy norm**: $\|v\|^2_\mathcal{C} := a_\mathcal{C}(v, v) \sim \|\nabla v\|^2_{L^2(y^\alpha,\mathcal{C})}$.
Trace on $\Omega \times \{0\}$: $\operatorname{tr}_\Omega H^1_\Gamma(y^\alpha, \mathcal{C}) = \mathbb{H}^s(\Omega)$, $\|\operatorname{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \leq C_{\operatorname{tr}_\Omega} \|w\|_{H^1_\Gamma(y^\alpha,\mathcal{C})}$.
Localized weak formulation: Given $f \in L^2(\Omega)$, find $\mathscr{U} \in H^1_\Gamma(y^\alpha, \mathcal{C})$ such that

$$a_\mathcal{C}(\mathscr{U}, v) = d_s \langle f, \operatorname{tr}_\Omega v \rangle \qquad \forall v \in H^1_\Gamma(y^\alpha, \mathcal{C}).$$

$$\boxed{u = \mathcal{L}^{-s} f = \operatorname{tr}_\Omega \mathscr{U}}$$

Localization II: sinc-Quadrature for Balakrishnan functional integral

$$\mathcal{L}^{-s} = c_B \int_0^\infty \lambda^{-s}(\lambda I + \mathcal{L})^{-1}\, \mathrm{d}\lambda = c_B \int_{-\infty}^\infty e^{(1-s)y}(e^y I + \mathcal{L})^{-1}\, \mathrm{d}y = c_B \int_{-\infty}^\infty e^{-sy}(I + e^{-y}\mathcal{L})^{-1}\, \mathrm{d}y \; .$$

$$c_B := \pi^{-1}\sin(\pi s)$$

- **Sinc Quadrature** for $\int_{-\infty}^{\infty}$ (A. Bonito, J.Pasciak et al. (2015-)) to obtain

$$u = \mathcal{L}^{-s} f \sim Q_k^{-s}(\mathcal{L}) f := c_B k \sum_{|j|\le K} \varepsilon_j^{2s} \left(I + \varepsilon_j^2 \mathcal{L}\right)^{-1} f \; ,$$

where

$$y_j := jK^{-1/2} = jk, \;\; k := 1/\sqrt{K}, \;\; \varepsilon_j := e^{-y_j/2} = e^{j/2\sqrt{K}}, \; |j| \le K \; .$$

- $2K+1$ **Decoupled, local reaction diffusion problems** in Ω: $\left(I + \varepsilon_j^2 \mathcal{L}\right) u_j = f$.
- **Exponential Convergence**
  [Bonito, Andrea; Lei, Wenyu; Pasciak, Joseph E.: J. Numer. Math. **27** (2019) 57-68]
  For every $0 \le \beta \le s$, ex. $b, C > 0$ such that for $f \in L^2(\Omega)$ and every $K \in \mathbb{N}$:

$$\|(\mathcal{L}^{-s} - Q_k^{-s}(\mathcal{L}))f\|_{D(\mathcal{L}^\beta)} \le C\exp(-b/k)\|f\|_{L^2(\Omega)} = C\exp(-b\sqrt{K})\|f\|_{L^2(\Omega)} \; .$$

CST-Extension: Analytic Regularity of $\mathscr{U} : (0,\infty) \to \mathbb{H}^s(\Omega)$

Theorem[Banjai, Melenk, Nochetto, Otarola, Salgado, Schwab: JFoCM **19**(4) (2019) 901-962]

Fix $0 \leq \tilde{\nu} < s$ and $0 \leq \nu < 1+s$. Then ex. $\kappa > 1$ s.t. $\forall \ell \in \mathbb{N}_0$

$$
\begin{aligned}
\|\partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2\ell-2\tilde{\nu},\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)! \|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}, \\
\|\nabla_{x'}\partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)! \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \\
\|\mathcal{L}\partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)! \|f\|_{\mathbb{H}^{1-s+\nu}(\Omega)}.
\end{aligned}
$$

If also $0 \leq \nu' < 1-s$ then

$$
\begin{aligned}
\|\mathcal{L}\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu',\gamma},\mathcal{C})} &\lesssim \|f\|_{\mathbb{H}^{1-s+\nu'}(\Omega)}, \\
\|\nabla_{x'}\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu',\gamma},\mathcal{C})} &\lesssim \|f\|_{\mathbb{H}^{-s+\nu'}(\Omega)}, \\
\|\mathscr{U}\|_{L^2(\omega_{\alpha-2\nu',\gamma},\mathcal{C})} &\lesssim \|f\|_{\mathbb{H}^{-1-s+\nu'}(\Omega)}.
\end{aligned}
$$

Constants in $\lesssim$ independent of $\mathscr{U}$, ℓ and f, and weight $\omega_{\beta,\gamma}$ defined by

$$
\omega_{\beta,\gamma}(y) = y^\beta e^{\gamma y}, \quad 0 \leq \gamma < 2\sqrt{\lambda_1}, \quad y \in (0,\infty), \quad \|v\|_{L^2(\omega_{\beta,\gamma},\mathcal{C})} := \left(\int_0^\infty \int_\Omega \omega_{\beta,\gamma}(y) |v(x',y)|^2 \, \mathrm{d}x' \, \mathrm{d}y \right)^{\frac{1}{2}}.
$$

CST-Extension: Truncation and hp-Semidiscretization on $(0,\infty)$ of $\mathscr{U}$

Truncation: $0 < \mathcal{Y} < \infty$ truncation parameter, $\mathcal{C}_{\mathcal{Y}} := (0,\mathcal{Y}) \times \Omega$ truncated cylinder.
With $\lambda_1 > 0$ first eigenvalue of $\mathcal{L}$, ex. $\mathcal{U} \in H^1_\Gamma(y^\alpha, \mathcal{C})$, $\mathcal{U}|_{(\mathcal{Y},\infty)\times\Omega)} = 0$, with

$$\|\nabla(\mathscr{U} - \mathcal{U})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

hp-FEM in $\mathcal{C}_{\mathcal{Y}}$: $\mathcal{G}^M = \{I_m\}_{m=1}^M$ (geometric) mesh in $(0,\mathcal{Y})$, $\boldsymbol{r} = \{r_m\}_{m=1}^M \in \mathbb{N}^M$ polynomial degrees.

$$\begin{aligned} S^{\boldsymbol{r}}((0,\mathcal{Y}),\mathcal{G}^M) &= \left\{v_M \in C[0,\mathcal{Y}] : v_M|_{I_m} \in \mathbb{P}_{r_m}(I_m), I_m \in \mathcal{G}^M, m = 1,\dots,M\right\}. \\ S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M) &= \left\{v_M \in S^{\boldsymbol{r}}((0,\mathcal{Y}),\mathcal{G}^M) : v_M(\mathcal{Y}) = 0\right\}. \end{aligned}$$

hp-Semidiscretizations: based on (closed, Hilbertian) tensor product spaces

$$\mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_{\mathcal{Y}}) := H^1_0(\Omega) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M) \subset H^1_\Gamma(y^\alpha,\mathcal{C}),$$

defined by Galerkin-Projection: Find

$$\mathscr{U}_M \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_{\mathcal{Y}}) \text{ s.t. } a_{\mathcal{C}}(\mathscr{U}_M,\varphi) = d_s\langle f, \operatorname{tr}_\Omega \varphi\rangle \qquad \forall \varphi \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_{\mathcal{Y}}).$$

Truncation and hp-Semidiscretization on $(0,\infty)$ of $\mathscr{U}$

Lemma [hp-Semidiscretization error bound]

Assume

- $f \in \mathbb{H}^{-s+\nu}(\Omega)$ for some $\nu > 0$,
- $2 \leq M \in \mathbb{N}$, $\mathcal{Y} \in (0,\infty)$ with $c_1 M \leq \mathcal{Y} \leq c_2 M$.

Geometric mesh $\mathcal{G}^M_{geo,\sigma}$ on $(0,\mathcal{Y})$: M elements, grading factor $\sigma \in (0,1)$:

$$\{I_i \,|\, i = 1,\ldots,M\}: \quad I_1 = [0, \mathcal{Y}\sigma^{M-1}], I_i = [\mathcal{Y}\sigma^{M-i+1}, \mathcal{Y}\sigma^{M-i}] \text{ for } i = 2,\ldots,M.$$

Then exist C, $b > 0$ (depending solely on s, $\mathcal{L}$, c_1, c_2, σ, ν) s.t. for any polynomial degree $\boldsymbol{r} \sim M$ holds

$$\|\nabla(\mathscr{U} - \mathscr{U}_M)\|_{L^2(y^\alpha,\mathcal{C})} \leq Ce^{-bM}\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}.$$

Remark: $\mathscr{U}_M \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y})$ extended by zero to $\{y > \mathcal{Y}\}$.

Diagonalization

Issue: $\mathscr{U}_M \in \mathbb{V}^{\boldsymbol{r}}_M(\mathcal{C}_\mathcal{Y})$ solution of $\mathcal{M} = O(M^2)$ many **coupled reaction-diffusion equations** in Ω.
Eigenvalue problem: Find $(v, \mu) \in S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M) \setminus \{0\} \times \mathbb{R}$ such that

$$\mu \int_0^{\mathcal{Y}} y^\alpha v'(y) w'(y)\, \mathrm{d}y = \int_0^{\mathcal{Y}} y^\alpha v(y) w(y)\, \mathrm{d}y \qquad \forall w \in S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M).$$

All eigenvalues μ are positive, and $S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M)$ has eigenbasis $(v_i)_{i=1}^{\mathcal{M}}$, with $\mathcal{M} := \dim S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M)$, such that, for i, $j \in \{1, \dots, \mathcal{M}\}$,

$$\int_0^{\mathcal{Y}} y^\alpha v_i'(y) v_j'(y)\, \mathrm{d}y = \delta_{i,j}, \qquad \int_0^{\mathcal{Y}} y^\alpha v_i(y) v_j(y)\, \mathrm{d}y = \mu_i \delta_{i,j}.$$

Write $\mathscr{U}_M(x', y) := \sum_{i=1}^{\mathcal{M}} U_i(x') v_i(y)$, test with $\varphi(x', y) = V(x') v_j(y)$, with $V \in H^1_0(\Omega)$.
$\Longrightarrow$ **System of *decoupled* reaction-diffusion problems in** Ω: for $i = 1, \dots, \mathcal{M}$, find

$$U_i \in H^1_0(\Omega) \text{ s.t. } a_{\mu_i,\Omega}(U_i, V) = d_s v_i(0) \langle f, V \rangle \qquad \forall V \in H^1_0(\Omega),$$

where

$$a_{\mu_i,\Omega}(U, V) := \mu_i a_\Omega(U, V) + \int_\Omega U V \,\mathrm{d}x', \qquad a_\Omega(U, V) := \int_\Omega \nabla V \cdot A(x') \nabla U \,\mathrm{d}x'.$$

Diagonalization

Singular Perturbations: range of eigenvalues μ_i

Lemma [properties of eigenpairs (μ_i, v_i)]

Assume

- $0 < \sigma < 1$, $2 \leq M \in \mathbb{N}$ arbitrary.
- $\mathcal{G}^M_{geo,\sigma}$ geometric mesh on $(0, \mathcal{Y})$, $\boldsymbol{r}$ a linear degree vector with slope $\mathfrak{s} \geq \mathfrak{s}_{min} > 0$.
- $c_1 M \leq \mathcal{Y} \leq c_2 M$ for some constants $0 < c_1 < c_2 < \infty$.

Then ex $C > 0$ (dep. on α, σ, c_1, c_2, $\mathfrak{s}_{min}$) s.t. for all $i = 1, ..., \mathcal{M}$ holds

$$\|v_i\|_{L^\infty(0,\mathcal{Y})} \leq C M^s, \qquad C^{-1} \left(\mathcal{Y} \mathfrak{s}^{-2} M^{-2} \sigma^M\right)^2 \leq \mu_i \leq C M^2.$$

Proof: Via weighted Markov-inequalities for polynomials. Appendix B in [Banjai, Melenk, Nochetto, Otarola, Salgado, Schwab: JFoCM **19**(4) (2019) 901-962]

Remark:

1. Normalization $\|v_i\|_{L^2(0,\mathcal{Y})} = 1$.
2. Range of μ_i from small ($\sim \sigma^M$, singular perturbation) to large ($\sim M^2$, regular perturbation)
3. Full discretization: **one common hp-FE space $S^q(\Omega, \mathcal{T})$ for all $U_i \in H^1_0(\Omega)$.**
4. Design of $S^q(\Omega, \mathcal{T}) \Longrightarrow$ ensure **robust exponential convergence of hp-FEM** in Ω for

$$-\mu_i \nabla \cdot (A(x') \nabla U_i) + U_i = d_s v_i(0) f \quad \text{in} \quad \Omega, \quad U_i|_{\partial\Omega} = 0 \ .$$

hp-FEM for Reaction-Diffusion in Ω

[LB,MM,CS 20]: hp-FEM for reaction-diffusion equations II:
Robust exponential convergence for multiple length scales in corner domains, ArXiv:2004.10517
[Melenk, Jens M]: hp-FEM for singular perturbations. Springer LNM **1796** (2002).
Model Reaction-Diffusion Problem: for $0 < \varepsilon \leq 1$

$$-\varepsilon^2 \nabla \cdot (A(x')\nabla u^\varepsilon) + c(x')u^\varepsilon = f \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0 \; .$$

Assume $\operatorname{ess\,inf}_{x' \in \Omega} c(x') \geq c_0 > 0$ and

A, c, and f are analytic on $\overline{\Omega}$ and A symmetric, pointwise uniformly positive definite.

Design $S^p(\Omega, \mathcal{T})$ for **robust exponential convergence** in $\|w\|_{\varepsilon^2,\Omega} := \left(\|w\|^2 + \varepsilon^2 \|\nabla w\|^2\right)^{1/2}$.
Geometric Boundary Layer Mesh design principles:

- **fixed, regular macro-triangulation** (of quadrilaterals): $\mathcal{T}^{\mathcal{M}} = \{K^{\mathcal{M}} \,|\, K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}\}$ of Ω (curvilinear quadrilateral patches $K^{\mathcal{M}} \subset \Omega$ w. usual compatibility conditions),
- **analytic patch maps**: $F_{K^{\mathcal{M}}} : \widehat{S} = (0,1)^2 \to K^{\mathcal{M}}$,
- Finite **catalog** $\mathfrak{P}$ **of reference patch mesh structures** $\check{\mathcal{T}} \in \mathfrak{P}$ in reference patch $\widehat{S}$.

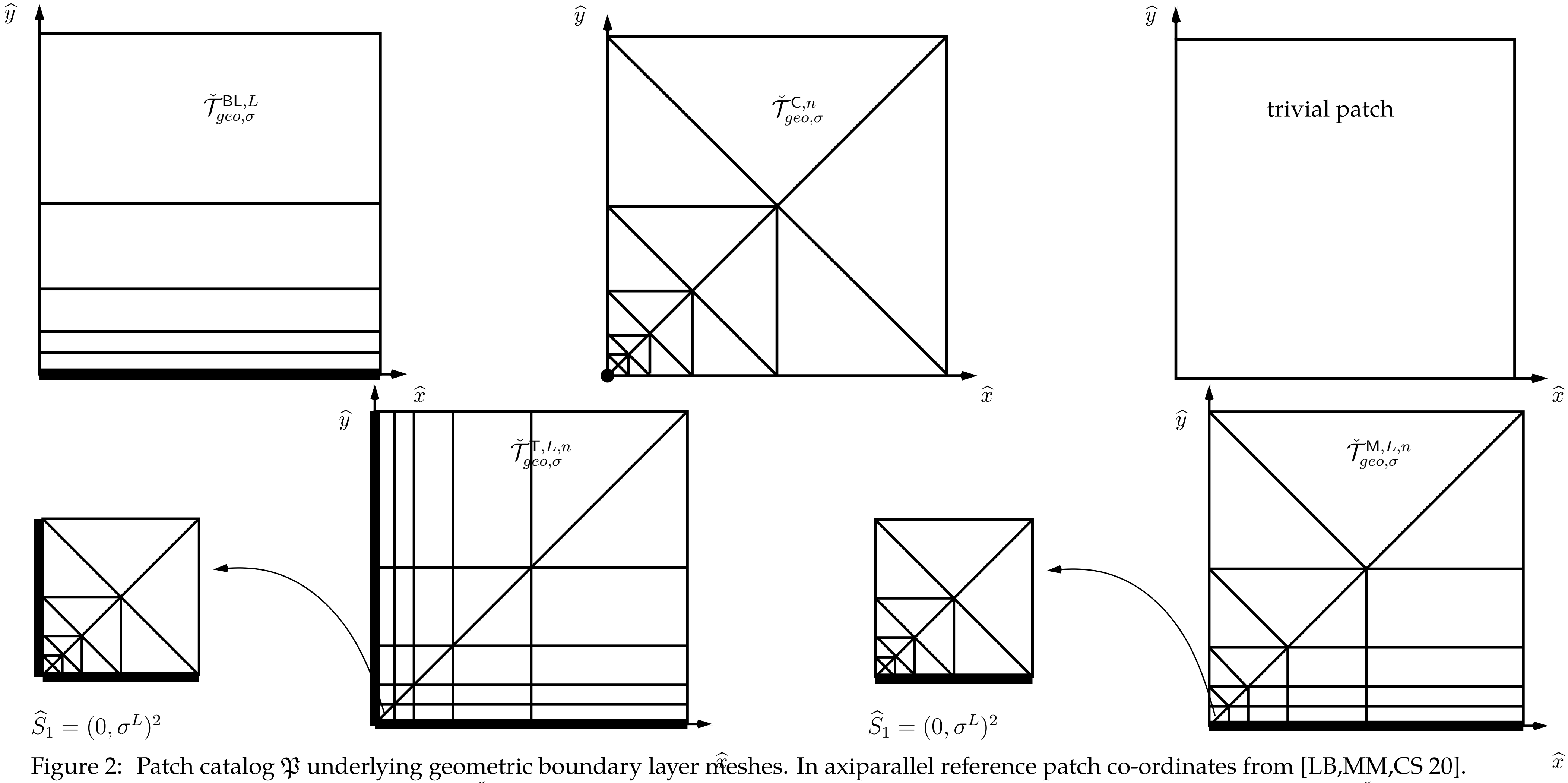

Figure 2: Patch catalog $\mathfrak{P}$ underlying geometric boundary layer meshes. In axiparallel reference patch co-ordinates from [LB,MM,CS 20].
Top row: reference **boundary layer patch** $\check{\mathcal{T}}^{\mathsf{BL},L}_{geo,\sigma}$ with L layers of geometric refinement towards $\{\widehat{y} = 0\}$; reference **corner patch** $\check{\mathcal{T}}^{\mathsf{C},n}_{geo,\sigma}$ with n layers of geometric refinement towards $(0,0)$; trivial patch.
Bottom row: reference **tensor patch** $\check{\mathcal{T}}^{\mathsf{T},L,n}_{geo,\sigma}$ with n layers of refinement towards $(0,0)$ and L layers of refinement towards $\{\widehat{x} = 0\}$ and $\{\widehat{y} = 0\}$; reference **mixed patch** $\check{\mathcal{T}}^{\mathsf{M},L,n}_{geo,\sigma}$ with L layers of refinement towards $\{\widehat{y} = 0\}$ and n layers of refinement towards $(0,0)$. Geometric entities shown in boldface indicate parts of $\partial\widehat{S}$ that are mapped to $\partial\Omega$. Patch meshes are transported into the curvilinear polygon Ω shown in Fig. 1 via analytic patch maps $F_{K^{\mathcal{M}}}$.

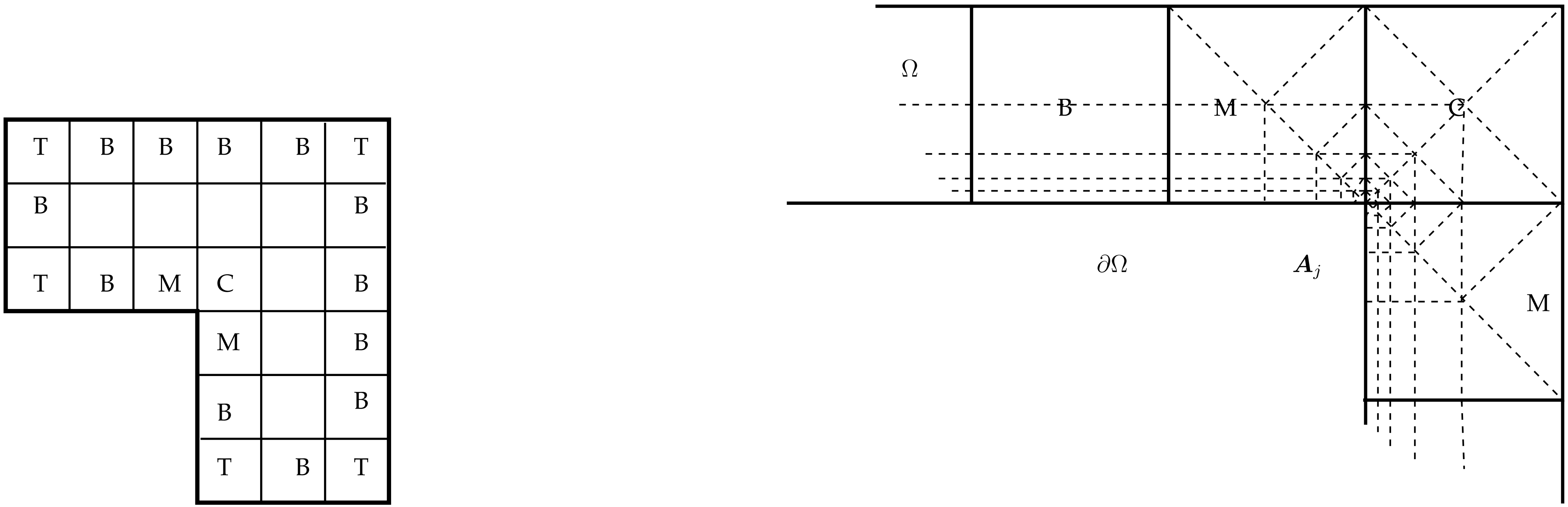

Figure 3: Patch arrangement in “L-shaped” domain Ω [LB,MM,CS 20].
Left panel: example of L-shaped domain decomposed into 27 patches
(T, B, M, C indicate tensor, boundary layer, mixed, corner patches, empty squares stand for trivial patches).
Right panel: Zoom-in near the reentrant corner $\boldsymbol{A}_j$. Solid lines indicate patch boundaries, dashed lines mesh lines.

Definition of hp-FE spaces $S_0^q(\Omega, \mathcal{T}^{L,n}_{geo,\sigma})$

Geometric Boundary Layer Meshes $\{\mathcal{T}^{L,n}_{geo,\sigma}\}_{L\geq 1, n\geq L}$:
Given $\sigma \in (0,1)$, L, $n \in \mathbb{N}_0$ with $n \geq L \geq 1$, $\mathcal{T}^{L,n}_{geo,\sigma}$ is called *geometric boundary layer mesh* if the following conditions hold:

1. $\mathcal{T}^{L,n}_{geo,\sigma}$ is obtained by refining each patch $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$ according to a finite "catalog" $\mathfrak{P}$ of **structured patch-refinement patterns** governed by the parameters σ, L, and n.
2. $\mathcal{T}^{L,n}_{geo,\sigma}$ is a regular triangulation of Ω, i.e., no irregular ("hanging") nodes.

For each patch $K^{\mathcal{M}} \in \mathcal{T}^{\mathcal{M}}$, exactly one of the following cases holds:

3. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \emptyset$. Reference patch is the **trivial patch**.
4. $K^{\mathcal{M}}$ satisfies $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \{\boldsymbol{A}_j\}$ for a vertex $\boldsymbol{A}_j$ of Ω.
   Reference patch is the corner patch $\check{\mathcal{T}}^{C,n}_{geo,\sigma}$. Additionally, $F_{K^{\mathcal{M}}}(0) = \boldsymbol{A}_j$.
5. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \overline{e}$ for an edge e of $K^{\mathcal{M}}$ and neither endpoint of e is a vertex of Ω. Reference patch is the **boundary layer patch** $\check{\mathcal{T}}^{\mathsf{BL},L}_{geo,\sigma}$, and the element maps $F_{K^{\mathcal{M}}}$ are such that $F_{K^{\mathcal{M}}}(\{\widehat{y}=0\}) \subset \partial\Omega$.
6. $\overline{K^{\mathcal{M}}} \cap \partial\Omega = \overline{e}$ for an edge e of $K^{\mathcal{M}}$ and exactly one endpoint of e is a vertex $\boldsymbol{A}_j$ of Ω.
   Reference patch is the **mixed layer patch** $\check{\mathcal{T}}^{\mathsf{M},L,n}_{geo,\sigma}$, and the element maps $F_{K^{\mathcal{M}}}$ are such that the edge $F_{K^{\mathcal{M}}}(\{\widehat{y}=0\}) \subset \partial\Omega$ and $F_{K^{\mathrm{M}}}(0) = \boldsymbol{A}_j$.
7. Exactly two edges of a macro-element K^{M} are situated on $\partial\Omega$. Reference patch is the **tensor patch** $\check{\mathcal{T}}^{\mathsf{T},L,n}_{geo,\sigma}$. Additionally, edges $\{\hat{y}=0\}$, $\{\hat{x}=0\}$ satisfy $F_{K^{\mathcal{M}}}(\{\widehat{y}=0\}) \subset \partial\Omega$, $F_{K^{\mathcal{M}}}(\{\widehat{x}=0\}) \subset \partial\Omega$, and $F_{K^{\mathcal{M}}}(0) = \boldsymbol{A}_j$ is a vertex of Ω.

Main Theorem on hp-FEM for Singular Perturbations [LB,MM,CS 20]

Theorem [Robust Exponential Convergence of hp-FEM on geometric boundary layer meshes]

- $\Omega \subset \mathbb{R}^2$ curvilinear polygon with J vertices,
- A, $c \geq c_0 > 0$, f analytic on $\overline{\Omega}$
- A uniformly symmetric positive definite.
- $\{\mathcal{T}^{L,n}_{geo,\sigma}\}_{L\geq 1, n\geq L}$ sequence of geometric boundary layer meshes,

Fix $c_1 > 0$. Then, ex. $C, b > 0$, $\beta \in [0,1)$ such that the following holds:
if $\varepsilon \in (0,1]$ and L satisfy the **(boundary layer) scale resolution condition**

$$\sigma^L \leq c_1 \varepsilon,$$

then, for any q, $n \geq L \in \mathbb{N}$, $u^\varepsilon \in H^1_0(\Omega)$ can be approximated from $S^q_0(\Omega, \mathcal{T}^{L,n}_{geo,\sigma})$ such that

$$\inf_{v \in S^q_0(\Omega, \mathcal{T}^{L,n}_{geo,\sigma})} \|u^\varepsilon - v\|_{\varepsilon^2,\Omega} \leq C q^9 \left[\varepsilon^\beta \sigma^{(1-\beta)n} + e^{-bq} \right],$$

$$N := \dim S^q_0(\Omega, \mathcal{T}^{L,n}_{geo,\sigma}) \leq C \left(L^2 q^2 \operatorname{card} \mathcal{T}^{\mathcal{M}} + n q^2 J \right).$$

Corollary: for $q \simeq n \simeq L \gtrsim |\ln \varepsilon|$: $\quad \inf_{v \in S^q_0(\Omega, \mathcal{T}^{L,n}_{geo,\sigma})} \|u^\varepsilon - v\|_{\varepsilon^2,\Omega} \leq C \exp(-b\sqrt[4]{N})$.

Exponential Convergence of CST-extension and hp-FEM

Tensor-Product hp-FEM in $\mathcal{C}_{\mathcal{Y}}$:

$$\mathbb{V}^{q,\boldsymbol{r}}_{L,M,n,\sigma}(\mathcal{C}_{\mathcal{Y}}) := S^q_0(\Omega, \mathcal{T}^{L,n}_{geo,\sigma}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M)$$

- Use hp-FE solver NGSOLVE in $2+1$ dimensional domain $\mathcal{C}_{\mathcal{Y}}$.
- hp-FEM in $(0,\mathcal{Y}) \Rightarrow$ "automatic" exponential consistency of element stiffness matrix quadratures with singular weight y^α

Theorem [Exponential Convergence of CST-hpFEM]

- $\Omega \subset \mathbb{R}^2$ curvilinear polygon with J vertices, $A \in L^\infty(\Omega, \mathsf{GL}(\mathbb{R}^2))$ unif. SPD, A, f analytic on $\overline{\Omega}$
- $\{\mathcal{T}^{L,n}_{geo,\sigma}\}_{L\geq 1, n\geq L}$ sequence of geometric boundary layer meshes, $q \simeq \boldsymbol{r} \simeq L \simeq M \simeq n \simeq \mathcal{Y} =: p$
- $\sigma \in (0,1)$ arbitrary, fixed.

Then the unique tensor product hp-FE solutions $\mathscr{U}^p_{TP} \in \mathbb{V}^{q,\boldsymbol{r}}_{L,M,n,\sigma}$ of the CST-extended, localized hp-FE approximation in $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0,\mathcal{Y})$ satisfy the error bound

$$\|u - \mathrm{tr}_\Omega\, \mathscr{U}^p_{TP}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}^p_{TP})\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \exp(-bp) \simeq \exp(-b'\sqrt[6]{N}) \;,$$

with $N := \dim \mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T}^{L,n}_{geo,\sigma}, \mathcal{G}^M_{geo,\sigma})$ denoting the overall number of degrees of freedom.

Exponential Convergence of sinc-BK hp-FEM

sinc-BK hp-FEM:
$K \in \mathbb{N}$, choose **sinc-parameters** $y_j := jk,\ k := 1/\sqrt{K},\ \varepsilon_j := e^{-y_j/2} = e^{j/2\sqrt{K}},\ |j| \leq K$.
Compute sinc-hpFE approximation

$$Q_k^{-s}(\mathcal{L}_{hp})f = c_B k \sum_{|j|\leq K} \varepsilon_j^{2s} w_j^{hp} \ ,$$

where hp-FE approximations $w_j^{hp} \in H_0^1(\Omega)$ solve

$$w_j^{hp} \in S_0^q(\Omega, \mathcal{T}_{geo,\sigma}^{L,n}) \quad \forall v \in S_0^q(\Omega, \mathcal{T}_{geo,\sigma}^{L,n}) : \ a_{\varepsilon_j^2,\Omega}(w_j^{hp}, v) = \langle f, v \rangle \ .$$

Theorem [Exponential Convergence of sinc-BK hpFEM]

- $\Omega \subset \mathbb{R}^2$ curvilinear polygon, J vertices, A, f analytic on $\overline{\Omega}$, A unif. SPD,
- $\{\mathcal{T}_{geo,\sigma}^{L,n}\}_{L\geq 1, n\geq L}$ sequence of geometric boundary layer meshes, $q \simeq \sqrt{K} \simeq L \simeq n \simeq \mathcal{Y} =: p$,
- $\sigma \in (0,1)$ arbitrary, fixed.

Then the sinc-BK hp-FEM approximations $Q_k^{-s}(\mathcal{L}_{hp})f \in S_0^q(\Omega, \mathcal{T}_{geo,\sigma}^{L,n})$ of $u \in \mathbb{H}^s(\Omega)$ satisfy

$$\|u - Q_k^{-s}(\mathcal{L}_{hp})f\|_{\mathbb{H}^s(\Omega)} \lesssim C \exp(-b' \sqrt[6]{N}) \ ,$$

with $N := Kp^4 \simeq O(p^6)$ overall degrees of freedom.

Exponential Convergence of sinc-BK hp-FEM

Corollary: $Q_k^{-s}(\mathcal{L}_{hp})f \in S_0^q(\Omega, \mathcal{T}_{geo,\sigma}^{L,n})$, and $\bar{n} := \dim(S_0^q(\Omega, \mathcal{T}_{geo,\sigma}^{L,n})) = O(p^4)$ implies

$$d_{\bar{n}}(\mathbb{H}^s(\Omega), \mathcal{L}^{-s}(\mathcal{A})) \lesssim \exp(-b\bar{n}^{1/4}) \ .$$

Here,

- $\mathcal{L}^{-s} : \mathbb{H}^{-s}(\Omega) \to \mathbb{H}^s(\Omega)$ solution map of fractional diffusion problem,
- $\mathcal{A} \subset L^2(\Omega) \subset\subset \mathbb{H}^{-s}(\Omega)$ set of bounded, (piecewise) analytic in $\overline{\Omega}$ functions.

Remark: Constructive bound, slightly worse than

$$d_{\bar{n}}(\mathbb{H}^s(\Omega), \mathcal{L}^{-s}(\mathcal{A})) \leq c \exp(-b\bar{n}^{1/2})$$

obtained from

$$V_{\bar{n}} := \operatorname{span}\{\mathcal{L}^{-s}(f_j) : j = 1, ..., O(p^2)\}$$

where $\sum_{1 \leq j \leq O(p^2)} c_j f_j \in \mathbb{P}_p(\Omega)$ is exponentially convergent approximation of f, i.e.

$$\left\| f - \sum_{1 \leq j \leq O(p^2)} c_j f_j \right\|_{\mathbb{H}^{-s}(\Omega)} \leq C \exp(-bp) \ .$$

[Melenk, J. M. On n-widths for elliptic problems. J. Math. Anal. Appl. **247** (2000), 272-289]

Conclusions

- hp-FE error analysis for spectral fractional Laplacian in curvilinear polygons Ω
- Exponential rates of convergence $C\exp(-b\sqrt[\alpha]{N})$ with $\alpha = 6$ in terms of NDOF N
- Analytic in $\overline{\Omega}$ data A, f without any boundary compatibility
- Resolution of the **boundary singularity** of u by *anisotropic, boundary fitted, geometrically refined triangulation* $\mathcal{T}$ *of* Ω
- Mesh design and construction of hp-interpolants via block-structured partitions from finite patch-catalog $\mathfrak{P}$
- Automatic geometric mesh generation by NETGEN, NGSOLVE (J. Schöberl)
- CST-Localization with a) one extra variable y, b) local, *degenerate* elliptic, divergence form operator
- CST+hp-Semidiscretization in $(0, \mathcal{Y}) \Rightarrow$ decoupling into $\mathcal{M}$ decoupled, local, singularly perturbed elliptic problems in Ω; Multiple length scales, *variational setting* (a-posteriori error estimation)
- Sinc-BK approximation: exponential convergence and decoupling into $2K + 1$ decoupled, local, singularly perturbed elliptic problems in Ω; Multiple length scales.
- Tensor Product hp-FE space in $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$ allows solution *without* numerical diagonalization.

Numerical Experiments

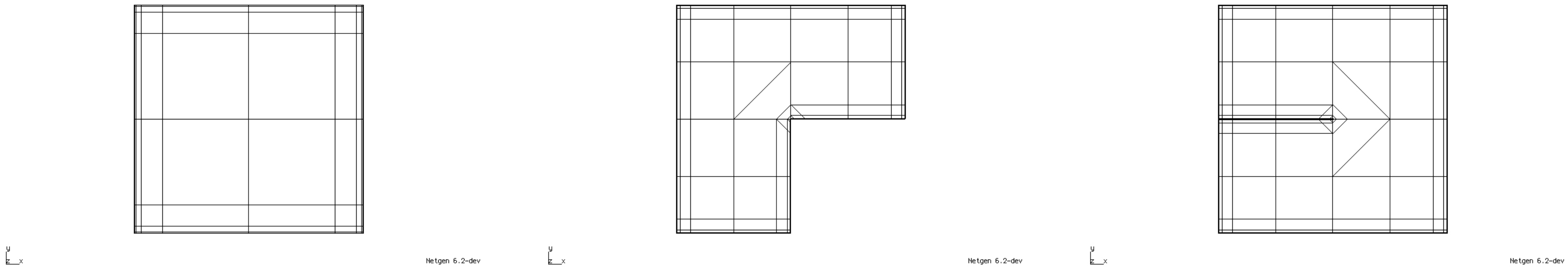

Figure 4: Examples of geometric boundary layer meshes for the three domains $\Omega = \Omega_j$, $j = 1, 2, 3$, ordered from left to right

- Experiments performed with hp-FE solver **NGSOLVE** (https://ngsolve.org/).
- Geom. Bdry. Layer meshes $\mathcal{T}^{L,n}_{geo,\sigma}$ in Ω *automatically generated* with NETGEN.

  [J. Schöberl: **NETGEN** – an advancing front 2d/3d-mesh generator based on abstract rules. J. Comput. Visual. Sci. **1** (1997) 41-52]
- Error Measure:

$$e(\tilde{u}) = \left| d_s \int_\Omega f(u^{\text{fine}} - \tilde{u}) \, \mathrm{d}x' \right|^{1/2},$$

$$\|u - \mathrm{tr}_\Omega \, \mathscr{U}^p\|^2_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}^p)\|^2_{L^2(y^\alpha, \mathcal{C})} = d_s \int_\Omega f(u - \mathrm{tr}_\Omega \, \mathscr{U}^p) \, \mathrm{d}x'.$$

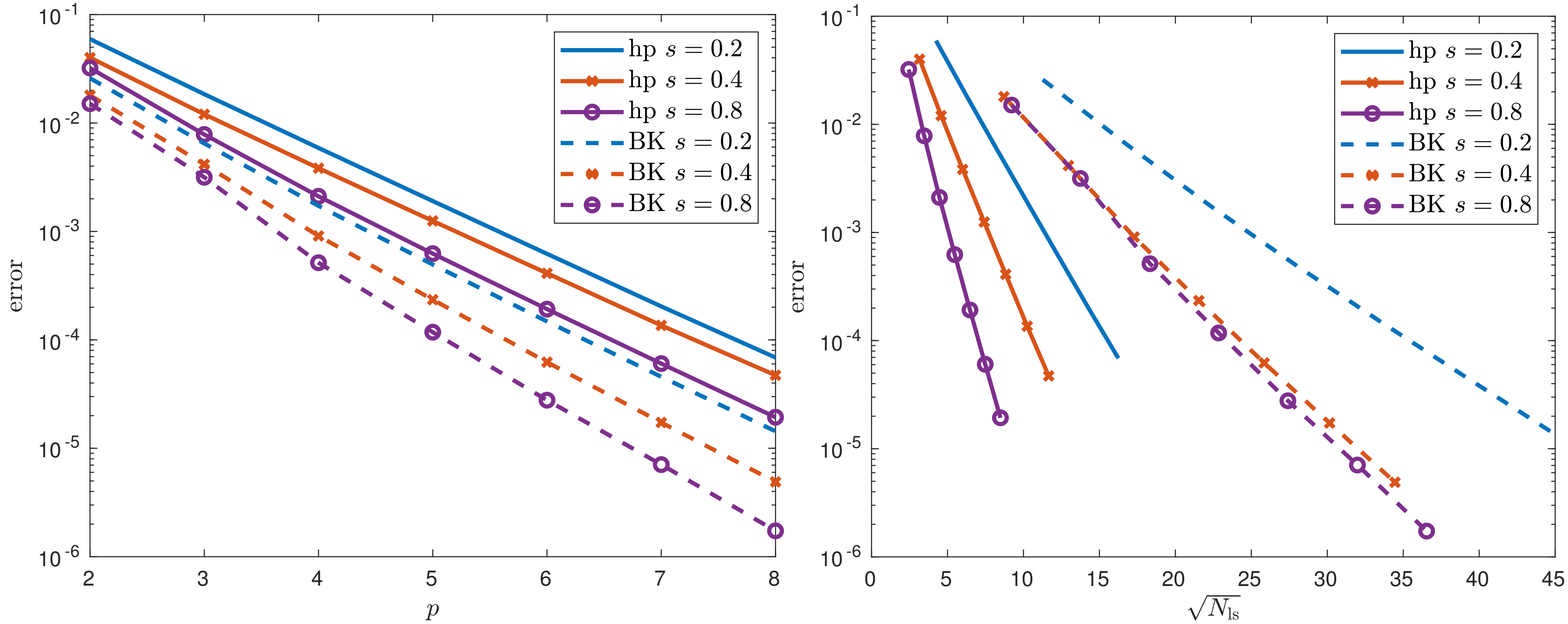

Figure 5: Error convergence of the CST-hpFEM and the sinc-BK hpFEM for the domain Ω_1, depicted versus the polynomial degree p and $N_{\mathrm{ls}}^{1/2}$, where N_{ls} denote the number of linear systems that need to be solved. Solid lines correspond to CST-hpFEM and dashed lines to the sinc-BK FEM. Results for $s = 0.2$, $s = 0.4$ and $s = 0.8$ are shown.

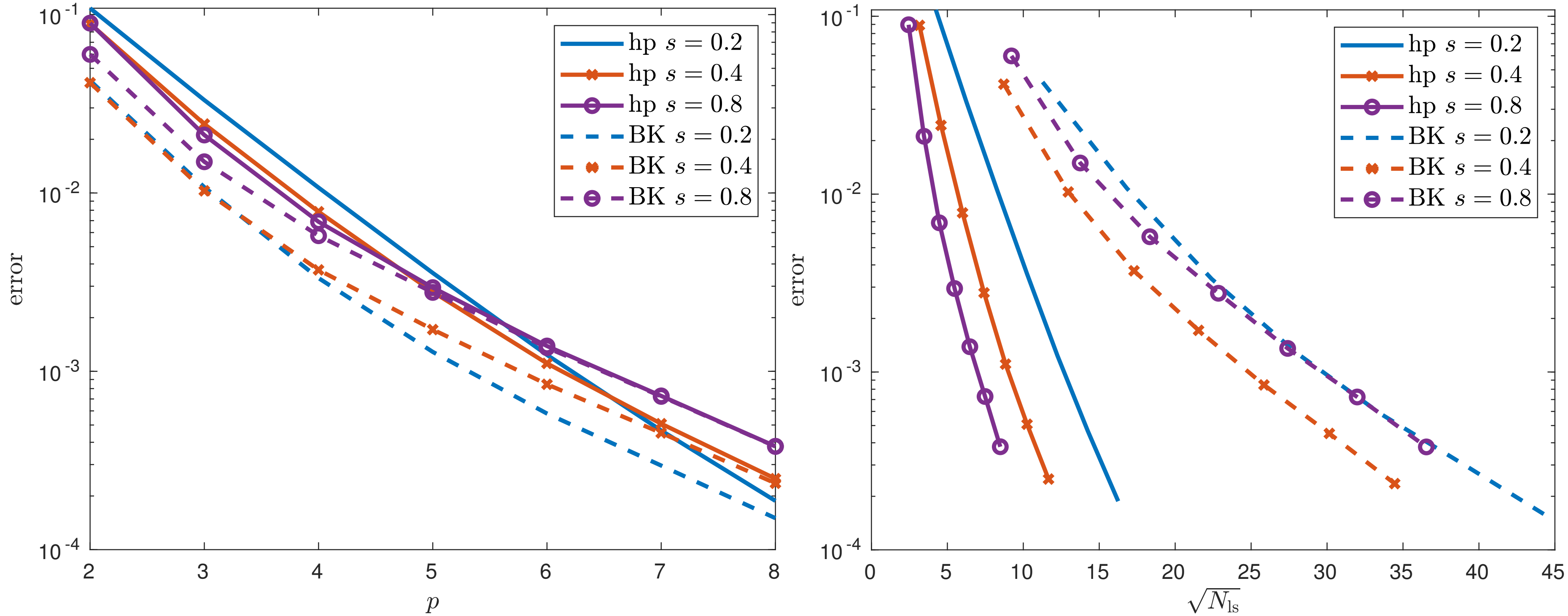

Figure 6: Error convergence of the CST-hpFEM and the sinc-BK hpFEM for L-shaped domain Ω_2, depicted versus the polynomial degree p and $N_{\text{ls}}^{1/2}$, the square root of the number of linear systems to be solved.

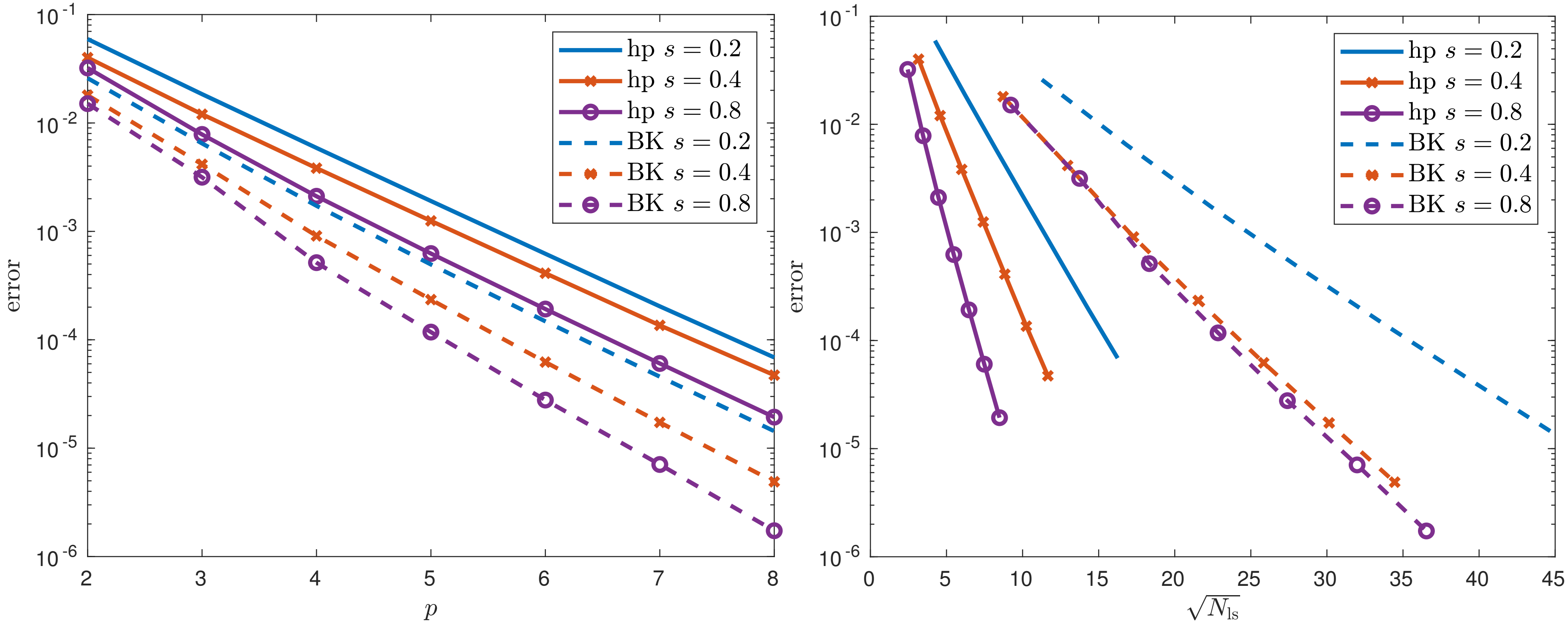

Figure 7: Error convergence of the CST-*hp*FEM and the sinc-BK *hp*FEM for the slit domain Ω_3, depicted versus the polynomial degree p and $N_{ls}^{1/2}$, the square root of the number of linear systems to be solved.

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Thank You.