Towards a coproduct for multiple q-zeta values

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Recent Advances in Anabelian Geometry and Related Topics
28 Apr - 02 May 2025
ICMS
Edinburgh Futures Institute, UK

Report on joint work in progress with:
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https://arxiv.org/abs/2504.19661

Multiple q-zeta values

Definition

For $s_1 \ldots, s_{l-1} \geqslant 0$ and $Q_1(t) \ldots, Q_{l-1}(t) \in \mathbb{Q}[t]$ and $s_l \geqslant 1$ and $Q_l(t) \in t\mathbb{Q}[t]$ we define

$$\zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = \sum_{0<n_1<\cdots<n_l} \frac{Q_1(q^{n_1}) \ldots Q_l(q^{n_l})}{(1-q^{n_1})^{s_1} \cdots (1-q^{n_l})^{s_l}} \in \mathbb{Q}[[q]].$$

- Recall, for natural numbers $s_1, ..., s_{l-1} \geqslant 1$ and $s_l \geqslant 2$ the sum

$$\zeta(s_1, ..., s_l) = \sum_{0<n_1<\ldots<n_l} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}}$$

  is called a multiple zeta value (MZV) of weight $s_1 + \ldots + s_l$ and depth l.
- If $s_l > 1$ and $s_1 \ldots, s_{l-1} \geqslant 1$, then we get q-analogues of multiple zeta values:

$$\lim_{q \to 1} (1-q)^{s_1+\cdots+s_l} \zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = Q_1(1) \ldots Q_l(1) \cdot \zeta(s_1, \ldots, s_l) \, .$$

Algebra of multiple q-zeta values

Definition

We define the algebra of multiple q-zeta values to be the $\mathbb{Q}$-algebra

$$\mathcal{Z}_q := \left\langle \zeta_q(s_1,\ldots,s_l;Q_1,\ldots,Q_l) \;\middle|\; l \geqslant 0,\ \deg(Q_j) \leqslant s_j \right\rangle_{\mathbb{Q}}.$$

- $\mathcal{Z}_q$ is an $\mathbb{Q}$-algebra, for example it is

  $$\zeta_q(s_1;Q_1)\cdot\zeta_q(s_2;Q_2) = \zeta_q(s_1,s_2;Q_1,Q_2)+\zeta_q(s_2,s_1;Q_2,Q_1)+\zeta_q(s_1+s_2;Q_1\cdot Q_2)\,,$$

  and clearly $\deg Q_1 \cdot Q_2 \leqslant s_1 + s_2$ if $\deg Q_j \leqslant s_j$ for $j = 1, 2$.
- a notion of weight and depth is defined using a spanning set of $\mathcal{Z}_q$. Caution, we have $\zeta_q(s;Q) = \zeta_q(s+1,(1-t)\cdot Q)$.
- we have conjectures for the weight, resp. weight and depth, graded pieces of $\mathcal{Z}_q$ similar to the conjectures of Zagier resp. of Broadhurst and Kreimer

Schlesinger-Zudilin model for multiple q-zeta values

We call a Schlesinger-Zudilin multiple q-zeta value the q-series

$$\zeta_q^{\mathsf{SZ}}(s_1,\ldots,s_l) = \zeta_q(s_1,\ldots,s_l;t^{s_1},\ldots,t^{s_l}).$$

We have

$$\begin{aligned}
\mathcal{Z}_q &= \left\langle \zeta_q^{\mathsf{SZ}}(s_1,\ldots,s_l) \,\middle|\, l \geqslant 0,\ s_1,\ldots,s_{l-1} \geqslant 0, s_l \geqslant 1, \right\rangle_{\mathbb{Q}}, \\
\mathcal{Z}_q^{\circ} &= \left\langle \zeta_q(s_1,\ldots,s_l;Q_1,\ldots,Q_l) \in \mathcal{Z}_q \,\middle|\, Q_1,\ldots,Q_l \in t\mathbb{Q}[t] \right\rangle_{\mathbb{Q}} \\
&= \left\langle \zeta_q^{\mathsf{SZ}}(s_1,\ldots,s_l) \,\middle|\, l \geqslant 0,\ s_1,\ldots,s_l \geqslant 1 \right\rangle_{\mathbb{Q}}.
\end{aligned}$$

Observe $\mathcal{Z}_q^{\circ} \subset \mathcal{Z}_q$ is a subalgebra. We set

$$\mathcal{Z}_{q,123} = \left\langle \zeta_q^{\mathsf{SZ}}(s_1,\ldots,s_l) \,\middle|\, l \geqslant 0,\ s_1,\ldots,s_l \in \{1,2,3\} \right\rangle_{\mathbb{Q}}.$$

Spanning Conjectures (Bachmann-K.)

Each of the following inclusions is an equality:

$$\mathcal{Z}_{q,123} \subseteq \mathcal{Z}_q^{\circ} \subseteq \mathcal{Z}_q.$$

Motivation

The final aim is to prove the conjecture $\mathcal{Z}_{q,123} = \mathcal{Z}_q$ along the path of Brown's proof for Hoffman's conjecture, i.e., the multiple zeta values $\zeta(s_1, ..., s_d)$ with $s_i \in \{2, 3\}$ span the spaces of all multiple zeta values. A major first step in this direction is the following conjecture:

Structure Conjectures (Bachmann-K. + Burmester + ...)

(i) $\mathcal{Z}_q$ is a free polynomial algebra

(ii) $\mathcal{Z}_q \cong \mathrm{qMF} \otimes \mathcal{H}_q$ for some commutative Hopf algebra $(\mathcal{H}_q, ш, \Delta_u)$

(iii) $(\mathcal{H}_q, ш, \Delta_u) = \mathcal{U}(\mathfrak{bm}_0)^\vee$ for some Lie algebra $(\mathfrak{bm}_0, \{\ ,\ \}_u)$

(iv) we have a coaction $\Delta_u : \mathcal{Z}_q \to \mathcal{H}_q \otimes \mathcal{Z}_q$

- understand the restriction of Δ_u to $\mathcal{Z}_{q,123}$
- find a multiple q-version of Zagier's theorem
- refined conjectures on the generation of the Lie algebra $\mathfrak{bm}_0$ would imply the beautiful dimension conjectures for $\mathcal{Z}_q$.
- understand the relations in between the 123-multiple q-zeta values

We worked out this circle of ideas for formal multiple zeta values, see `https://arxiv.org/abs/2406.13630`.

Post-Lie algebra

Definition 1 (Post-Lie algebra)

A *post-Lie algebra* $\left(\mathfrak{g}, [\_,\_], \rhd\right)$ is a Lie algebra $(\mathfrak{g}, [\_,\_])$ together with a bilinear map $\rhd\colon\ \mathfrak{g} \bigotimes \mathfrak{g} \to \mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$

$$x \rhd [y, z] = [x \rhd y, z] + [y, x \rhd z] \tag{1}$$

$$[x, y] \rhd z = x \rhd (y \rhd z) - (x \rhd y) \rhd z - y \rhd (x \rhd z) + (y \rhd x) \rhd z. \tag{2}$$

- Post Lie algebras are currently under research in various fields of mathematics (e.g. Zentralblatt search found "post-Lie algebra" in 103 documents)
- We will present and study further examples that are motivated by multiple zeta values or its q-analogs in below.

Proposition 2

Given a post-Lie algebra $\left(\mathfrak{g}, [\_,\_], \rhd\right)$, then $(\mathfrak{g}, \{\_,\_\})$ with

$$\{x, y\} := x \rhd y - y \rhd x + [x, y]$$

is also a Lie algebra. We call this Lie bracket the *post-Lie bracket* .

Extension of the post Lie product

Definition 3

Let $(\mathfrak{g}, [\_,\_], \rhd)$ be a post-Lie algebra. We extend the post-Lie product to a bilinear pairing $\rhd\colon \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ via

$$A \rhd \mathbf{1} = (A \mid \mathbf{1}) \tag{3}$$

$$\mathbf{1} \rhd A = A \tag{4}$$

$$xA \rhd y = x \rhd (A \rhd y) - (x \rhd A) \rhd y \tag{5}$$

$$A \rhd BC = \big(A_{(1)} \rhd B\big)\big(A_{(2)} \rhd C\big) \tag{6}$$

for all $A, B, C \in \mathcal{U}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$.

Definition 3 yields a unique and well-defined bilinear product on $\mathcal{U}(\mathfrak{g})$.

Remark

Let $x, y, z \in \mathfrak{g}$ and $xy, yx \in \mathcal{U}(\mathfrak{g})$, then (2) corresponds by (5) to the identity

$$[x, y] \rhd z = (xy - yx) \rhd z.$$

Theorem (Ebrahimi-Fard, Munthe-Kaas, Lundervold)

Let $(\mathfrak{g}, [\ ,\ ], \rhd)$ *be a post-Lie algebra.*

(i) *The Grossman-Larson product on* $\mathcal{U}(\mathfrak{g})$ *given by*

$$A \circledast B := A_{(1)}(A_{(2)} \rhd B) \qquad \text{for } A, B \in \mathcal{U}(\mathfrak{g}) \tag{7}$$

is associative.

(ii) $\big(\mathcal{U}(\mathfrak{g}), \circledast, \Delta\big)$ *is a Hopf algebra.*

It is $(\mathcal{U}(\mathfrak{g}), \mathrm{conc}, \Delta, \rhd)$ a post-Hopf algebra and $\big(\mathcal{U}(\mathfrak{g}), \circledast, \Delta\big)$ its subadjacent Hopf algebra.

Theorem (Ebrahimi-Fard, Munthe-Kaas, Lundervold)

Let $\mathrm{g} = (\mathfrak{g}, [\_,\_], \rhd)$ *be a post-Lie algebra and let* $\bar{\mathrm{g}} = (\mathfrak{g}, \{\_,\_\})$ *be the related Lie algebra from Proposition 2. Then there is an isomorphism of Hopf algebras*

$$(\mathcal{U}(\bar{\mathrm{g}}), \mathrm{conc}, \Delta) \cong (\mathcal{U}(\mathrm{g}), \circledast, \Delta)$$
$$a_1 a_2 \cdots a_n \mapsto a_1 \circledast a_2 \circledast \ldots a_n.$$

Details can be found here `https://arxiv.org/abs/1410.6350v2`.

Remark

(i) Let $a, b \in \mathfrak{g}$. Since $\Delta(a) = a \otimes \mathbf{1} + \mathbf{1} \otimes a$ we get

$$a \circledast b = ab + a \rhd b$$

and thus by symmetry the post-Lie bracket from Proposition 2 satisfies

$$\{a, b\} = a \circledast b - b \circledast a. \tag{8}$$

(ii) If $\Delta(a) = a \otimes a$, then

$$a \circledast b = a(a \rhd b).$$

Remark 4

Let $A, b_1 \cdots b_m \in \mathcal{U}(\mathfrak{g})$. We can calculate $A \circledast b_1 \ldots b_m$ by applying (6) iteratively

$$A \circledast b_1 \cdots b_m = A_{(1)}(A_{(2)} \rhd b_1)(A_{(3)} \rhd b_2) \cdots (A_{(m+1)} \rhd b_m). \tag{9}$$

Then, each factor $A_{i+1} \rhd b_i$ can be determined iteratively using (5).

Assume that $(\mathfrak{g}, [\ ,\ ], \rhd)$ is a graded post-Lie algebra, i.e., we have $\mathfrak{g} = \bigoplus_{n \geqslant 0} \mathfrak{g}_n$ with each $\mathfrak{g}_n$ finite dimensional and $[\mathfrak{g}_n, \mathfrak{g}_m],\ \mathfrak{g}_n \rhd \mathfrak{g}_m \subset \mathfrak{g}_{n+m}$ for all $m, n \geqslant 0$. The grading on $\mathfrak{g}$ induces a grading on the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\overline{\mathfrak{g}})$ as Hopf algebras.
We may summarize our discussion in the commutative diagrams:

$$\begin{tikzcd}
\big(\mathcal{U}(\mathfrak{g})^\vee, ⧢, \Delta_{\mathrm{dec}}\big) \arrow[r, leftrightsquigarrow, "\text{graded dual}"] \arrow[d, twoheadrightarrow, "\text{mod products}"'] & \big(\mathcal{U}(\mathfrak{g}), \mathrm{conc}, \Delta_{⧢}\big) \\
\big(\mathrm{indec}(\mathcal{U}(\mathfrak{g})^\vee), \delta\big) \arrow[r, leftrightsquigarrow, "\text{graded dual}"] & \big(\mathfrak{g}, [\ ,\ ]\big). \arrow[u, hookrightarrow]
\end{tikzcd} \tag{10}$$

Here, $\mathrm{indec}(\mathcal{U}(\mathfrak{g})^\vee)$ is the Lie coalgebra of the indecomposables elements of $\mathcal{U}(\mathfrak{g})^\vee$.
We analyse the change of the picture (10) by altering the Lie bracket with a post-Lie structure:

$$\begin{tikzcd}
\big(\mathcal{U}(\mathfrak{g})^\vee, ⧢, \Delta_{\circledast}\big) \arrow[r, leftrightsquigarrow, "\text{graded dual}"] \arrow[d, twoheadrightarrow, "\text{mod products}"'] & \big(\mathcal{U}(\mathfrak{g}), \circledast, \Delta_{⧢}\big) \\
\big(\mathrm{indec}(\mathcal{U}(\mathfrak{g})^\vee), \delta_{\circledast}\big) \arrow[r, leftrightsquigarrow, "\text{graded dual}"] & \big(\mathfrak{g}, \{\ ,\ \}\big). \arrow[u, hookrightarrow]
\end{tikzcd} \tag{11}$$

Free Lie algebras

Notation 5

- $V = \{v_0, v_1, \ldots\}$ is a countable set, whose elements v_i we call letters
- V^* is the set of words w with letters in V
- $\mathbf{1}$ denotes the empty word, we let $\mathbf{1} \in V^*$
- $\mathbb{Q}\langle V\rangle$ is the non-commutative free algebra equipped with concatenation
- $\mathrm{Lie}(V) \subset \mathbb{Q}\langle V\rangle$ is the free Lie Algebra on the set V
- $\mathcal{U}(\mathrm{Lie}(V)) = (\mathbb{Q}\langle V\rangle, \mathrm{conc}, \Delta_{⧢})$ is the universal enveloping algebra

We fix a duality pairing $(\,|\,)$ and identify $V^\vee$ with V via $v \mapsto (v|\,)$.
The map dual to $\Delta_{⧢}$ is the usual shuffle product ⧢, which is recursively defined on V^* by $\mathbf{1} ⧢ w = w = w ⧢ \mathbf{1}$ and

$$v_i w_1 ⧢ v_j w_2 = v_i(w_1 ⧢ v_j w_2) + v_j(v_i w_1 ⧢ w_2)$$

for $v_i, v_j \in V$ and $w, w_1, w_2 \in V^*$ and extended by $\mathbb{Q}$-linearity.

Calculating the coproduct $\Delta_{\circledast}$

A naive way determine the coproduct $\Delta_{\circledast}$ is to calculate tables of a large number of $\circledast$-products and then use the duality

$$\big(\Delta_{\circledast}(w) \mid f \otimes g\big) = \big(f \circledast g \mid w\big)$$

for all $w \in V^*$ and $f, g \in V^*$ to collect all the coefficients of $\Delta_{\circledast}(w)$. In fact it suffices to calculate

$$\big(\lhd^{\mathrm{irr}}(w) \mid f \otimes v\big) = \big(f \rhd v \mid w\big)$$

for all $w \in V^*$, $f \in V^*$ and $v \in V$.

Theorem 6

Let V be an alphabet and $\big(\mathrm{Lie}(V), [\ ,\ ], \rhd\big)$ be a graded post-Lie algebra. Then, we have for and $A \in V^*$

$$\Delta_{\circledast}(A) = \sum_{A = A_1 \cdots A_n} (A_1 \otimes \mathbf{1}) \sqcup\!\sqcup_{\bullet} \lhd^{\mathrm{irr}}(A_2) \sqcup\!\sqcup_{\bullet} \cdots \sqcup\!\sqcup_{\bullet} \lhd^{\mathrm{irr}}(A_n).$$

Here, $\lhd^{\mathrm{irr}}$ is the dual map to the triangle map $\rhd$ restricted to letters in the right factors. The product $\sqcup\!\sqcup_{\bullet}$ on $\mathbb{Q}\langle V\rangle^{\otimes 2}$ is the shuffle product on the left factor and concatenation on the right factor.

A nonassociative point of view on free post-Lie products

Recall a set with a binary operation is called a magma. We consider here the free magma $M(V)$ generated by a set V with operation $\star : M(V) \times M(V) \to M(V)$. The $\mathbb{Q}$-vector space $M(V)_{\mathbb{Q}}$ spanned by $M(V)$ is nothing else than the free noncommutative, nonassociative algebra generated by V.

A derivation is a linear map $d : M(V)_{\mathbb{Q}} \to M(V)_{\mathbb{Q}}$ such that $d(a \star b) = da \star b + a \star db$. Let $(M(V)_{\mathbb{Q}}, \star)$ be the free magma on V. Assume, we are given another map

$$\begin{aligned} \rhd : M(V)_{\mathbb{Q}} \times V &\to M(V)_{\mathbb{Q}} \\ (t, v) &\mapsto t \rhd v, \end{aligned} \tag{12}$$

then for each $t \in M(V)$ we extend $t \rhd$ to a derivation on $(M(V)_{\mathbb{Q}}, \star)$. This allows us to view $\rhd$ as a bilinear pairing on $M(V)_{\mathbb{Q}}$.

Notation 7 (Non-commutative setup)

Let $(M(V), \star, \rhd)$ as before. Denote by $\big(\mathbb{Q}\langle M(V)\rangle,\ \cdot,\ \Delta\big)$ the free associative, noncommutative $\mathbb{Q}$-algebra generated by $M(V)$ with its natural coproduct. We extend the pairing $\rhd$ on $M(V)_{\mathbb{Q}}$ further to a bilinear pairing

$$\rhd\colon \mathbb{Q}\langle M(V)\rangle \times \mathbb{Q}\langle M(V)\rangle \to \mathbb{Q}\langle M(V)\rangle$$

via the requirements

$$A \rhd \mathbf{1} = (A \mid \mathbf{1}) \tag{13}$$

$$\mathbf{1} \rhd A = A \tag{14}$$

$$(x \cdot A) \rhd y = x \rhd (A \rhd y) - (x \rhd A) \rhd y \tag{15}$$

$$A \rhd (B \cdot C) = \big(A_{(1)} \rhd B\big)\big(A_{(2)} \rhd C\big) \tag{16}$$

for all $A, B, C \in \mathbb{Q}\langle M(V)\rangle$ and $x, y \in M(V)$.

From magmas to Lie algebras

The identity on V extends naturally to homomorphism

$$\begin{aligned} \mathrm{Lie}: \quad M(V)_{\mathbb{Q}} &\to \mathrm{Lie}(V)_{\mathbb{Q}} \\ a \star b &\mapsto [a, b], \end{aligned}$$

whose kernel is the ideal I_{Lie} generated by $x \star x$ and $(x \star y) \star z + (y \star z) \star x + (z \star x) \star y$. We use the same notation for an element $x \in M(V)_{\mathbb{Q}}$ as well as for its class $x \in \mathrm{Lie}(V)$.

Proposition

The above described extension of a map $\rhd : M(V)_{\mathbb{Q}} \times V \to M(V)_{\mathbb{Q}}$ determines a post-Lie algebra $\big(\mathrm{Lie}(V), [\ ,\ ], \rhd\big)$, if and only if $\rhd$ descends to a bilinear pairing on $\mathrm{Lie}(V)$ and for all $x, y, z \in M(V)_{\mathbb{Q}}$

$$\mathrm{Lie}\Big(\big(x \star y - x \cdot y + y \cdot x\big) \rhd z\Big) = 0$$

Application 1: Ihara bracket and Goncharov coproduct

As a motivation we consider a Lie algebra, which is well-known in the theory of multiple zeta values.

Definition

Keep the notations 5 and 7. Let $V_0 \subset V = \{v_0, v_1, v_2, ...\}$ be a subset and consider

$$\begin{aligned} \rhd_I : M(V)_{\mathbb{Q}} \times V &\to M(V)_{\mathbb{Q}} \\ (t, v) &\mapsto \begin{cases} 0 & v \in V_0 \\ v \star t & v \in V \backslash V_0 \, . \end{cases} \end{aligned}$$

We denote by $\rhd_I$ also its extensions to a derivation on $M(V)_{\mathbb{Q}}$ and to $\mathbb{Q}\langle M(V) \rangle$.

Lemma

The pairing $\rhd_I$ descends to $\mathrm{Lie}(V)$.

Proof: Obviously, it is necessary and sufficient for $\rhd$ to descend to a pairing on $\mathrm{Lie}(V)$, that $I_{\mathrm{Lie}} \rhd M(V)_{\mathbb{Q}} \subseteq I_{\mathrm{Lie}}$ and $M(V)_{\mathbb{Q}} \rhd I_{\mathrm{Lie}} \subseteq I_{\mathrm{Lie}}$. The first condition follows by definition of $\rhd_I$ and the second holds as $t \rhd_I$ is a derivation for all $t \in M(V)_{\mathbb{Q}}$ $\square$

Lemma 8

We have for $v \in V \backslash V_0$

$$(t_1 \cdot ... \cdot t_n) \rhd_I v = (((v \star t_1) \star t_2)...) \star t_n \, .$$

Proof. We prove the claim by induction. Assume it holds for $n-1$ factors, then by construction and using (16)

$$\begin{aligned}
(t_1 \cdot ... \cdot t_n) \rhd_I v &= t_1 \rhd_I \Big((t_2 \cdot ... \cdot t_n) \rhd_I v \Big) - \Big(\big(t_1 \rhd_I (t_2 \cdot ... \cdot t_n) \big) \rhd_I v \Big) \\
&= (((t_1 \rhd_I v) \star t_2)...) \star t_n + \sum_{i=2}^{n} \Big((((v \star t_2)...) \star (t_1 \rhd_I t_i))... \Big) \star t_n \\
&\qquad - \sum_{i=2}^{n} \Big((((v \star t_2)...) \star (t_1 \rhd_I t_i))... \Big) \star t_n \\
&= (((v \star t_1) \star t_2)...) \star t_n .
\end{aligned}$$

$\square$

Theorem

The triple $\left(\mathrm{Lie}(V), [\_,\_], \rhd_I\right)$ *is a post-Lie algebra.*

Proof. We have by definition of $\rhd_I$

$$(t_1 \star t_2) \rhd_I v = v \star (t_1 \star t_2)$$

and by Lemma 8

$$(t_1 \cdot t_2 - t_2 \cdot t_1) \rhd_I v = (v \star t_1) \star t_2 - (v \star t_2) \star t_1$$

Applying the Lie-map gives by means of the Jacobi relation

$$\begin{aligned} &\mathrm{Lie}\Big(v \star (t_1 \star t_2) - \big((v \star t_1) \star t_2 - (v \star t_2) \star t_1\big)\Big) \\ &= [v, [t_1, t_2]] - [[v, t_1], t_2] + [[v, t_2], t_1] \\ &= 0. \end{aligned}$$

$\square$

We write $\{\_,\_\}_I$ for the induced post Lie bracket and call $\{\_,\_\}_I$ the Ihara bracket. The universal algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to the free non-commutative algebra $\mathbb{Q}\langle V\rangle$ and the extension 3 yields a bilinear pairing

$$\rhd_I : \mathbb{Q}\langle V\rangle \times \mathbb{Q}\langle V\rangle \to \mathbb{Q}\langle V\rangle.$$

Because of Remark 4 it suffices to understand $A \rhd_I v$ for $A \in \mathbb{Q}\langle V\rangle$ and $v \in V$.

Proposition 9

For all $A \in V^*$ and $v \in V$, we have for $\rhd_I$ given by Definition 3

$$A \rhd_I v = \chi(A,v)\, S\big(A_{(1)}\big) v A_{(2)},$$

where

$$\chi(A,v) = \begin{cases} \delta_{A,\mathbf{1}}, & \text{if} \quad v \in V_0, \\ 1, & \text{else}, \end{cases}$$

Proof. By Lemma 8 and the definitions of the coproduct Δ and the antipode S for $\mathbb{Q}\langle V\rangle$ we have for $v \in V \backslash V_0$ and for $A = a_1 \cdots a_n \in \mathcal{U}(\mathrm{Lie}(V))$, where $a_1, \ldots, a_n \in \mathrm{Lie}(V)$, that

$$A \rhd_I v = [\ldots[[v,a_1],a_2],\ldots,a_n] = S\big(A_{(1)}\big) v A_{(2)}.$$

□

explicit Grossman-Larson product for the Ihara bracket

Theorem 10

Let $A \in \mathbb{Q}\langle V\rangle$ and $w = w_1 v_{i_1} \cdots w_d v_{i_d} w_{d+1} \in V^*$ with $w_1, \ldots, w_{d+1} \in V_0^*$ and $v_{i_1}, \ldots, v_{i_d} \in V \backslash V_0$. Then

$$A \circledast_{\mathsf{I}} w = A_{(1)} w_1 S\big(A_{(2)}\big) v_{i_1} A_{(3)} w_2 \cdots w_d S\big(A_{(2d)}\big) v_{i_d} A_{(2d+1)} w_{d+1}.$$

Proof. We deduce from (7) and several applications of (6) that

$$\begin{aligned} A \circledast_{\mathsf{I}} w &= A_{(1)}(A_{(2)} \rhd_I w) \\ &= A_{(1)}\big(A_{(2)} \rhd_I w_1 v_{i_1}\big) \cdots \big(A_{(d+1)} \rhd_I w_d v_{i_d}\big)\big(A_{(d+2)} \rhd_I w_{d+1}\big) \\ &= A_{(1)} w_1 \big(A_{(2)} \rhd_I v_{i_1}\big) \cdots w_d \big(A_{(d+1)} \rhd_I v_{i_d}\big) w_{d+1}. \end{aligned}$$

Now recall Proposition 9, i.e. we have for $v_i \in V \backslash V_0$ that

$$A \rhd_I v_i = S\big(A_{(1)}\big) v_i A_{(2)},$$

which in turn proves the claimed formula. $\square$

A variation of Goncharov's coproduct

Proposition 11

For $a_1, \ldots, a_n \in V$, we have

$$\lhd_I^{\mathrm{irr}}(a_1 \cdots a_n) = \sum_{j=1}^{n} \mathbb{I}\big(S(a_1 \cdots a_{j-1}) \sqcup\!\sqcup\, a_{j+1} \cdots a_n;\, a_j\big) \otimes a_j .$$

Here $\mathbb{I} : \mathbb{Q}\langle V \rangle \times V \to \mathbb{Q}\langle V \rangle$ is the $\mathbb{Q}$-linear map defined for words $w \in V^*$ and a letter $v \in V$ by

$$\mathbb{I}(w; v) = \chi(w, v)\, w.$$

With the the explicit formula for the reduced triangle map in Proposition 11 and Theorem 6, we are able to describe the dual coproduct Δ_I for the Grossman-Larson product $\circledast_{\mathrm{I}}$ corresponding to the Ihara bracket.

Theorem

For $a_1, \ldots, a_n \in V$, we have

$$\Delta_{\mathrm{I}}(a_1 \cdots a_n) = \sum_{\substack{0 \leqslant k \leqslant n \\ 0 \leqslant i_1 < j_1 \leqslant i_2 < j_2 \leqslant \cdots \leqslant i_k < j_k \leqslant n}} \Big(a_1 \cdots a_{i_1} \sqcup\!\sqcup\, \mathbb{I}\big(S(a_{i_1+1} \cdots a_{j_1-1}) \sqcup\!\sqcup\, a_{j_1+1} \cdots a_{i_2}; a_{j_1}\big) \sqcup\!\sqcup \cdots$$

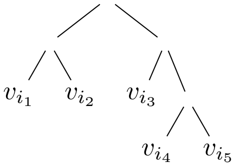
$$\sqcup\!\sqcup\, \mathbb{I}\big(S(a_{i_k+1} \cdots a_{j_k-1}) \sqcup\!\sqcup\, a_{j_k+1} \cdots a_n; a_{j_k}\big)\Big) \otimes a_{j_1} \cdots a_{j_k}.$$

Remark

In fact Δ_{I} equals precisely the Goncharov coproduct in the case $V = \{x_0, x_1\}$ and $V_0 = \{x_0\}$.

Application 2: The ari bracket of Ecalle

Let $V = \{v_0, v_1, v_2, ...\}$ and $V_0 = \{v_0\}$. There is a one-to-one correspondence of $M(V)$ and planar, rooted, binary trees with leaves labelled by elements of V. For example the tree



corresponds to $((v_{i_1} \star v_{i_2}) \star (v_{i_3} \star (v_{i_4} \star v_{i_5})))$. If we order the leaves of such a tree, e.g. from left to the right, then we may consider a tree t with d leaves as a map $V^d \to M(V)$. More generally we can even restrict ourselves to subsets of leaves with a fixed property.

Elements as function on indices

In the following we write $t(\mathbf{k})$, whenever we consider the element $t \in M(V)$ as a function

$$t : V^d \to M(V)$$
$$(v_{k_1}, \ldots, v_{k_d}) \mapsto t(v_{k_1}, \ldots, v_{k_d}).$$

For example, if $t = v_1 \star ((v_0 \star v_1) \star (v_4 \star v_0))$, then $t = t(v_1, v_0, v_1, v_4, v_0)$ and $t(v_7, v_2, v_8, v_9, v_{13})$ equals $v_7 \star ((v_2 \star v_8) \star (v_9 \star v_{13}))$. Even more generally we view such t as a function on the indices

$$t : \mathbb{N}^d \to M(V)$$
$$\mathbf{k} \mapsto t(\mathbf{k}),$$

where $t(\mathbf{k})$ is given by $t(v_{k_1}, \ldots, v_{k_d})$.

Notation

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- $|\mathbf{k}| = k_1 + ... + k_d$ for $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d$
- $\mathbf{l} \leqslant \mathbf{k}$ holds for $\mathbf{k}$, $\mathbf{l} \in \mathbb{N}^d$, if $l_i \leqslant k_i$ for all $i = 1, ..., d$
- we set $\binom{0}{-1} = \binom{-1}{0} = 0$ and $\binom{-1}{-1} = 1$

Definition

For $\mathbf{k} = (k_1, \ldots, k_d)$, $\mathbf{l} = (l_1, \ldots, l_d) \in \mathbb{N}^d$ we define their ari multiplicity by

$$\mathrm{m}_{\mathbf{k},\mathbf{l}} := (-1)^{|\mathbf{k}|+|\mathbf{l}|} \cdot \binom{k_1 - 1}{l_1 - 1} \cdots \binom{k_d - 1}{l_d - 1}.$$

Observe that, if $\mathbf{l} > \mathbf{k}$, then $\mathrm{m}_{\mathbf{k},\mathbf{l}} = 0$. Also $\mathrm{m}_{\mathbf{k},\mathbf{l}} = 0$, if $k_i = 0$ and $l_i \neq 0$ or if $k_i > 0$ and $l_i = 0$.

Definition 12

Keep the previous notation and consider

$$\begin{aligned} \rhd_a : M(V) \times V &\to M(V)_{\mathbb{Q}} \\ (t(\mathbf{k}), v_a) &\mapsto \begin{cases} 0 & v_a = v_0 \\ \sum_{\mathbf{l} \in \mathbb{N}^{\ell(\mathbf{k})}} \mathrm{m}_{\mathbf{k},\mathbf{l}} \;\; v_{a+|\mathbf{k}-\mathbf{l}|} \star t(\mathbf{l}) & v_a \neq v_0 \end{cases} \end{aligned}$$

and denote $\rhd_a$ also its extensions to a derivation on $M(V)_{\mathbb{Q}}$ and to $\mathbb{Q}\langle M(V)\rangle$.

Lemma

The pairing $\rhd_a$ on $M(V)_{\mathbb{Q}}$ descends to a bilinear pairing $\rhd_a$ on $\mathrm{Lie}(V)$.

Proof: We need to check $I_{\mathrm{Lie}} \rhd_a M(V)_{\mathbb{Q}} \subseteq I_{\mathrm{Lie}}$. $\square$

Lemma 13

We have for $v_a \neq v_0$

$$(t_1(\mathbf{k}_1) \cdot \ldots \cdot t_n(\mathbf{k}_n)) \rhd_a v_a$$
$$= \sum_{1 \leqslant \mathbf{l}_1, \ldots, \mathbf{l}_n} \Big(\prod_{i=1}^{n} \mathrm{m}_{\mathbf{k}_i, \mathbf{l}_i} \Big) \cdot \Big(\Big(\Big(v_{a + \sum\limits_{i=1}^{n} (|\mathbf{k}_i - \mathbf{l}_i|)} \star t_1(\mathbf{l}_1) \Big) \star t_2(\mathbf{l}_2) \Big) \ldots \Big) \star t_n(\mathbf{l}_n) \, .$$

Idea of proof. Using the general properties of $\rhd$, we obtain an equivalence of Lemma 13 and the following identity for the ari multiplicities :
Let $\mathbf{k}, \mathbf{l}, \mathbf{n} \in \mathbb{N}^d$ and $a, b \in \mathbb{N}$, then we have

$$\sum_{\mathbf{n} \leqslant \mathbf{r} \leqslant \mathbf{k}} (-1)^{|\mathbf{k}| + |\mathbf{r}|} \, m_{\mathbf{k} - \mathbf{n} + \mathbf{1}, \mathbf{r} - \mathbf{n} + \mathbf{1}} \cdot m_{a + |\mathbf{k}| - |\mathbf{r}|, b} = m_{a, b - |\mathbf{k} - \mathbf{n}|} \cdot$$

This identity for binomial coefficients can be proven by standard methods. $\square$

Theorem

Let $\rhd_a$ be given by Definition 12, then $\big(\mathrm{Lie}(V), [\_,\_], \rhd_a\big)$ is a post-Lie algebra.

Proof. Set $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2)$ and $\mathbf{l} = (\mathbf{l}_1, \mathbf{l}_2)$. We have by definition of $\rhd_a$ that

$$(t_1(\mathbf{k}_1) \star t_2(\mathbf{k}_2)) \rhd_a v = \sum_{1 \leqslant \mathbf{l} \leqslant \mathbf{k}} \mathrm{m}_{\mathbf{k},\mathbf{l}}\ v_{a+|\mathbf{k}-\mathbf{l}|} \star (t_1(\mathbf{l}_1) \star t_2(\mathbf{l}_2))$$

and by Lemma 13

$$\begin{aligned}
&(t_1(\mathbf{k}_1) \cdot t_2(\mathbf{k}_2) - t_2(\mathbf{k}_2) \cdot t_1(\mathbf{k}_1)) \rhd_a v \\
&= \sum_{1 \leqslant \mathbf{l} \leqslant \mathbf{k}} \mathrm{m}_{\mathbf{k},\mathbf{l}} \Big((v_{a+|\mathbf{k}-\mathbf{l}|} \star t_1(\mathbf{l}_1)) \star t_2(\mathbf{l}_2) - (v_{a+|\mathbf{k}-\mathbf{l}|} \star t_2(\mathbf{l}_2)) \star t_1(\mathbf{l}_1) \Big)
\end{aligned}$$

Applying the Lie-map gives by means of the Jacobi relation for each $\mathbf{l} = (\mathbf{l}_1, \mathbf{l}_2)$

$$\begin{aligned}
&\mathrm{Lie}\Big(v_{a+|\mathbf{k}-\mathbf{l}|} \star (t_1(\mathbf{l}_1) \star t_2(\mathbf{l}_2)) - \big((v_{a+|\mathbf{k}-\mathbf{l}|} \star t_1(\mathbf{l}_1)) \star t_2(\mathbf{l}_2) - (v_{a+|\mathbf{k}-\mathbf{l}|} \star t_2(\mathbf{l}_2)) \star t_1(\mathbf{l}_1)\big) \Big) \\
&= [v_{a+|\mathbf{k}-\mathbf{l}|}, [t_1(\mathbf{l}_1), t_2(\mathbf{l}_2)]] - [[v_{a+|\mathbf{k}-\mathbf{l}|}, t_1(\mathbf{l}_1)], t_2(\mathbf{l}_2)] + [[v_{a+|\mathbf{k}-\mathbf{l}|}, t_2(\mathbf{l}_2)], t_1(\mathbf{l}_1)] \\
&= 0.
\end{aligned}$$

$\square$

Definition

We write $\{\_,\_\}_a$ for the induced post Lie bracket and call $\{\_,\_\}_a$ the Ecalle ari bracket.

The universal algebra $\mathcal{U}(\mathrm{Lie}(V))$ is isomorphic to the free non-commutative algebra $\mathbb{Q}\langle V\rangle$ and the extension of $\rhd_a$, which we described in Definition 3, yields a bilinear pairing

$$\rhd_a : \mathbb{Q}\langle V\rangle \times \mathbb{Q}\langle V\rangle \to \mathbb{Q}\langle V\rangle.$$

Remark

We call the induced post Lie bracket the ari bracket, as it translates into the ARI Lie bracket of polynomial bimoulds using a certain map $\mathbb{Q}\langle V\rangle \to \mathbb{Q}[X_1, X_2, ..., Y_1, Y_2, ...]$. For more details we refer to the thesis of A. Burmester.

Proposition 14

For all $A(\mathbf{k}) \in \mathbb{Q}\langle V\rangle$ and $v_i \in V$, we have

$$A(\mathbf{k}) \rhd_a v_i = \begin{cases} 0 & \text{if} \quad v_i = v_0 \\ \sum_{\mathbf{l}} m_{\mathbf{k},\mathbf{l}}\, A(\mathbf{l}_{(1)}) v_{i+|\mathbf{k}-\mathbf{l}|} A(\mathbf{l}_{(2)}) & \text{if} \quad v_i \neq v_0. \end{cases}$$

Here we use the notation $\Delta(A(\mathbf{k})) = A_{(1)}(\mathbf{k}) \otimes A_{(2)}(\mathbf{k}) = A(\mathbf{k}_{(1)}) \otimes A(\mathbf{k}_{(2)})$.

Proposition 15

Let $A(\mathbf{k}) \in \mathcal{U}(\mathfrak{g})$ and $w = w_1 v_{i_1} \cdots w_d v_{i_d} w_{d+1} \in V^*$ with $w_1 = v_0^{j_1}, \ldots, w_{d+1} = v_0^{j_d}$ and $v_{i_1}, \ldots, v_{i_d} \in V \backslash \{v_0\}$. Then

$$A(\mathbf{k}) \circledast_a w = \sum_{\mathbf{l} \in \mathbb{N}^{\ell(\mathbf{k})}} m_{\mathbf{k},\mathbf{l}}\, A(\mathbf{k}_{(1)}) w_1 S\big(A(\mathbf{l}_{(2)})\big) v_{i_1 + |\mathbf{k}_{(2)} + \mathbf{k}_{(3)}| - |\mathbf{l}_{(2)} + \mathbf{l}_{(3)}|} A(\mathbf{l}_{(3)}) w_2 \cdots$$
$$w_d S\big(A(\mathbf{l}_{(2d)})\big) v_{i_d + |\mathbf{k}_{(2d)} + \mathbf{k}_{(2d+1)}| - |\mathbf{l}_{(2d)} + \mathbf{l}_{(2d+1)}|} A(\mathbf{l}_{(2d+1)}) w_{d+1}.$$

Some observations

Proposition

(i) It is $\big(\mathbb{Q}\langle v_1, v_2, v_3, ...\rangle, \circledast_a, \Delta\big)$ a sub Hopf algebra of $\mathcal{H}_a = \big(\mathbb{Q}\langle V\rangle, \circledast_a, \Delta\big)$.
(ii) It is $\big(\mathbb{Q}\langle x_0, x_1\rangle, \circledast_{\mathrm{I}}, \Delta\big) \cong \big(\mathbb{Q}\langle v_0, v_1\rangle, \circledast_a, \Delta\big)$ a sub Hopf algebra of $\mathcal{H}_a$.

Proposition

- Let $i \geqslant 0$, then the space

$$\mathfrak{o}_a(v_i) = \Big\{w \in \mathrm{Lie}(V \backslash \{v_0\}) \,\Big|\, w \rhd_a v_i = 0\Big\}$$

  is a Lie subalgebra of $(\mathrm{Lie}(V), \{\ ,\ \}_a)$.
- For all $w \in \mathfrak{o}_a(v_1)$ and for all $b \in \mathrm{Lie}(v_0, v_1)$ we have $\{w, b\}_a = 0$.

Corollary

Consider $\mathcal{U}(\mathfrak{o}_a(v_1))$ as a Hopf subalgebra of $\mathcal{H}_a$. Then for any $w \in \mathcal{U}(\mathfrak{o}_a(v_1))$ and any $b \in \mathbb{Q}\langle v_0, v_1\rangle$, we have

$$w \circledast_a b = b \circledast_a w.$$

Proposition 16

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $d \geqslant 1$, we have

$$\lhd_a^{\mathrm{irr}}(v_{k_1} \ldots v_{k_d}) = \sum_{i=1}^{d} \sum_{\substack{\mathbf{l}=(l_1,\ldots,\widehat{l_i},\ldots,l_d)\in\mathbb{N}^{d-1} \\ \mathbf{l}=\varnothing \text{ or } |\mathbf{l}|\leqslant k_i-1}}$$

$$\mathrm{m}_{\widehat{\mathbf{k}}^i+\mathbf{l},\widehat{\mathbf{k}}^i}\left(S(v_{k_1+l_1} \ldots v_{k_{i-1}+l_{i-1}}) \sqcup\!\sqcup (v_{k_{i+1}+l_{i+1}} \ldots v_{k_n+l_n})\right) \otimes v_{k_i-|\mathbf{l}|}.$$

Observe the case $\mathbf{l} = \varnothing$ only occurs if $d = 1$ and then

$$\lhd_a^{\mathrm{irr}}(v) = \mathbf{1} \otimes v$$

for any letter $v \in V$.

If $d \geqslant 2$ and we have $k_i = 0$ for some i, the summation index of the sum is empty and hence the sum vanishes by convention. So for any word w with at least two letters, the letter v_0 will never occur as a right factor in $\lhd_a^{\mathrm{irr}}(w)$. In particular, $\lhd_a^{\mathrm{irr}}(v_0^m) = 0$ for all $m \geqslant 2$.

Having a formula for the reduced triangle map, we can use Theorem 6 to give an explicit formula for the coproduct Δ_a dual to the Grossman-Larson product $\circledast_a$.

Theorem

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we have

$$\Delta_a(v_{k_1} \cdots v_{k_d}) = \sum_{\substack{0 \leqslant n \leqslant d \\ 0 \leqslant i_1 < j_1 \leqslant i_2 < j_2 \leqslant \cdots \leqslant i_n < j_n \leqslant d}} \sum_{\substack{\mathbf{l} = (\mathbf{l}_1, \ldots, \mathbf{l}_n) \\ \mathbf{l}_s = (l_{i_s+1}, \ldots, \widehat{l_{j_s}}, \ldots, l_{i_{s+1}}) \\ \mathbf{l}_s = \varnothing \text{ or } |\mathbf{l}_s| \leqslant k_{j_s} - 1}} \mathrm{m}_{\widehat{\mathbf{k}}^{j_1, \ldots, j_n} + \mathbf{l}, \widehat{\mathbf{k}}^{j_1, \ldots, j_n}}$$

$$\Big(v_{k_1} \cdots v_{k_{i_1}} \sqcup S(v_{k_{i_1+1}+l_{i_1+1}} \cdots v_{k_{j_1-1}+l_{j_1-1}}) \sqcup v_{k_{j_1+1}+l_{j_1+1}} \cdots v_{k_{i_2}+l_{i_2}} \sqcup$$

$$\cdots \sqcup v_{k_{j_n+1}+l_{j_n+1}} \cdots v_{k_d+l_d} \Big) \otimes v_{k_{j_1}-|\mathbf{l}_1|} \cdots v_{k_{j_n}-|\mathbf{l}_n|},$$

where we formally set $i_{n+1} = d$.

By the discussion after Proposition 16, a v_0 at the s-th position in the right factor corresponds to $i_s + 1 = i_{s+1}$ and $\mathbf{l}_s = \varnothing$.

Proposition 17

Let $B_{123} = \mathbb{Q}\langle v_1, v_2, v_3\rangle \subset \mathbb{Q}\langle V\rangle$, then

$$\Delta_a : B_{123} \longrightarrow \mathbb{Q}\langle V\rangle \otimes B_{123}.$$

In particular, Δ_a of a 1-2-3 word has only 1-2-3 words as right factors.

By observation we found the following conjecture, which we checked then numerically for a large number of words.

Conjecture

For $k \geqslant 0$ we define the k-level of a word $v_{k_1} \cdots v_{k_d}$ by

$$\text{k-level}(v_{k_1} \cdots v_{k_d}) = d - \#\{j \mid k_j = k\},$$

and extend this to an increasing filtration on $\mathbb{Q}\langle V\rangle$. For $w \in \mathbb{Q}\langle V\rangle$, write as usual $\Delta_a(w) = w_{(1)} \otimes w_{(2)}$. Then, we always have

$$\text{k-level}(w_{(2)}) \leqslant \text{k-level}(w).$$

If $w \in \mathbb{Q}\langle v_1, v_2, v_3\rangle$, then conjecturally the right tensor product factors of $\Delta_a(w)$ have not more indicies distinct to 2 than w. This is a another indication that there might be a generalization of Brown's result on a generating set for multiple zeta values.

Application 3: The uri bracket

Recall $\mathrm{m}_{\mathbf{k},\mathbf{l}} = (-1)^{|\mathbf{k}|+|\mathbf{l}|} \cdot \binom{k_1-1}{l_1-1} \cdots \binom{k_d-1}{l_d-1}$ for $\mathbf{k}, \mathbf{l} \in \mathbb{N}^d$. Set

$$\mathcal{C}^s(n) = \{\mathbf{a} \in \mathbb{N}_{>0}^s \mid |\mathbf{a}| = n\}.$$

For any threshold $a \in \mathbb{N}_{>0}$ we have a indicator function on $\mathcal{C}^s(n)$, which is given by

$$\mathrm{ind}_a(\mathbf{a}) = \min_{1 \leqslant j \leqslant \ell(\mathbf{a})} \{j \mid a_1 + \cdots + a_j \geqslant a\}. \tag{17}$$

By convention we set $\mathrm{ind}_a(\mathbf{a}) = 0$, if either $\mathbf{a} = \varnothing$, $a > |\mathbf{a}|$.

Definition 18

Given a composition $\boldsymbol{\alpha}$ and a threshold a, then their uri multiplicity is given by

$$\mu_{a,\boldsymbol{\alpha}} = B_1(\ell(\boldsymbol{\alpha}), \mathrm{ind}_a(\boldsymbol{\alpha})),$$

where B_k is the k-th Bernoulli number and

$$B_1(m,n) = \frac{1}{m!} \sum_{k=0}^{n-1} \binom{m}{k} B_k.$$

Definition

Keep the previous notation and consider

$$\rhd_u : M(V) \times V \to M(V)_{\mathbb{Q}}$$

$$(t(\mathbf{k}), v_a) \mapsto \begin{cases} 0 & v_a = v_0 \\ \sum\limits_{\substack{1 \leqslant \mathbf{l} \leqslant \mathbf{k} \\ \alpha \in \mathcal{C}(a+|\mathbf{k}|-|\mathbf{l}|)}} \mathrm{m}_{\mathbf{k},\mathbf{l}}\, \mu_{a,\alpha}\; v_{\alpha_1} \star (v_{\alpha_2} \star (\ldots \star (v_{\alpha_{r+1}} \star t(\mathbf{l})))) & v_a \neq v_0 \end{cases}$$

and denote $\rhd_u$ also its extensions to a derivation on $M(V)_{\mathbb{Q}}$ and to $\mathbb{Q}\langle M(V)\rangle$.

Lemma

The pairing $\rhd_u$ on $M(V)_{\mathbb{Q}}$ descends to a bilinear pairing $\rhd_u$ on $\mathrm{Lie}(V)$.

Proof: We need to check $I_{\mathrm{Lie}} \rhd_u M(V)_{\mathbb{Q}} \subseteq I_{\mathrm{Lie}}$. $\square$

Lemma 19

Let $a > 0$, then

$$\big(t_1(\mathbf{k}_1)\cdot t_2(\mathbf{k}_2)\big) \rhd_u v_a = \sum_{\substack{1\leqslant \mathbf{l}_1 \leqslant \mathbf{k}_1 \\ 1\leqslant \mathbf{l}_2 \leqslant \mathbf{k}_2}} m_{\mathbf{k}_1,\mathbf{l}_1} m_{\mathbf{k}_2,\mathbf{l}_2} \sum_{\alpha\in\mathcal{C}(a+|\mathbf{k}_2|-|\mathbf{l}_2|)} \mu_{a,\alpha} \sum_{i=1}^{\ell(\alpha)} \sum_{\beta\in\mathcal{C}(\alpha_i+|\mathbf{k}_1|-|\mathbf{l}_1|)} \mu_{\alpha_i,\beta}$$

$$v_{\alpha_1}\star\Big(\ldots\star\Big(v_{\alpha_{i-1}}\star\Big(\underbrace{\big(v_{\beta_1}\star\big(v_{\beta_2}\star\big(\ldots\star\big(v_{\beta_{\ell(\beta)}}\star t_1(\mathbf{l}_1)\big)\big)\big)\big)}_{i\text{-th position}}\star\Big(v_{\alpha_{i+1}}\ldots\star\big(v_{\alpha_{\ell(\alpha)}}\star t_2(\mathbf{l}_2)\big)\Big)\Big)\Big)\Big)$$

Idea of proof. Both sides are a linear combination of elements in M(V) and thus we need to compare both sides by comparing their respective coefficients: For $\mathbf{k},\mathbf{n}\in\mathbb{N}^d$, $a\in\mathbb{N}$ and $\beta\in\mathbb{N}^s$, we have

$$\sum_{\mathbf{n}\leqslant\mathbf{r}\leqslant\mathbf{k}} (-1)^{|\mathbf{k}|+|\mathbf{r}|}\,\mathrm{m}_{\mathbf{k}-\mathbf{n}+1,\mathbf{r}-\mathbf{n}+1} \sum_{\rho\in\mathcal{C}^s(a+|\mathbf{k}-\mathbf{r}|)} \mathrm{m}_{\rho,\beta}\ \mu_{a,\rho} = \mathrm{m}_{a,|\beta|-|\mathbf{k}-\mathbf{n}|}\ \mu_{|\beta|-|\mathbf{k}-\mathbf{n}|,\beta}\,.$$

$\square$

Theorem 20

If the Bernoulli numbers satisfy the threshold shuffle identities, then for the above $\rhd_u$ the triple $\Big(\mathrm{Lie}(V), [\_,\_], \rhd_u\Big)$ is a post-Lie algebra.

Idea of proof: Extracting the image of the Lie-map gives linear combination of elements in $\mathbb{Q}\langle V\rangle$ on each side of the desired equality. Thus we need to compare both sides by comparing their respective coefficients. We have implemented these threshold shuffle identities and checked them for thousands of cases in various weights and depths. $\square$

Definition

We write $\{\_,\_\}_u$ for the induced post Lie bracket and call $\{\_,\_\}_u$ the uri bracket.

Remark

The uri bracket is related to the (expected) Lie bracket

$$\mathsf{uri}(\_,\_) = \mathsf{ganit}_{\mathsf{pic}}\Big(\mathsf{ari}\big(\mathsf{ganit}_{\mathsf{poc}}(\_), \mathsf{ganit}_{\mathsf{poc}}(\_)\big)\Big)$$

on symmetril, polynomial moulds.

Open problems

- prove that the Bernoulli numbers satisfy the threshold shuffle identities
- extend Lemma 19 to products with more than two factors
- find a closed formula for $\lhd_u$ and thereby a closed formula for Δ_u

A nice model for multiple q-zeta values is parametrized by the quotient of $\big(\mathbb{Q}\langle V\rangle, ⧢\big)$ by an involution. The order of summation is chosen in such a way that the coproduct Δ_u dual to $\circledast_u$ descends (experimentally).

Future directions

(i) Prove that Burmester's $\mathfrak{bm}_0$ is a sub Lie algebra of $\big(\mathbb{Q}\langle V\rangle, \{\_,\_\}_u\big)$.

(ii) What is the extension of Zagier's theorem we need for a proof of the 1-2-3 conjecture?

Identities for Bernoulli numbers via threshold functions on compositions

Recall

$$\mathcal{C}^s(n) = \{\mathbf{a} \in \mathbb{N}_{>0}^s \mid |\mathbf{a}| = n\}.$$

For any threshold $a \in \mathbb{N}_{>0}$ we have a indicator function on $\mathcal{C}^s(n)$, which is given by

$$\mathrm{ind}_a(\mathbf{a}) = \min_{1 \leqslant j \leqslant \ell(\mathbf{a})} \{j \mid a_1 + \cdots + a_j \geqslant a\}. \tag{18}$$

By convention we set $\mathrm{ind}_a(\mathbf{a}) = 0$, if either $\mathbf{a} = \varnothing$, $a > |\mathbf{a}|$.
In the following we want to study some properties of $\mathrm{ind}_a(\mathbf{a})$. Given any formal power series

$$\mathbf{B}(x,y) = \sum_{m,n \geqslant 0} B(m,n) x^m y^n,$$

with the normalizing condition $B(1,1) = 1$, we can define multiplicities by

$$\mu^B_{a,\boldsymbol{\alpha}} = B\left(\ell(\boldsymbol{\alpha}), \mathrm{ind}_a(\boldsymbol{\alpha})\right), \qquad a \in \mathbb{N}, \boldsymbol{\alpha} \in \mathcal{C}. \tag{19}$$

Here, we set $\mu^B_{a,\boldsymbol{\alpha}} = 0$ if $\mathrm{ind}_a(\boldsymbol{\alpha}) = 0$.

We extend the multiplicity $\mu^B_{a,\_}$ linearly in the second argument, i.e., for a formal sum of indices $n_1\boldsymbol{\alpha}_1 + n_2\boldsymbol{\alpha}_2$ we set

$$\mu^B_{a,n_1\boldsymbol{\alpha}_1+n_2\boldsymbol{\alpha}_2} = n_1\,\mu^B_{a,\boldsymbol{\alpha}_1} + n_2\,\mu^B_{a,\boldsymbol{\alpha}_2}.$$

The concatenation of two indices $\mathbf{k}, \mathbf{l}$ is given by the index $(k_1, ..., k_{\ell(\mathbf{k})}, l_1, ..., l_{\ell(\mathbf{l})})$. We denote it by $(\mathbf{k}, \mathbf{l})$ and we extend this pairing bilinear. We combine these convention with the shuffle product of indices, for example

$$\mu^B_{2,(4,(3,2)\sqcup\!\sqcup(2))} = \mu^B_{2,(4,2\,(3,2,2)+(2,3,2))} = 2\mu^B_{2,(4,3,2,2)} + \mu^B_{2,(4,2,3,2)}.$$

By a decomposition of an index $\mathbf{k}$ we understand a pair of indices $\mathbf{k}_1, \mathbf{k}_2$ such that $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2)$, where we allow either $\mathbf{k}_1$ or $\mathbf{k}_2$ to be the empty index, i.e., $\mathbf{k} = (\varnothing, \mathbf{k}) = (\mathbf{k}, \varnothing)$ is allowed in this context. Given an index $\mathbf{k} = (k_1, k_2, \ldots, k_d)$ its reversed index is given by $\overline{\mathbf{k}} = (-1)^d\,(k_d, \ldots, k_2, k_1)$.

Definition 21

We say a multiplicity $\mu^B_{a,\boldsymbol{\alpha}}$ given by (19) satisfies the *threshold shuffle identities* if the following equalities hold

(i) For all $\boldsymbol{\sigma} \in \mathbb{N}_{>0}^{\ell(\boldsymbol{\sigma})}$, $\boldsymbol{\theta} \in \mathbb{N}_{>0}^{\ell(\boldsymbol{\theta})}$ and all $d_1, d_2 \in \mathbb{N}_{>0}$ we have that

$$\mu^B_{t,\boldsymbol{\sigma}\sqcup\boldsymbol{\theta}} = \sum_{\substack{(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=\boldsymbol{\sigma}\\(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=\boldsymbol{\theta}}} \mu^B_{|\boldsymbol{\sigma}_2|-d_1,\boldsymbol{\sigma}_2}\,\mu^B_{t,(\boldsymbol{\sigma}_1\sqcup\boldsymbol{\theta}_1,|\boldsymbol{\sigma}_2|-d_1,\boldsymbol{\theta}_2)} + \mu^B_{|\boldsymbol{\theta}_2|-d_2,\boldsymbol{\theta}_2}\,\mu^B_{t,(\boldsymbol{\sigma}_1\sqcup\boldsymbol{\theta}_1,|\boldsymbol{\theta}_2|-d_2,\boldsymbol{\sigma}_2)},$$

where $t = |\boldsymbol{\sigma}| + |\boldsymbol{\theta}| - d_1 - d_2$,

(ii) For all $\boldsymbol{\sigma} \in \mathbb{N}_{>0}^{\ell(\boldsymbol{\sigma})}$, $\boldsymbol{\tau} \in \mathbb{N}_{>0}^{\ell(\boldsymbol{\tau})}$, $\boldsymbol{\theta} \in \mathbb{N}_{>0}^{\ell(\boldsymbol{\theta})}$ and all $d_1, d_2 \in \mathbb{N}_{>0}$ we have that

$$0 = \sum_{\substack{(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=\boldsymbol{\sigma}\\(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)=\boldsymbol{\tau}\\(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=\boldsymbol{\theta}}} \mu^B_{|\boldsymbol{\sigma}_2|+|\boldsymbol{\tau}_1|-d_1,\boldsymbol{\sigma}_2\sqcup\overline{\boldsymbol{\tau}_1}}\,\mu^B_{t,(\boldsymbol{\sigma}_1\sqcup\overline{\boldsymbol{\theta}_2},|\boldsymbol{\sigma}_2|+|\boldsymbol{\tau}_1|-d_1,\boldsymbol{\tau}_2\sqcup\overline{\boldsymbol{\theta}_1})}$$
$$+\mu^B_{|\boldsymbol{\tau}_2|+|\boldsymbol{\theta}_1|-d_2,\boldsymbol{\tau}_2\sqcup\overline{\boldsymbol{\theta}_1}}\,\mu^B_{t,(\boldsymbol{\sigma}_1\sqcup\overline{\boldsymbol{\theta}_2},|\boldsymbol{\tau}_2|+|\boldsymbol{\theta}_1|-d_2,\boldsymbol{\sigma}_2\sqcup\overline{\boldsymbol{\tau}_1})},$$

where $t = |\boldsymbol{\sigma}| + |\boldsymbol{\tau}| + |\boldsymbol{\theta}| - d_1 - d_2$.

Observe that the identities in (i) and (ii) are homogeneous with respect to scaling of each $B(\ell(\boldsymbol{\alpha}), \mathrm{ind}_a(\boldsymbol{\alpha}))$ by a factor $t^{\ell(\boldsymbol{\alpha})-1}$.

The examples we calculated with PARI/GP indicate that the uri multiplicities are in fact the only non trivial multiplicities as in (19) satisfying the threshold shuffle identities up to scaling. More precisely, for $t \in \mathbb{R} \backslash \{0\}$ set

$$\mathbf{B}_t(x,y) = \frac{y^2 x e^{xt}}{(1-y)(e^{xyt}-1)} = \sum_{m,n \geqslant 0} B_t(m,n) x^m y^n,$$

and define a family of multiplicities as in (19) by

$$\mu^t_{a,\boldsymbol{\alpha}} = B_t\left(\ell(\boldsymbol{\alpha}), \mathrm{ind}_a(\boldsymbol{\alpha})\right), \qquad a \in \mathbb{N}, \boldsymbol{\alpha} \in \mathcal{C}.$$

Of course, the case $t = 1$ gives the uri multiplicities.

Conjecture

We conjecture the following holds:

(i) For all $t \neq 0$ the multiplicities $\mu^t_{a,\boldsymbol{\alpha}}$ satisfy the threshold shuffle identities from Definition 21.

(ii) If a multiplicity $\mu^{\tilde{B}}_{a,\boldsymbol{\alpha}}$ given by (19) satisfies the threshold shuffle identities from Definition 21, then $\mu^{\tilde{B}}_{a,\boldsymbol{\alpha}} = \mu^{B}_{a,\boldsymbol{\alpha}}$ for all $a \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathcal{C}$ and

$$\mathbf{B}(x,y) = \begin{cases} xy & \text{if } \tilde{B}(2,1) = 0, \\ \mathbf{B}_{2\tilde{B}(2,1)}(x,y) & \text{if } \tilde{B}(2,1) \neq 0. \end{cases}$$