Topology

Prospects in Mathematics
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Plan

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What is topology?

Topology is the study of geometric objects ("spaces") up to continuous deformation.
You can think about metric spaces or surfaces or manifolds or knots or topological spaces or spaces with group actions or ...
Typically we study spaces via numerical or algebraic *invariants*:

$$\text{spaces} \to \text{numbers or groups or rings or ...}$$
$$X \mapsto I(X)$$

such that some notion of sameness is preserved.
This notion of sameness might be homeomorphism or homotopy equivalence or stable homotopy equivalence or equivariant homotopy equivalence or diffeomorphism or ...
If $I(X) \neq I(Y)$ then X and Y are not the same; we use invariants to distinguish spaces and tackle classification problems.

Examples of topological invariants

- Euler characteristic
  $V - E + F = 2$ for convex polyhedra
  $V - E + F = 2 - 2g$ for closed orientable surfaces, where g is the number of “holes”
- Jones polynomial for knots
- fundamental group π_1
- homology groups H_*
- K-theory, cobordism, ...

Some areas in topology

- general topology
- algebraic topology
- differential topology
- geometric topology
- applied topology

Example: homotopy associativity

In algebra, we often consider a set, say X, with an associative binary operation

$$X \times X \to X, \qquad (a, b) \mapsto a \bullet b.$$

Associativity is the condition $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in X$. This condition on triples means that we don't need to write any brackets in longer strings.
In topology, it is natural to weaken this notion to homotopy associativity.
When we do so a hierarchy of higher structures appears.

So now X is a space and we have a continuous binary operation $X \times X \to X$. We can consider the two maps $X \times X \times X \to X$,

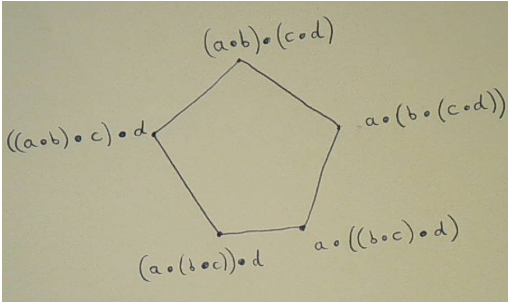
$$(a, b, c) \mapsto (a \bullet b) \bullet c, \qquad (a, b, c) \mapsto a \bullet (b \bullet c),$$

and require that these are *homotopic*, that is, there is a continuous deformation from one to the other.

If you have studied composition of based loops then you have seen a very important example of this.
In defining π_1, one passes to homotopy classes and thus obtains a strictly associative operation, but it's also interesting to consider the structure without passing to homotopy classes.

A higher associativity condition

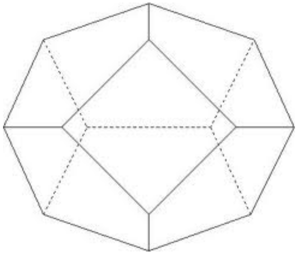
When we use the binary operation to combine 4 inputs, we are now led to consider a pentagon.



Given 4 inputs, the information from the homotopy for triples allows us to define a continuous map from the boundary of a pentagon to X. Asking this map to extend over the interior of the pentagon is a higher homotopy associativity condition.

And so on...

What about 5 inputs? There's a condition involving the figure:



And one can continue to more and more inputs. If all the higher associativity conditions are satisfied, we have an A_∞-space. It has a multiplication which is homotopy associative in the strongest possible sense.

A_∞-structures first arose in the study of loop spaces in topology. They are now important in many areas of mathematics, including algebra, geometry and mathematical physics.

You can play the same kind of game with other algebraic conditions and there are lots of mathematical tools for understanding these important homotopy invariant versions of algebraic structures.

My former student Gemma Halliwell did a PhD working with generalisations of these structures (derived A_∞-structures).

Topology in the UK

- Aberdeen: Mark Grant, Richard Hepworth, Jarek Kedra, Ran Levi, Assaf Libman, Simona Paoli, Irakli Patchkoria
  - algebraic and geometric topology
  - applied topology, including neuro-topology
- Cambridge: Oscar Randal-Williams, Jacob Rasmussen
  - algebraic and differential topology
  - low dimensional topology, knot theory, tilings
  - topological data analysis
- Durham: Jeff Giansiracusa, John Hunton, Andrew Lobb, Mark Powell, Dirk Schuetz
  - algebraic and differential topology
  - low dimensional topology, knot theory
- Edinburgh: Clark Barwick, Tom Leinster, Jonathan Pridham
  - homotopy theory, algebraic and differential topology
  - higher category theory, magnitude

Topology in the UK

- Oxford: Martin Bridson, Christopher Douglas, Andras Juhasz, Marc Lackenby, Vidit Nanda, Ulrike Tillmann
  - algebraic and geometric topology, low dimensional topology
  - topological data analysis
- Sheffield: Neil Strickland, Sarah Whitehouse, Simon Willerton
  - algebraic topology, homotopical algebra
  - magnitude
- Southampton: Jelena Grbic, Ian Leary, Stephen Theriault
  - unstable homotopy theory
  - topological data analysis
- Warwick: Emanuele Dotto, John Greenlees, Saul Schleimer, Goncalo Tabuada, Karen Vogtmann
  - algebraic and geometric topology, equivariant homotopy theory

Topology in the UK

- Belfast: Dave Barnes
- Cardiff: Ulrich Pennig
- Kent: Constanze Roitzheim
- Lancaster: Andrey Lazarev
- Liverpool: Jon Woolf
- Manchester: Peter Symonds
- Queen Mary: Michael Farber

See also the British Topology webpage.