Discrete Painlevé equations and pencils of quadrics in $\mathbb{P}^3$

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Discretization in
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Discrete Painleve equations of symmetry type $E_8^{(1)}$

Configurations of ten (ordered) points in $\mathbb{P}^2$:

$$M = GL(3,\mathbb{C})\backslash \left\{ \begin{pmatrix} x_1 & x_2 & \dots & x_9 & x \\ y_1 & y_2 & \dots & y_9 & y \\ z_1 & z_2 & \dots & z_9 & z \end{pmatrix} \right\} /(\mathbb{C}^*)^{10}$$

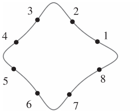
Points $p_i = [x_i : y_i : z_i]$, $i = 1, \dots, 9$ are *parameters*, point $p = [x : y : z]$ is the dependent variable. The action of the affine Weyl group $W(E_8^{(1)})$ on M by Cremona transformations is generated by

$$s_i : p_i \leftrightarrow p_{i+1}, \quad i = 1, 2, \dots, 8$$

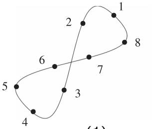
and s_0, the Cremona inversion based at p_1, p_2, p_3. *Discrete Painlevé equation*: action of a translation from $W(E_8^{(1)})$.

Configurations of eight points in $\mathbb{P}^1 \times \mathbb{P}^1$

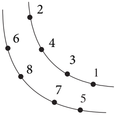
Left part of the Sakai table.



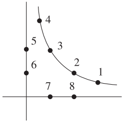
$eP(E_8^{(1)})$



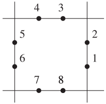
$qP(E_8^{(1)})$



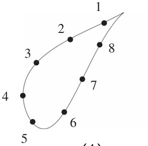
$qP(E_7^{(1)})$



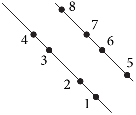
$qP(E_6^{(1)})$



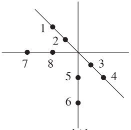
$qP(D_5^{(1)})$



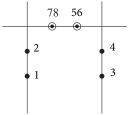
$dP(E_8^{(1)})$



$dP(E_7^{(1)})$



$dP(E_6^{(1)})$



$dP(D_4^{(1)})$

Example: qP$_{III}$ vs a QRT recurrence

$$\mathrm{qP}_{III}: \qquad y_{n+1}y_{n-1} = \frac{y_n^2 - 1}{1 - c_n^{-2}y_n^2}, \quad c_n = c_0 q^{2n}.$$

A non-autonomous version of a *QRT recurrence*

$$y_{n+1}y_{n-1} = \frac{y_n^2 - 1}{1 - c^{-2}y_n^2},$$

which can be put as $f : \mathbb{C}^2 \to \mathbb{C}^2$ (a *QRT root*),

$$f: \quad (x, y) \mapsto (\widetilde{x}, \widetilde{y}) = \left(y, \frac{y^2 - 1}{x(1 - c^{-2}y^2)} \right).$$

Inverse map:

$$f^{-1}: (\widetilde{x}, \widetilde{y}) \mapsto (x, y) = \left(\frac{\widetilde{x}^2 - 1}{\widetilde{y}(1 - c^{-2}\widetilde{x}^2)}, \widetilde{x} \right).$$

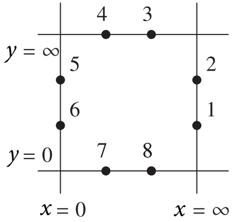
QRT as a birational map

Lift f to $\mathbb{P}^1 \times \mathbb{P}^1$. Then it has four indeterminacy points

$$p_1 = (\infty, c), \quad p_2 = (\infty, -c), \quad p_5 = (0, 1), \quad p_6 = (0, -1),$$

while f^{-1} has four indeterminacy points

$$p_3 = (c, \infty), \quad p_4 = (-c, \infty), \quad p_7 = (1, 0), \quad p_8 = (-1, 0).$$



QRT as a birational map

The eight singular points define a *pencil of biquadratic curves* in $\mathbb{P}^1 \times \mathbb{P}^1$, which are invariant under the map f:

$$C_\mu : \quad \left\{ c^{-2}x^2y^2 - x^2 - y^2 + 1 - \mu xy = 0 \right\}.$$

(Note that C_∞ is the union of four lines from the previous picture.)

Singularity confinement for f:

$$\begin{array}{ccccccc} \{y=-c\} & \to & (-c,\infty) & \to & (\infty,c) & \to & \{x=c\}, \\ \{y=c\} & \to & (c,\infty) & \to & (\infty,-c) & \to & \{x=-c\}, \\ \{y=-1\} & \to & (-1,0) & \to & (0,1) & \to & \{x=1\}, \\ \{y=1\} & \to & (1,0) & \to & (0,-1) & \to & \{x=-1\}. \end{array}$$

From a pencil of biquadratic curves to a QRT map

One can construct f starting with the pencil C_μ.

- For a given (x, y), determine μ such that $(x, y) \in C_\mu$.
- Define the *vertical switch* i_1 and the *horizontal switch* i_2 as the second intersection point of C_μ with the line $x = \mathrm{const}$, resp. $y = \mathrm{const}$. One computes:

$$i_1(x,y) = \Big(x, \frac{x^2-1}{y(1-c^{-2}x^2)}\Big), \quad i_2(x,y) = \Big(\frac{y^2-1}{x(1-c^{-2}y^2)}, y\Big).$$

- Define the *QRT map* $F = i_1 \circ i_2$. If the pencil C_μ is symmetric under $s(x,y) = (y,x)$, define the *QRT root* $f = s \circ i_2 = i_1 \circ s$, so that $F = f^2$.

From a QRT recurrence to qP_{III}

One can consider qP_{III} as a sequence of maps of the type f, but for which (some of) the points $p_1, \ldots, p_8$ depend on n through $c = c_n = c_0 q^{2n}$ (*de-autonomization*).

Main requirement (which singles out the evolution $c_n = c_0 q^{2n}$): the same singularity confinement patterns.

No algebraic integrals of motion! However, universally accepted as an integrable system:

- vanishing algebraic entropy
- isomonodromic structure (hence, monodromy data serve as transcendental integrals of motion)

A 3D map with a similar singularity confinement

Consider a birational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given by

$$\Phi = A \circ \sigma,$$

where A is a linear projective automorphism of $\mathbb{P}^3$, and $\sigma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is the *Cremona inversion*

$$\sigma : \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \mapsto \begin{bmatrix} 1/X_1 \\ 1/X_2 \\ 1/X_3 \\ 1/X_4 \end{bmatrix} = \begin{bmatrix} X_2X_3X_4 \\ X_1X_3X_4 \\ X_1X_2X_4 \\ X_1X_2X_3 \end{bmatrix}.$$

Algebraic geometry of Cremona inversion

The critical set and the indeterminacy set:

$$\mathcal{C}(\sigma) = \bigcup_{i=1}^{4} \Pi_i, \qquad \mathcal{I}(\sigma) = \bigcup_{1 \le i < j \le 4} \ell_{ij},$$

where $\Pi_i = \{X_i = 0\}$ are the coordinate planes and $\ell_{ij} = \Pi_i \cap \Pi_j$ are lines. Use also the four points

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Singularity confinement patterns:

$$\sigma: \quad \Pi_i \to e_i \to \Pi_i, \quad i = 1, \ldots, 4.$$

Map Φ with longer singularity confinement patterns

Consider $\Phi = A \circ \sigma$. Setting $a_i := A(e_i)$, we have:

$$\Phi : \quad \Pi_i \to a_i, \quad e_i \to A(\Pi_i).$$

Suppose

$$\Phi(a_i) = e_i \quad \Leftrightarrow \quad A \circ \sigma \circ A(e_i) = e_i, \quad i = 1, \ldots, 4,$$

then have the following singularity patterns:

$$\Phi : \quad \Pi_i \ \to \ a_i \ \to \ e_i \ \to \ A(\Pi_i).$$

The above condition is satisfied if

$$A = A_q = \begin{pmatrix} -1 & q & 1 & q \\ q & -1 & q & 1 \\ 1 & q & -1 & q \\ q & 1 & q & -1 \end{pmatrix}.$$

$q = 1$: integrability

If $q = 1$, the family of quadrics through eight points e_i and $a_i = Ae_i$ is two-dimensional, spanned by two pencils

$$\mathcal{Q}_\lambda = \big\{X \in \mathbb{P}^3 : Q_0(X) - \lambda Q_1(X) = 0\big\},$$

$$\mathcal{P}_\mu = \big\{X \in \mathbb{P}^3 : Q_0(X) - \mu Q_2(X) = 0\big\},$$

where

$$Q_0(X) = (X_1 + X_3)(X_2 + X_4), \quad Q_1(X) = (X_1 - X_3)(X_2 - X_4),$$

$$Q_2(X) = X_1^2 + X_2^2 - X_3^2 - X_4^2.$$

Theorem. *If $q = 1$, both pencils $\{\mathcal{Q}_\lambda\}$, $\{\mathcal{P}_\mu\}$ are invariant under Φ. In other words, Φ has two rational integrals*

$$\lambda = \frac{Q_1(X)}{Q_0(X)} \quad \textit{and} \quad \mu = \frac{Q_2(X)}{Q_0(X)}.$$

$q = 1$: action of Φ on fibers $\mathcal{Q}_\lambda$

Parametrize each $\mathcal{Q}_\lambda$ by $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ so that two families of straight line generators of $\mathcal{Q}_\lambda$ are given by $\{x = \mathrm{const}\}$, resp. $\{y = \mathrm{const}\}$ (*pencil-adapted coordinates* on $\mathbb{P}^3$):

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} x + \lambda^{-1} xy \\ y + 1 \\ x - \lambda^{-1} xy \\ y - 1 \end{bmatrix}$$

Thus,

$$x = \frac{X_1 + X_3}{X_2 - X_4}, \quad y = \frac{X_2 + X_4}{X_2 - X_4}, \quad \lambda = \frac{Q_0(X)}{Q_1(X)}.$$

In these coordinates:

$$\Phi : \quad \widetilde{x} = y, \quad \widetilde{y} = \frac{y^2 - 1}{x(1 - \lambda^{-2} y^2)}, \quad \widetilde{\lambda} = \lambda.$$

Each $\mathcal{Q}_\lambda$ is invariant, and in pencil-adapted coordinates Φ acts on $\mathcal{Q}_\lambda$ as a λ-dependent 2D QRT root.

Definition of 3D QRT maps

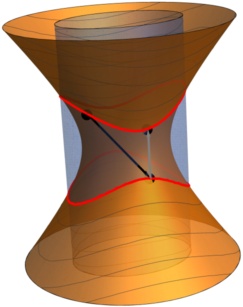
- For any $X \in \mathbb{P}^3$, determine λ and μ so that $X \in \mathcal{Q}_\lambda \cap \mathcal{P}_\mu$.
- Let $\ell_1(X), \ell_2(X)$ be two straight line generators of $\mathcal{Q}_\lambda$ through X.
- Denote by $i_1(X)$, $i_2(X)$ the second intersection points of $\ell_1(X)$, $\ell_2(X)$ with $\mathcal{P}_\mu$. Birational involutions $i_1, i_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ are called *3D QRT involutions defined by the pencils* $\{\mathcal{Q}_\lambda\}$ *and* $\{\mathcal{P}_\mu\}$.
- The *3D QRT map defined by the pencils* $\{\mathcal{Q}_\lambda\}$ *and* $\{\mathcal{P}_\mu\}$ is $F = i_1 \circ i_2$.
- If both $\mathcal{Q}_\lambda$ and $\mathcal{P}_\mu$ are symmetric w.r.t. a linear projective involution s on $\mathbb{P}^3$, then $\Phi = i_1 \circ s = s \circ i_2$ is called the *3D QRT root defined by the pencils* $\{\mathcal{Q}_\lambda\}$ *and* $\{\mathcal{P}_\mu\}$, as one has $F = \Phi \circ \Phi$.

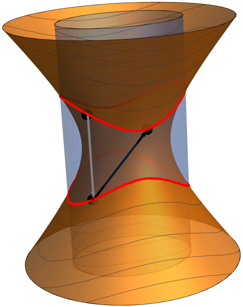
$q = 1$: Φ is a 3D QRT root

Theorem. *If $q = 1$, then the map $\Phi = A \circ \sigma$ is the 3D QRT root*

$$\Phi = i_1 \circ s = s \circ i_2,$$

where $s(X_1, X_2, X_3, X_4) = (X_2, X_1, X_4, X_3)$.





$q \neq 1$: a 3D Painlevé equation q-P_{III}

Map $\Phi_q = A_q \circ \sigma$ has for any q exactly the same singularity confinement patterns:

$$\Phi_q : \quad \Pi_i \to a_i \to e_i \to A_q(\Pi_i).$$

But:

The family of quadrics through eight points e_i and $a_i = A_q(e_i)$ is one-dimensional, namely, the pencil $\mathcal{Q}_\lambda$. Map Φ_q has no rational integrals. It sends each $\mathcal{Q}_\lambda$ to $\mathcal{Q}_{q^2\lambda}$. In pencil-adapted coordinates:

$$\Phi_q : \quad \widetilde{x} = y, \quad \widetilde{y} = \frac{y^2 - 1}{x(1 - \lambda^{-2}y^2)}, \quad \widetilde{\lambda} = q^2\lambda.$$

This is equivalent to q-P_{III}:

$$y_{n+1}y_{n-1} = \frac{y_n^2 - 1}{1 - \lambda_n^{-2}y_n^2}, \qquad \lambda_n = \lambda_0 q^{2n}.$$

General scheme

Input data.

1. A pencil $\{C_\mu\}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with the base points $s_1, \dots, s_8 \in \mathbb{P}^1 \times \mathbb{P}^1$, and the corresponding QRT map $f = i_1 \circ i_2$.
2. One distinguished biquadratic curve C_∞ of the pencil.

Goal.

- Construct a discrete Painlevé equation as a de-autonomization of f along the fiber C_∞.

General scheme

Construction [J. Alonso, Yu.S., K. Wei '24].

1. Let $\mathcal{Q}_0 = \{X_1X_2 - X_3X_4 = 0\} \subset \mathbb{P}^3$. Recall that $\mathcal{Q}_0$ is the image of the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$, via

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni \big([x_1 : x_0], [y_1 : y_0]\big) \mapsto [x_1y_0 : x_0y_1 : x_1y_1 : x_0y_0] \in \mathcal{Q}_0.$$

2. Let $S_1, \ldots, S_8$ be the images of the base points $s_1, \ldots, s_8$ under Segre embedding.
3. To each biquadratic curve

$$C_\mu : \big\{a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4x^2 + a_5y^2 + a_6xy + a_7x + a_8y + a_9 = 0\big\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

there corresponds its *Segre lift*, the quadric

$$\mathcal{P}_\mu : \big\{a_1X_3^2 + a_2X_1X_3 + a_3X_2X_3 + a_4X_1^2 + a_5X_2^2 + a_6X_3X_4 + a_7X_1X_4 + a_8X_2X_4 + a_9X_4^2 = 0\big\} \subset \mathbb{P}^3.$$

(Actually, C_μ can be identified with $\mathcal{Q}_0 \cap \mathcal{P}_\mu$.)

General scheme

Construction (contunued).

4. Construct the *pencil of quadrics* $\{\mathcal{Q}_\lambda\}$ *in* $\mathbb{P}^3$ *spanned by* $\mathcal{Q}_0$ *and* $\mathcal{P}_\infty$. The base curve of $\{\mathcal{Q}_\lambda\}$ is $\mathcal{Q}_0 \cap \mathcal{P}_\infty$, the image of C_∞ under Segre embedding. Its intersection with the base curve of $\{\mathcal{P}_\mu\}$ consists of $S_1, \ldots, S_8$.
5. Consider *3D QRT involutions* i_1, i_2 on $\mathbb{P}^3$ defined by intersections of generators ℓ_1, ℓ_2 of $\mathcal{Q}_\lambda$ with the quadrics $\mathcal{P}_\mu$. On each quadric $\mathcal{Q}_\lambda$, the map $\Phi = i_1 \circ i_2$ induces a λ-deformation of the original QRT map f.

Problem. It is not necessarily the case that $\ell_1(X)$, $\ell_2(X)$ depend on X rationally.

Counterexample. Let $\mathcal{Q}_\lambda$ be the pencil

$$\mathcal{Q}_\lambda = \big\{X_1^2 + X_2^2 + X_3^2 - \lambda X_4^2 = 0\big\}.$$

In the affine patch with $X_4 = 1$, we have $X_1^2 + X_2^2 + X_3^2 = \lambda$. Generators are given by $(X_1 + tV_1, X_2 + tV_2, X_3 + tV_3)$, where

$$[V_1 : V_2 : V_3] = \left[\frac{-X_1X_2 \pm i\sqrt{\lambda}X_3}{X_1^2 + X_3^2} : 1 : \frac{-X_2X_3 \mp i\sqrt{\lambda}X_1}{X_1^2 + X_3^2}\right].$$

Thus, for any fixed λ, directions of generators are rational functions on $\mathbb{P}^3$ with coefficients depending on $\sqrt{\lambda} = \sqrt{X_1^2 + X_2^2 + X_3^2}$. As functions of X, they are non-rational.

Resolution of the issue of rationality

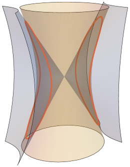
Proposition. For a point X from $\mathcal{Q}_\lambda$, generators $\ell_1(X)$, $\ell_2(X)$ are rational functions of X and of $\sqrt{\Delta(\lambda)}$, where $\Delta(\lambda)$ is the characteristic polynomial of the pencil $\mathcal{Q}_\lambda$.

In particular, if $\Delta(\lambda)$ is a complete square, they are rational functions of X and of λ. In the latter case, generators $\ell_1(X)$ and $\ell_2(X)$ are rational functions of $X \in \mathbb{P}^3$, and involutions i_1, i_2 are birational maps $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

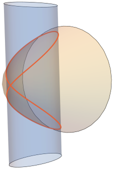
This is the case for 7 out of 13 projective classes of pencils of quadrics.

Projective classification of pencils of quadrics in $\mathbb{P}^3$

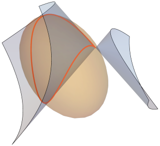
Six cases where $\Delta(\lambda)$ is not a complete square:



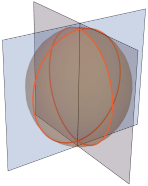
$\Delta(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)$



$\Delta(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)(\lambda-\lambda_3)$



$\Delta(\lambda)=(\lambda-\lambda_1)^3(\lambda-\lambda_2)$



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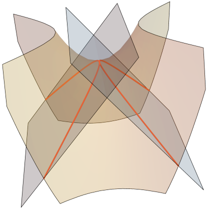

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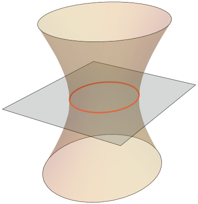
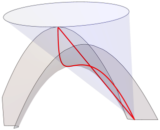


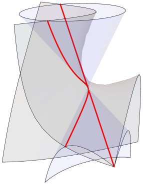
$\Delta(\lambda)=(\lambda-\lambda_1)^3(\lambda-\lambda_2)$

Projective classification of pencils of quadrics in $\mathbb{P}^3$

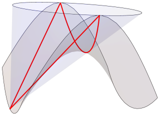
... and seven cases where $\Delta(\lambda)$ is a complete square:



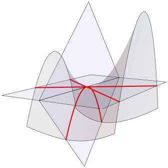
$\Delta(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2$



$\Delta(\lambda)=(\lambda-\lambda_1)^4$



$\Delta(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2$



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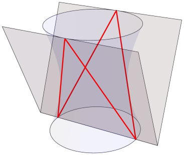
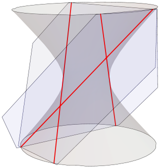


$\Delta(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2$



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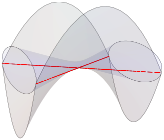


$\Delta(\lambda)=(\lambda-\lambda_1)^4$

General scheme if $\Delta(\lambda)$ is a complete square

Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius automorphism fixing the set

$$\mathrm{Sing}(\mathcal{Q}) := \big\{\lambda \in \mathbb{P}^1 : \ \mathcal{Q}_\lambda \ \text{is degenerate}\big\}.$$

Let a birational map L on $\mathbb{P}^3$ be defined by imposing on each $\mathcal{Q}_\lambda$ one of the formulas (depending on the case at hand):

$$[X_1 : X_2 : X_3 : X_4] \ \mapsto \ [X_1X_4 : X_2X_4 : X_3X_4 - \big(\sigma(\lambda)-\lambda\big)P_\infty(X) : X_4^2]$$

or

$$[X_1 : X_2 : X_3 : X_4] \ \mapsto \ [X_1X_2 + \big(\sigma(\lambda)-\lambda\big)P_\infty(X) : X_2^2 : X_2X_3 : X_2X_4].$$

Then:

a) L preserves the pencil $\{\mathcal{Q}_\lambda\}$, and maps each $\mathcal{Q}_\lambda$ to $\mathcal{Q}_{\sigma(\lambda)}$;
b) The maps $L \circ i_1, L \circ i_2$ have the same singularity confinement properties as the QRT involutions i_1, i_2.

The map $F = (L \circ i_1) \circ (L \circ i_2)$ is called the 3D discrete Painlevé equation obtained by the de-autonomization of the QRT map along the fiber C_∞.

d-Painlevé vs. q-Painlevé

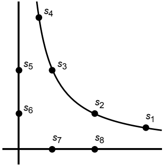
If $\Delta(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2$, can normalize $\lambda_1 = 0$, $\lambda_2 = \infty$, then $\sigma(\lambda) = q\lambda$, resulting in a q-Painlevé equation.

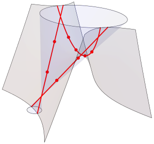
If $\Delta(\lambda) = (\lambda - \lambda_1)^4$, can normalize $\lambda_1 = \infty$, then $\sigma(\lambda) = \lambda + \delta$, resulting in a d-Painlevé equation.

Remark. Note that q, resp. δ are arbitrary, i.e., do not depend on the point configuration!

Example 1: qP($E_6^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1X_2 - X_3X_4 - \lambda X_3(X_4 - X_3).$$

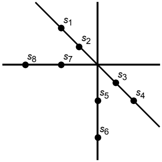
- Base curve: two lines $\{X_1 = X_3 = 0\}$, $\{X_2 = X_3 = 0\}$, and conic $\{X_1X_2 - X_3X_4 = 0, X_3 = X_4\}$.
- Characteristic polynomial: $\Delta(\lambda) = (\lambda + 1)^2$.
- Map L linear projective:

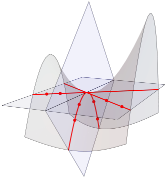
$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : X_3 : q^{-1}(X_4 - X_3) + X_3]$$

with $\sigma(\lambda) = -1 + q(\lambda + 1)$.

Example 2: dP($E_6^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1 X_2 - X_3 X_4 - \lambda(X_1 + X_2)X_4.$$

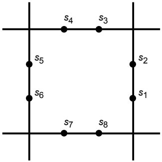
- Base curve: two lines $\{X_1 = X_4 = 0\}$, $\{X_2 = X_4 = 0\}$, and conic $\{X_1 X_2 - X_3 X_4 = 0, X_1 + X_2 = 0\}$.
- Characteristic polynomial: $\Delta(\lambda) = 1$.
- Map L linear projective:

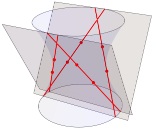
$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : X_3 - \delta(X_1 + X_2) : X_4]$$

with $\sigma(\lambda) = \lambda + \delta$.

Example 3: qP($D_5^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1X_2 - X_3X_4 - \lambda X_3X_4.$$

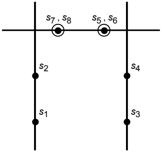
- Base curve: four lines $\{X_1 = X_3 = 0\}$, $\{X_1 = X_4 = 0\}$, $\{X_2 = X_3 = 0\}$, $\{X_2 = X_4 = 0\}$.
- Characteristic polynomial: $\Delta(\lambda) = (\lambda + 1)^2$.
- Map L linear projective:

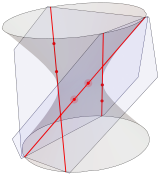
$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : q^{-1}X_3 : X_4]$$

with $\sigma(\lambda) = -1 + q(\lambda + 1)$.

Example 4: dP($D_4^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1 X_2 - X_3 X_4 - \lambda X_1 X_4.$$

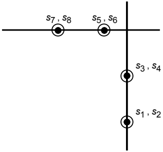
- Base curve: two lines $\{X_1 = X_3 = 0\}$, $\{X_2 = X_4 = 0\}$, and a double line $\{X_1 = X_4 = 0\}$.
- Characteristic polynomial: $\Delta(\lambda) = 1$.
- Map L linear projective:

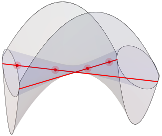
$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : X_3 - \delta X_1 : X_4]$$

with $\sigma(\lambda) = \lambda + \delta$.

Example 5: dP($A_3^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1 X_2 - X_3 X_4 - \lambda X_4^2.$$

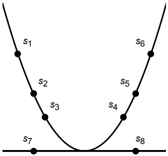
- Base curve: two double lines $\{X_1 = X_4 = 0\}$, $\{X_2 = X_4 = 0\}$.
- Characteristic polynomial: $\Delta(\lambda) = 1$.
- Map L linear projective:

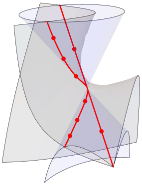
$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1 : X_2 : X_3 - \delta X_4 : X_4]$$

with $\sigma(\lambda) = \lambda + \delta$.

Example 6: alternative dP($E_7^{(1)}$) [Nagao' 2017]

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1X_2 - X_3X_4 - \lambda(X_1^2 - X_2X_4).$$

- Base curve: twisted cubic $\{[t : t^2 : t^3 : 1]\}$ and its tangent line $\{X_1 = X_4 = 0\}$.
- Characteristic polynomial: $\Delta(\lambda) = 1$.
- Map L birational of degree 2:

$$L : [X_1 : X_2 : X_3 : X_4] \mapsto [X_1X_4 : X_2X_4 : X_3X_4 + \delta(X_2X_4 - X_1^2) : X_4^2]$$

with $\sigma(\lambda) = \lambda + \delta$.

General scheme if $\Delta(\lambda)$ is not a complete square

In the six cases when $\Delta(\lambda)$ is *not* a complete square:

- instead of λ, work with $\mathcal{R}$, the Riemann surface of $\sqrt{\Delta(\lambda)}$, a double cover of $\mathbb{P}$ branched at four or two points, with the holomorphic universal covering $\mathbb{C} \to \mathcal{R}$, $\nu \mapsto \lambda(\nu)$;
- instead of $\mathbb{P}^3$, work with $\mathcal{X}$, the double cover of $\mathbb{P}^3$ branched along the singular quadrics $\mathcal{Q}_{\lambda_i}$, where λ_i are branching points of $\mathcal{R}$.

General scheme if $\Delta(\lambda)$ is not a complete square

- Define a birational map L on $\mathcal{X}$ by imposing on each $\mathcal{Q}_{\lambda(\nu)}$ one of the formulas (depending on the case at hand):

$$[X_1 : X_2 : X_3 : X_4] \mapsto [X_1 X_4 : X_2 X_4 : X_3 X_4 - \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big) P_\infty(X) : X_4^2]$$

or

$$[X_1 : X_2 : X_3 : X_4] \mapsto [X_1 X_2 + \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big) P_\infty(X) : X_2^2 : X_2 X_3 : X_2 X_4],$$

where $\widehat{\nu} = \nu + 2\delta$. Then L maps each $\mathcal{Q}_{\lambda(\nu)}$ to $\mathcal{Q}_{\lambda(\nu+2\delta)}$ and fixes the base curve $\mathcal{Q}_0 \cap \mathcal{P}_\infty$ pointwise.

- Factorize L into “triangular” maps:

$$L = L_1 \circ R_2 = L_2 \circ R_1,$$

where, in the pencil-adapted coordinates

$$L_1 : (x, y, \nu) \mapsto (x, \widetilde{y}, \nu + \delta), \quad R_2 : (x, y, \nu) \mapsto (\widetilde{x}, y, \nu + \delta),$$
$$L_2 : (x, y, \nu) \mapsto (\widetilde{x}, y, \nu + \delta), \quad R_1 : (x, y, \nu) \mapsto (x, \widetilde{y}, \nu + \delta).$$

General scheme if $\Delta(\lambda)$ is not a complete square

We set

$$\nu_n = \nu_0 + 2n\delta \quad \text{for} \quad n \in \tfrac{1}{2}\mathbb{Z}.$$

Definition. A 3D Painlevé map is given by

$$F = R_1 \circ i_1 \circ L_1 \circ R_2 \circ i_2 \circ L_2,$$

or, in coordinates,

$$(x_n, y_n, \nu_{2n-1/2}) \stackrel{L_2}{\to} (x, y_n, \nu_{2n}) \stackrel{i_2}{\to} (\widetilde{x}, y_n, \nu_{2n}) \stackrel{R_2}{\to} (x_{n+1}, y_n, \nu_{2n+1/2})$$
$$\stackrel{L_1}{\to} (x_{n+1}, y, \nu_{2n+1}) \stackrel{i_1}{\to} (x_{n+1}, \widetilde{y}, \nu_{2n+1}) \stackrel{R_1}{\to} (x_{n+1}, y_{n+1}, \nu_{2n+3/2}).$$

Thus, variables associated to the discrete Painlevé equations known from the literature, parametrize in our formulation the quadrics with half-integer indices:

$$(x_n, y_n, \nu_{2n-1/2}) \in Q_{\lambda(\nu_{2n-1/2})}, \quad (x_{n+1}, y_n, \nu_{2n+1/2}) \in Q_{\lambda(\nu_{2n+1/2})}.$$

e-Painlevé vs. q-Painlevé vs. d-Painlevé

If $\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, the Riemann surface $\mathcal{R}$ is a torus (elliptic curve), resulting in a e-Painlevé equation.

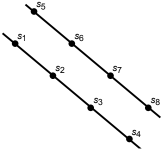
If $\Delta(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)(\lambda - \lambda_3)$, can normalize $\lambda_1 = \infty$, $\lambda_2 = 1$, $\lambda_3 = -1$. Then $\mathcal{R}$ is a cylinder (one of the periods is ∞). We uniformize $\sqrt{\lambda^2 - 1}$ via $\lambda = \frac{1}{2}(e^\nu + e^{-\nu}) = \frac{1}{2}(z + z^{-1})$. The shift in the variable z is $z \mapsto qz$, where $q = e^\delta$, resulting in a q-Painlevé equation.

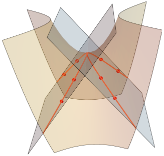
If $\Delta(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$, can normalize $\lambda_1 = \infty$, $\lambda_2 = 0$. Then $\mathcal{R}$ is a plane (both torus periods are ∞). We uniformize $\sqrt{\lambda}$ via $\lambda = \nu^2$, where $\nu \in \mathbb{C}$, resulting in a d-Painlevé equation.

Remark. Note that δ, resp. q are arbitrary, i.e., do not depend on the point configuration!

Example 7: dP ($E_7^{(1)}$)

Point configurations:





- Pencil of quadrics:

$$Q(\lambda) = X_1X_2 - X_3X_4 - \lambda(X_1 + X_2)(X_1 + X_2 - X_4).$$

- Base curve: two conics, $\{X_1X_2 - X_3X_4 = 0, X_1 + X_2 = 0\}$ and $\{X_1X_2 - X_3X_4 = 0, X_1 + X_2 - X_4 = 0\}$.
- Base point configuration:

$$\begin{aligned} S_i &= [a_i : -a_i : -a_i^2 : 1], \quad i = 1, \ldots, 4, \\ S_i &= [a_i : 1 - a_i : a_i(1 - a_i) : 1], \quad i = 5, \ldots, 8. \end{aligned}$$

- Characteristic polynomial: $\Delta(\lambda) = 4\lambda - 1$.
- Uniformization of $\sqrt{1-4\lambda}$:

$$\lambda = \frac{1-\nu^2}{4}, \quad \nu \in \mathbb{C}.$$

- Pencil-adapted coordinates (x, y, ν) on double cover of $\mathbb{P}^3$:

$$x = \frac{(1+\nu)X_1 - (1-\nu)X_2}{2X_4}, \quad y = \frac{(1+\nu)X_2 - (1-\nu)X_1}{2X_4}.$$

- Base points in coordinates (x, y) on $Q_{\lambda(\nu)}$:

$$\begin{aligned} s_i(\nu) &= (a_i, -a_i), \quad i = 1, \ldots, 4, \\ s_i(\nu) &= \left(a_i + \frac{\nu-1}{2}, -a_i + \frac{\nu+1}{2}\right), \quad i = 5, \ldots, 8. \end{aligned}$$

- Painlevé deformation map:

$$L: \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \mapsto \begin{bmatrix} X_1 X_4 \\ X_2 X_4 \\ X_3 X_4 + \delta(\nu+\delta)(X_1+X_2)(X_1+X_2-X_4) \\ X_4^2 \end{bmatrix}.$$

- In pencil-adapted coordinates:

$$L: (x, y, \nu) \mapsto \Big(x + \frac{\delta}{\nu}(x+y), y + \frac{\delta}{\nu}(x+y), \nu + 2\delta\Big).$$

Factorizations: $L = L_1 \circ R_2 = L_2 \circ R_1$ with

$$L_1 = R_1 : (x, y, \nu) \mapsto \Big(x, y + \frac{\delta}{\nu}(x+y), \nu + \delta\Big),$$

$$L_2 = R_2 : (x, y, \nu) \mapsto \Big(x + \frac{\delta}{\nu}(x+y), y, \nu + \delta\Big).$$

References

1. J. Alonso, Yu.S., K. Wei. *Discrete Painlevé equations and pencils of quadrics in* $\mathbb{P}^3$. `arXiv:2403.11349`

2. J. Alonso, Yu.S. *Discrete Painlevé equations and pencils of quadrics in* $\mathbb{P}^3$*. 2. Pencils with branching generators.* (In preparation)