To further our analysis recall that the bosonic delta functions

$$\delta^{2k}(C\cdot\tilde{\lambda})\,\delta^{2(n-k)}(C_{\perp}\cdot\lambda)$$

forces λ to be inside C. We can ~~s~~use ~~answers~~ the $GL(2)$ subgroup of $GL(k)$ to ~~P~~ "gauge fix" C to the form.

$$C = k\left\{\begin{bmatrix} & & & \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \end{bmatrix}\right\} k' = k-2.$$

This implies that we went ahead and solved momentum conservation. as a result we are left with.

$$\int \frac{dC^{k\times n}}{GL(k)\prod_{i=1}^{n} M_i(C)}\,\delta^{2k}(C\cdot\tilde{\lambda})\,\delta^{2\times(n-k)}(C_{\perp}\cdot\lambda)\,\delta^{4k}(C\cdot\eta)$$

$$= \frac{\delta^{4}(\sum_i \lambda_i\tilde{\lambda}_i)\,\delta^{8}(\sum_i \lambda_i\eta_i)}{\prod_{i=1}^{n}\langle i\, i{+}1\rangle} \times \int \frac{dC'}{GL(k')\prod_{i=1}^{n} M_i(C')}\,\delta^{4k'}(C'\cdot W)\,\delta^{4k'}(C'\cdot\chi)$$

where $W_i^A = \begin{pmatrix}\lambda_i^{\alpha}\\ \mu_i^{\dot\alpha}\end{pmatrix}$ where μ are linearly related to $\tilde{\lambda}$ and χ^I linearly related to η^I via.

$$\tilde{\lambda}_i = \frac{\langle i{+}1\, i\rangle \mu_{i-1} + \langle i\, i{-}1\rangle \mu_{i+1} + \langle i{-}1\, i{+}1\rangle \mu_i}{\langle i{-}1\, i\rangle\langle i\, i{+}1\rangle} \quad (A.6)$$

$$\eta_i = \frac{\langle i{+}1\, i\rangle \chi_{i-1} + \langle i\, i{-}1\rangle \chi_{i+1} + \langle i{-}1\, i{+}1\rangle \chi_i}{\langle i{-}1\, i\rangle\langle i\, i{+}1\rangle}$$

For more detail of this transformation see 1410.0621. The new variables $W_i^A = \begin{pmatrix}\lambda_i^{\alpha}\\ \mu_i^{\dot\alpha}\end{pmatrix}$ are called momentum twistors. We will come to their inherent nature shortly. Let's first consider the form of the amplitude in these variables.

Recall that the 6-pt amplitude with $N=6$ $K=3$ is given by the sum over residues of.

$$\{M_1=0\} \quad \{M_3=0\} \quad \{M_5=0\}$$

In terms of our momentum twistor integral we now have. $K=1$ $N=6$

$$\{M_1=0\} = A_{MHV} \times \int \frac{dC_2}{C_2}\frac{dC_3}{C_3}\frac{dC_4}{C_4}\frac{dC_5}{C_5}\frac{dC_6}{C_6}\,\delta^4(C_2W_2+C_3W_3+C_4W_4+C_5W_5+C_6W_6)\,\delta^4(C_2X_2+C_3X_3+C_4X_4+C_5X_5+C_6X_6)$$

$$\{M_3=0\} = A_{MHV} \times \int \frac{dC_1}{C_1}\frac{dC_2}{C_2}\frac{dC_4}{C_4}\frac{dC_5}{C_5}\frac{dC_6}{C_6}\,\delta^4(C_1W_1+C_2W_2+C_4W_4+C_5W_5+C_6W_6)\,\delta^4(C_1X_1+C_2X_2+C_4X_4+C_5X_5+C_6X_6)$$

$$\{M_5=0\} = A_{MHV} \times \int \frac{dC_1}{C_1}\frac{dC_2}{C_2}\frac{dC_3}{C_3}\frac{dC_4}{C_4}\frac{dC_6}{C_6}\,\delta^4(C_1W_1+C_2W_2+C_3W_3+C_4W_4+C_6W_6)\,\delta^4(C_1X_1+C_2X_2+C_3X_3+C_4X_4+C_6X_6)$$

Where $A_{MHV} \equiv \dfrac{\delta^4(P)\,\delta^8(\lambda\eta)}{\prod_{i=1}\langle i\, i+1\rangle}$. We see that we have a five-fold integral with a degree 4 delta-function. It's really a ~~three~~ four-fold integral since we can use $GL(1)$ symmetry to fix one of the C to 1. Then the delta function can be used to fix C. For example for $\{M_1=0\}$, let's set $C_2=1$

$$\int \frac{dC_3}{C_3}\frac{dC_4}{C_4}\frac{dC_5}{C_5}\frac{dC_6}{C_6}\,\delta^4(W_2+C_3W_3+C_4W_4+C_5W_5+C_6W_6)\,\delta^4(X_2+C_3X_3+C_4X_4+C_5X_5+C_6X_6)$$

The delt function localizes on ~~two~~ to

$$C_3=\frac{\langle 4562\rangle}{\langle 3456\rangle} \quad C_4=\frac{\langle 5623\rangle}{\langle 3456\rangle} \quad C_5=\frac{\langle 6234\rangle}{\langle 3456\rangle} \quad C_6=\frac{\langle 2345\rangle}{\langle 3456\rangle}$$

where now $\langle ijkl\rangle \equiv \frac{1}{4!}\epsilon_{ABCD}W_i^AW_j^BW_k^CW_l^D$

due to the identity

$$\langle 3456\rangle W_2^A + \langle 4562\rangle W_3^A + \langle 5623\rangle W_4^A + \langle 6234\rangle W_5^A + \langle 2345\rangle W_6^A = 0$$

The amplitude is just (after doing the same for the other two)

$\{M_1=0\} + \{M_3=0\} + \{M_5=0\}$

$$
= A_{MHV} \times \left\{ \frac{\left(\langle 3456\rangle \chi_2 + \langle 4562\rangle \chi_3 + \langle 5623\rangle \chi_4 + \langle 6234\rangle \chi_5 + \langle 2345\rangle \chi_6\right)^4}{\langle 4562\rangle\langle 5623\rangle\langle 6234\rangle\langle 2345\rangle\langle 3456\rangle} \right.
$$

$$
+ \frac{\left(\langle 1245\rangle \chi_6 + \langle 2456\rangle \chi_1 + \langle 4561\rangle \chi_2 + \langle 5612\rangle \chi_4 + \langle 6124\rangle \chi_5\right)^4}{\langle 1245\rangle\langle 2456\rangle\langle 4561\rangle\langle 5612\rangle\langle 6124\rangle}
$$

$$
\left. + \frac{\left(\langle 1234\rangle \chi_6 + \langle 2346\rangle \chi_1 + \langle 3461\rangle \chi_2 + \langle 4612\rangle \chi_3 + \langle 6123\rangle \chi_4\right)^4}{\langle 1234\rangle\langle 2346\rangle\langle 3461\rangle\langle 4612\rangle\langle 6123\rangle} \right\}
$$

This is remarkably simple!! Let's do some observation.

Comment 1: We see that some poles are repeated between different terms.

$\{M_1=0\} \to \langle 4562\rangle \quad \langle 6234\rangle$

$\{M_3=0\} \to \langle 6124\rangle \quad \langle 2456\rangle$

$\{M_5=0\} \to \langle 2346\rangle \quad \langle 4612\rangle$

Each time they repeat, their ordering have a relative minus sign

$$\langle 4562\rangle = -\langle 2456\rangle$$

Comment 2: The poles of the form $\langle i\, i{+}1\; j\, j{+}1\rangle$ are not repeated.

Comment 3: You can do the same with. $\{M_2=0\}$ $\{M_4=0\}$ $\{M_6=0\}$

To make things even more simpler (pretty) let's "bosonize" the fermionic twistors χ_i^I and combine with ω_i^A into a five-component twistor

$$\mathcal{W}_i^{\mathcal{A}} = \begin{pmatrix} \omega_i^A \\ \chi_i^I \phi_I \end{pmatrix}$$ where ϕ_I $I=1,2,3,4.$ is a grassmann odd variable

Define

$$[ijklm] = \int d^4\Phi \frac{\langle ijklm\rangle^4}{\langle 0ijkl\rangle\langle 0jklm\rangle\langle 0klmi\rangle\langle 0lmij\rangle\langle 0mijk\rangle} \quad (A.5)$$

Where now the five-bracket is defined with respect to the five component momentum twistors

$$\langle ijklm\rangle = \frac{1}{5!}\epsilon_{ABCDE} W_i^A W_j^B W_k^C W_l^D W_m^E$$

While the "reference twistor" $W_0 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$.

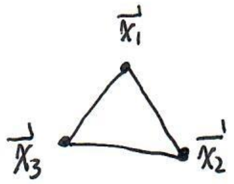
You can check that our amplitude can be compatly written as

$$A_{MHV} \times \left\{ [34562] + [12456] + [12346] \right\}$$

What are these square brackets ? and what are the momentum twistors ? It's time to address this !

§ Simplices in P^d §

Let's consider the area (volume) of simplex in 2-dimensions, i.e a triangle.



$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \vec{X}_1 & \vec{X}_2 & \vec{X}_3 \end{vmatrix}$$ where $\vec{X}_1, \vec{X}_2, \vec{X}_3$ are two component vectors.

We can write this projectively in P^2 by introducing 3-vector $X_i = \begin{pmatrix}1\\ \vec{X}_i\end{pmatrix}$ then

$$A = \frac{\langle X_1 X_2 X_3\rangle}{(0\cdot X_1)(0.X_2)(0.X_3)}$$ where the zero vector $(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

This is a formula for the area of the triangle in terms of its vertices. we can also use the boundaries (lines) to determine the area

Define $\omega_{1I} \equiv \epsilon_{IJK} X_1^J X_2^K \quad \omega_{2I} = \epsilon_{IJK} X_2^J X_3^K \quad \omega_{3I} = \epsilon_{IJK} X_3^J X_1^K$

The $\omega_{1I} X^I = 0$ ~~defines~~ corresponds to the boundary associated with line (X_1, X_2)

$\omega_{2I} X^I = 0$ — — — — — — (X_2, X_3)

$\omega_{3I} X^I = 0$ — — — $(X_3\, X_1)$

The inside of the triangle can be expressed as

$$\omega_1 \cdot X \geq 0$$
$$\omega_2 \cdot X \geq 0$$
$$\omega_3 \cdot X \geq 0$$

Using the ωs, the area is now

$$A = \frac{1}{2} \frac{\langle \omega_1 \omega_2 \omega_3 \rangle^2}{\langle 0 \omega_1 \omega_2 \rangle \langle 0 \omega_2 \omega_3 \rangle \langle 0 \omega_3 \omega_1 \rangle}.$$

We can immediately see that pushing this to P^3, the volume would be

$$\frac{\langle \omega_1 \omega_3 \omega_2 \omega_4 \rangle^3}{\langle 0 \omega_1 \omega_2 \omega_3 \rangle \langle 0 \omega_2 \omega_3 \omega_4 \rangle \langle 0 \omega_3 \omega_4 \omega_1 \rangle \langle 0 \omega_4 \omega_1 \omega_2 \rangle}.$$

In P^4

$$\frac{\langle \omega_1 \omega_2 \omega_3 \omega_4 \omega_5 \rangle^4}{\langle 0 \omega_1 \omega_2 \omega_3 \omega_4 \rangle \langle 0 \omega_2 \omega_3 \omega_4 \omega_5 \rangle \langle 0 \omega_3 \omega_4 \omega_5 \omega_1 \rangle \langle 0 \omega_4 \omega_5 \omega_1 \omega_2 \rangle \langle 0 \omega_5 \omega_1 \omega_2 \omega_3 \rangle}$$

This is precisely the form in (A.5). Thus the 6-point amplitude can be written as

$$A_{6pt.} = A_{MHV} \int d\phi^4 \; \mathrm{Simp}^{P^4}(23456) + \mathrm{Simp}^{P^4}(12456) + \mathrm{Simp}^{P^4}(12346)$$

Where Simp^{P^4} is the volume of a simplex in P^4 with boundaries defined via a set of 5 momentum twistors. That is, the amplitude is a ~~functi~~ polytope!

§ Momentum Twistors §

Let's now look at what momentum twistors are, $W_i = \begin{pmatrix} \lambda_i \\ \mu_i \end{pmatrix}$ where μ_i is defined in (A.6). Firstly (A.6) can be solved with

$$\mu_i^{\dot\alpha} = \Big(\sum_{j=1}^{i-1} P_j^{\alpha\dot\alpha}\Big)\lambda_{i\alpha} \quad \text{(show this ! a fun exercise)}$$

Equivalently we can write.

$$\mu_i^{\dot\alpha} = y_j^{\alpha\dot\alpha}\lambda_{i\alpha} \qquad \text{where } y_j = \sum_{j=1}^{i-1} P_j \quad \text{or } y_j - y_{j+1} = P_j$$

$$= y_{j+1}^{\alpha\dot\alpha}\lambda_{i\alpha}.$$

This is twistor's ~~indcident~~ incident relation, which defines spacetime points $\{y_i\}$ which are null separated

$$(y_j - y_{j+1})^2 = P_i^2 = 0$$

These relation are ~~better~~ easier to understand in ~~embedded~~ embedding space. A four-dimensional vector can be embedded as a point on six-dimensional projective null plane.

$$y^{\mu} \to \underset{\text{"null"}}{y^M y^N \eta_{MN} = 0} \qquad \underset{\text{"projective"}}{\rho y^M \sim y^M} \qquad M = 0,1,2,3,4,5$$

In bi-spinor formalism

$$y^{\alpha\dot\alpha} \to \epsilon_{ABCD}\, y^{AB} y^{CD} = 0 \quad (A.7) \qquad \rho\, y^{AB} \sim y^{AB} \quad \text{where } y^{AB} = -y^{BA} \qquad A,B = 1,2,3,4.$$

The solution to A.7 is simply

$$y^{AB} = W_1^{[A} W_2^{B]} \qquad \text{where } W_1, W_2 \text{ is the bi-twistor that defines } y.$$

More precisely $y^{[AB} W_1^{C]} = y^{[AB} W_2^{C]} = 0$

If we have two points (y_a, y_b) defined through two pairs of twistors. $y_a^{AB} = \omega_{a1}^{[A}\omega_{a2}^{B]}$ $\quad y_b^{AB} = \omega_{b1}^{[A}\omega_{b2}^{B]}$

Then the distanstance between two-points will be proportional to

$$(y_a - y_b)^2 \propto \epsilon_{ABCD}\,\omega_{a_1}^{A}\omega_{a_2}^{B}\omega_{b1}^{C}\omega_{b2}^{D} = \langle a1, a2, b1, b_2\rangle$$

(up to some thing called infinity twistor)

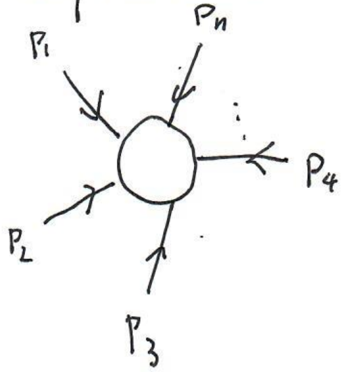
Note that since $y_j - y_{j+1} = P_j \Rightarrow (y_j - y_{j+1})^2 = 0$ so consequtive points are null separated, that is, ~~the p~~ there must be an overlap between $\langle j\,j_1 j_2$ twistors for y_j, and y_{j+1}. Since we had $u_j = y_j\lambda_j = y_{j+1}\lambda_j$

This tells us

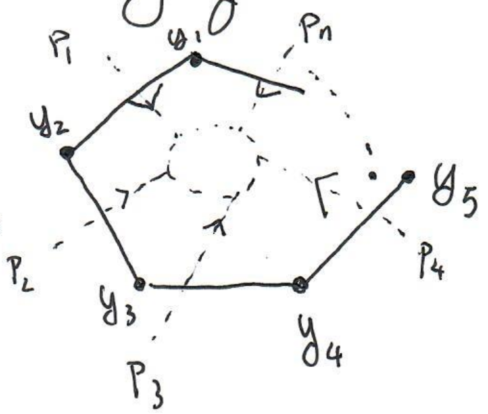
$$y_j \Rightarrow (\omega_{j-1}, \omega_j) \quad y_{j+1} \Rightarrow (\omega_j, \omega_{j+1})$$

Thus y_j, y_{j+1} shares the twistor ω_j

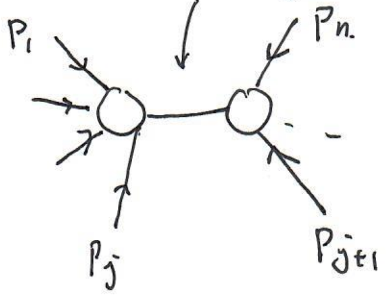
Let us summarize what we have established so far. The amplitude is a function of momenta statisfying momentum conservation

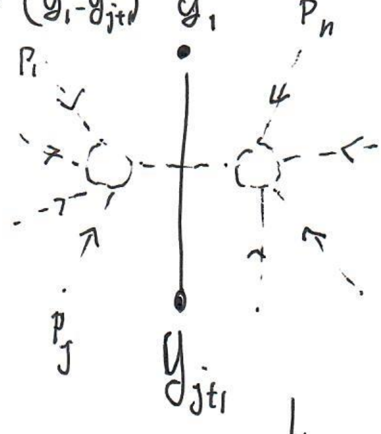


this data can be equivalently encoded in in a closed polygon with vertices y_j



Then a propagator $\frac{1}{(P_1+P_2+\cdots P_j)^2}$ is equal to $\frac{1}{(y_1 - y_{j+1})^2}$





In general propagators take the form $\frac{1}{(y_i - y_j)^2} \propto \frac{1}{\langle i-1\, i\, j-1\, j\rangle}$

But wait, these are precisely the boundaries associated with.
same form as the
simplices that built the 6-pt amplitude which did not cancel in pairs in the sum! It should now be obvious that the amplitude is simply a polytope with boundaries given by all pairs of $(i\,j) \rightarrow \langle i\, i{+}1\, j\, j{+}1 \rangle$. ~~These boundaries are a reflection~~ c. Said in another way, the amplitude is associated with a space carved out by all possible singularities i.e the propagators!

It will be convenient to denote the homogeneous coordinate of P^5 as Y^A, and the polytope is characterized by

$$\langle Y\, i\, i{+}1\, j\, j{+}1 \rangle \geq 0$$

where again $\langle Y i\, i{+}1\, j\, j{+}1 \rangle \equiv \epsilon_{ABCDE}\, Y^A W_i^B W_{i+1}^C W_j^D W_{j+1}^E$

The "volume-form" or more precisely, the canonical form which has logarithmic singularity at its boundaries. This is the simplest geometry that is refered to as the Amplituhedron.

Aftermath:

1. What we have given is the tree amplituhedron for $K=1$. For $K>1$ we are no longer in P^5 but ~~a~~ rather in $Y_a^{\mathcal{I}}$ where $a=1,2,\cdots K$ and $\mathcal{I}=1,2,\ldots 4+K$. the space is then defined as

$$\langle Y_1 Y_2 \cdots Y_K\, i\, i{+}1\, j\, j{+}1\rangle \geq 0$$

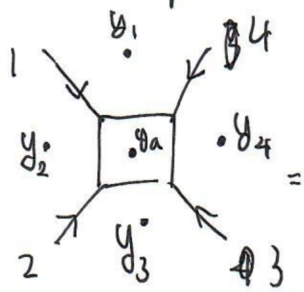
Where now the momentum twistor is $4+K$ components with

$$\mathcal{W}_i^{\mathcal{A}} = \left(\begin{array}{c} W_i^A \\ \phi_1\cdot\eta_i \\ \phi_2\cdot\eta_i \\ \vdots \\ \phi_K\cdot\eta_i \end{array}\right)\Bigg\}\, 4+K$$

that is we ~~at~~ have K grassmann odd ϕ

Note that the space is defined as a polynomial in Y coordinates, no longer linear. This is why it's called "hedron" and note amplitupolytope.

2. At loop level the integrand involves an extra y_a for each loop. For example:



$$= \int \frac{d^4 l}{l^2 (l-P_1)^2 (l-P_1-P_2)^2 (l+P_4)^2} = \int \frac{d^4 y_a}{(y_a-y_1)^2 (y_a-y_2)^2 (y_a-y_3)^2 (y_a-y_4)^2}$$

$$\sim \int \frac{d^4 W_A\, d^4 W_B}{GL(2)\ \langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

where (W_A, W_B) is the twistor pair for y_a, and the MOD $GL(2)$ is because any $GL(2)$ rotation of the two twistors.

~~It's~~ One can guess the space is now

$$\langle Y_1 \cdots Y_K\, i\, i{+}1\, j\, j{+}1\rangle \geq 0 \quad (B.1)$$

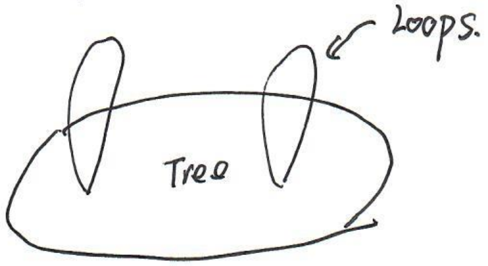
$$\langle Y_1 \cdots Y_K\, A_\ell B_\ell\, i\, i{+}1\rangle \geq 0 \quad (B.2) \qquad (1312.2007)$$

$$\langle Y_1 \cdots Y\, A_\ell B_\ell\, A_\ell B_\ell\rangle \geq 0 \quad (B.3) \qquad (1704.05069)$$

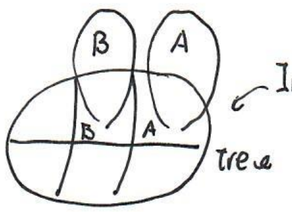
3. We can seperate the inequality in two parts

Tree region (B.1)   Loop-region (B.2) (B.3)

Then the geometry can be viewed as "loop fibers" on top of tree



As it turns out, the tree-region can be dissected into "chambers" for which in each chamber the loop form is homogeneous (analyticaly takes the same form)



In chamber A the loop-form is the same function of external data. throughout.. It is distinct from chamber B. But at the boundary, these special kinematic configuration the two-matches.

(2306.00951)

4. ~~Amplituhedron also gives the 3-dimensional~~

One can derive a 3-d projected geometry by imposing the constraint on momentum twistors $\omega_i^A \omega_j^B \Omega_{AB} = 0$ where Ω_{AB} is the $Sp(4)$ invariant metric.

The result is the ABJM amplitude. (2306.00951) (2204.08297)

5. Similar geometry was defined for correlation functions. the correlahedron.

(2405.20292)
(1701.00453)