here we use the notation $\langle \lambda \rho \rangle \equiv \epsilon_{\alpha\beta} \lambda^\alpha \rho^\beta = -\langle \rho \lambda \rangle$
$[\tilde{\lambda} \tilde{\rho}] \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\alpha}} \tilde{\rho}^{\dot{\beta}} = -[\tilde{\rho} \tilde{\lambda}]$

You can check that indeed $\epsilon^\mu \cdot P_\mu = \epsilon^{\alpha\dot{\alpha}} P_{\alpha\dot{\alpha}} = \epsilon^{\alpha\dot{\alpha}} \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = 0$

the auxilary spinors $(\rho, \tilde{\rho})$ just represent the "gauge" ambiguity in defining them.

All in all, we see that we are asking for a function in $\lambda, \tilde{\lambda}$ space.

$$f(P_i, \epsilon_i) \Longrightarrow f(\lambda_i, \tilde{\lambda}_i)$$

Now before moving on, let's consider the most important amplitude in this story, the three-point amplitude

§ The tale of 3-pt amplitude §

A general 3-pt amplitude is given by. (due to Lorentz inv

$$f(1^{h_1} 2^{h_2} 3^{h_3}) = \langle 12\rangle^a \langle 23\rangle^b \langle 31\rangle^c [12]^{\bar{a}} [23]^{\bar{b}} [31]^{\bar{c}}$$

where $\{h_1\, h_2\, h_3\}$ are the helicities of legs 1, 2, 3 and we must have

$$h_1 = -\frac{a}{2} + \frac{\bar{a}}{2} - \frac{c}{2} + \frac{\bar{c}}{2}$$
$$h_2 = -\frac{a}{2} + \frac{\bar{a}}{2} - \frac{b}{2} + \frac{\bar{b}}{2}$$
$$h_3 = -\frac{b}{2} + \frac{\bar{b}}{2} - \frac{c}{2} + \frac{\bar{c}}{2}$$

to match the helicity weights on both sides

How ever recall that the amplitude has support on momentum conservation.

$$f(\lambda, \tilde{\lambda}) = (\cdots) \, \delta^4 \left(\sum_j \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \right)$$

Comment 1: the constraint is a quadrtic constraint on the kinematic data.

Comment 2: At · 3-pts since $(P_1 + P_2)^2 = P_3^2 = 0$

$$\rightarrow 2 P_1 \cdot P_2 = \langle 12 \rangle [21] = 0$$

So we have that either $\langle 12 \rangle = 0$ or $[12] = 0$ (if the momenta is complexified)

If $\langle 12 \rangle = 0$ then λ_1 and λ_2 are proportional to each other, i.e.

$\lambda_1^\alpha = a_1 \chi^\alpha$, $\lambda_2^\alpha = a_2 \chi^\alpha$ where χ^α is some 2-dimension vector. Then from momentum conservation

$$-\lambda_3^\alpha \tilde{\lambda}_3^{\dot{\alpha}} = \lambda_1^\alpha \tilde{\lambda}_1^{\dot{\alpha}} + \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}} = \chi^\alpha \left(a_1 \tilde{\lambda}_1^{\dot{\alpha}} + a_2 \tilde{\lambda}_2^{\dot{\alpha}} \right)$$

So this immediately tells us that λ_3 is also proportional to χ^α.

So in conclusion at three points, either

All $\langle ij \rangle = 0$ or $[ij] = 0$

So given a set of helicities the amplitude is completely determined.

Exp 1: $f(1^- 2^- 3^+) = \left\{ \dfrac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} , \cancel{\dfrac{[13][23]}{[12]^3}} \right\}$

← diverges when $\tilde{\lambda}_1 // \tilde{\lambda}_3 // \tilde{\lambda}_2$

Exp 2: $f(1^{-2} 2^{-2} 3^{+2}) = \dfrac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2}$

Comment 1. Note that for vector $f(1^- 2^- 3^+)$ is anti-symmetric in $1 \leftrightarrow 2$ exchange which violates spin-statistics. This tells us spin-1 cannot self-interact UNLESS there are multiple spin-1s.

$$f(1^{-a} 2^{-b} 3^{c}) = \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} g^{abc}$$

where a, b labels the species of the spin-1 and g^{abc} are totally anti-symmetric.

Comment 2. Gravity = $(YM)^2$!!

§ Grassmannian incoming ! §

The momentum conservation constraint on my kinematic data is manifested as LINEAR constraints ~~[illegible]~~. This motivates us to rewrite momentum conservation delta function as linear in λ and $\tilde{\lambda}$.

Proposal: The 3-pt amplitude of $f_3(1^+ 2^+ 3^-) = \frac{[12]^3}{[13][23]} \delta^4(p)$

is the same as

$$\int \frac{dC_1\, dC_2}{C_1 C_2} \delta^2(C \cdot \tilde{\lambda})\, \delta^4(C_\perp \cdot \lambda)$$

where C is a 1×3 matrix and $C_\perp$ is the complement of C which is 2×3. Said in another way, C is a vector in 3-dim and $C_\perp$ is the 2-plane that is orthogonal to that vector.

More explicitly. $C = \{C_i\} = \{c_1, c_2, 1\}$

$$C_\perp = \{C_{\perp a i}\} = \begin{Bmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{Bmatrix} \quad \text{where}$$

Indeed $C \cdot C_\perp = \sum_{i=1}^{3} C_i C_{ai} = 0$ for $a = 1, 2$.

So more explicitly

$$\frac{[12]^3}{[13][23]} \delta^4(p) = \int \frac{dc_1}{c_1} \frac{dc_2}{c_2} \, \delta^2(c_1 \tilde{\lambda}_1^{\dot\alpha} + c_2 \tilde{\lambda}_2^{\dot\alpha} + \tilde{\lambda}_3^{\dot\alpha}) \, \delta^2(\lambda_1^\alpha - c_1 \lambda_3^\alpha) \, \delta^2(\lambda_2^\alpha - c_2 \lambda_3^\alpha) \quad (**)$$

Thus we see that the last two delta functions precisely enforces that λs are proportional. Let us check:

Step 1: use the first delta function to solve for c_1, c_2

$$c_1 \tilde{\lambda}_1^{\dot\alpha} + c_2 \tilde{\lambda}_2^{\dot\alpha} + \tilde{\lambda}_3^{\dot\alpha} = 0 \quad (*)^{\dot\alpha}$$

$$\rightarrow (*)^{\dot\alpha} \tilde{\lambda}_{1\dot\alpha} = c_2 [21] + [31] = 0$$

$$(*)^{\dot\alpha} \tilde{\lambda}_{2\dot\alpha} = c_1 [12] + [32] = 0$$

$$\text{So } \delta^2(c_1 \tilde{\lambda}_1^{\dot\alpha} + c_2 \tilde{\lambda}_2^{\dot\alpha} + \tilde{\lambda}_3^{\dot\alpha}) = \frac{1}{[21]} \delta\left(c_2 + \frac{[31]}{[21]}\right) \delta\left(c_1 + \frac{[32]}{[12]}\right)$$

$$\text{So } (**) = \frac{[21]}{[31][32]} \delta^2\left(\lambda_1 + \frac{[32]}{[12]} \lambda_3\right) \delta^2\left(\lambda_2 + \frac{[31]}{[12]} \lambda_3\right)$$

$$= \frac{[21]^3}{[31][32]} \delta^4(p) \quad (*)$$

where in the last equality we've identify the constraint $\sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i$ with 2 linear constraint by projecting the former onto $\tilde{\lambda}_2$ and $\tilde{\lambda}_1$ basis.

We would like to write the formula in a more invariant form, i.e so the we don't need choose

$C_1 = (c_1, c_2, 1)$ but rather $C_1 = (c_1, c_2, c_3)$

Let us assume a $GL(1)$ symmetry where $C_1 \to \rho C_1$

If such a symmetry exists, then we can consider $C_1 = (c_1, c_2, 1)$ as a gauge fixed version of general C_1 where $c_3 = 1$.

If it were so, then we can write our formula as

$$\int \frac{d^3 C_1}{c_1 c_2 c_3} \; \delta^2(C_1 \tilde{\lambda}) \, \delta^{2\times 2}(C_{1\perp} \lambda)$$

The delta functions has simple meaning: $\tilde{\lambda}$ as a 2-plane: $\begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$ is in $\tilde{C}_\perp$ and hence $\sum_i \lambda_i \tilde{\lambda}_i = 0$!!

Now let's try to see if the $GL(1)$ weight works out. Note that our $GL(1)$ scaling is with respect to C_1, however our integral formula is in terms of C_1 and $C_{1\perp}$ which is a bit inconvenient. Let's remedy this.

Recal the condition $\delta^{2\times2}(C_{1\perp} \cdot \lambda)$ forces λ to be inside C_1 this says that C_1 must be proportional to λ. i.e

$$C_1 = (c_1, c_2, c_3) = \alpha(\lambda_1^1, \lambda_2^1, \lambda_3^1) = \beta(\lambda_1^2, \lambda_2^2, \lambda_3^2)$$

where α, β is just some proportionality factors. Thus we have

$$\delta^{2\times2}(C_{1\perp} \cdot \lambda) = \int d^2\rho \prod_{i=1}^{3} \delta^2(\rho^\alpha c_i - \lambda_i^\alpha)$$

where $\rho^1 = \alpha$ and $\rho^2 = \beta$. You can see on the RHS we have 6 delta-functions sans two integrations, matching the 4 delta functions on the RHS

Next, we consider a fourier-transform $\lambda^\alpha \to \mu^\alpha$

$$\int \prod_{j=1}^{3} d^2\lambda_j \, e^{i(\lambda_j \cdot \mu_j)} \times \int \frac{dc_1 \, dc_2 \, dc_3}{c_1 \, c_2 \, c_3} \, \delta^2(C \cdot \tilde{\lambda}) \int d\rho^2 \prod_{k=1}^{3} \delta^2(\rho c_k - \lambda_k)$$

$$= \int \frac{dc_1 \, dc_2 \, dc_3}{c_1 \, c_2 \, c_3} \, \delta^2(C \cdot \tilde{\lambda}) \int d^2\rho \, e^{i\rho^\alpha (C \cdot \mu)_\alpha}$$

$$= \int \frac{dc_1 \, dc_2 \, dc_3}{c_1 \, c_2 \, c_3} \, \delta^4(C \cdot Z) \qquad \text{where } Z_i^A = \begin{pmatrix} \mu_i^\alpha \\ \tilde{\lambda}_i^{\dot{\alpha}} \end{pmatrix}$$

Now we immediately see that things are not GL(1) inv. Since the integration measure is of the form $\frac{dx}{x}$ it's scale inv. However the bosonic delta-function breaks the symmetry. The cure is clear! Just extend the four-component bosonic spinors to fermionic ones.

$$Z_i^A \to \mathcal{Z}_i = \begin{pmatrix} Z_i^A \\ \eta_i^I \end{pmatrix} \qquad \text{where } I = 1,2,3,4$$

That is, we embed the 3-pt amplitude in a super-amplitude with $\mathcal{N}=4$ supersymmetry. (see 1103.3477 for an overview of superamplitudes and tree-amplitudes)

That is, the three-pt amplitude for $\mathcal{N}=4$ SYM can be written as

$$\mathcal{A}_3^{k=1} = \int \frac{d^3C}{GL(1)\, M_1 M_2 M_3} \, \delta^{4|4}(C \cdot \mathcal{Z})$$

Here, I used the notation M_i to represent the i-th minor of the 1×3 matrix C.

Ofcourse there is another solution for 3-pt amplitudes, those whose $\tilde{\lambda}$ are proportional to each other. Since this amounts to interchanging λ and $\tilde{\lambda}$ through out the previous analysis, you can deduce the correct formula is utilizing a 2×3 C, ie.

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix}$$

and

$$A_3^{k=2} = \int \frac{d^6 C}{GL(2)\, M_1 M_2 M_3} \delta^{4\times 2|4\times 2}(C\cdot Z)$$

where now the minors are

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix}$$

M_1 M_2. e.t.c.

Comments:

1. We see in this formulation, the kinematic data is just a spectator, appearing only in the universal delta function whose form is identical between the two types of 3-pt amplitude.

2. The difference between the two distinct amplitudes are captured by the number of rows in C, i.e, k.

$k=1$ $[C]_{1\times 3} \rightarrow A_3([ij])$

$k=2$ $[C]_{2\times 3} \rightarrow A_3(\langle ij\rangle)$

In the future, we will use the notation C_{ai} where $a=1,2,..,k$ and $i=1,2,\cdots,n$ the number of particles.

3. Interchanging $\lambda \leftrightarrow \tilde{\lambda}$ (parity) amounts to $k \leftrightarrow (n-k)$

We are now ready for a bold proposal: the n-point amplitude is related to the integral

$$\int \frac{d^{n\times k}C}{GL(k).\, M_1 M_2 \cdots M_n}\, \delta^{4\times k\,|\,4\times k}(C\cdot \mathcal{Z}) \qquad (A.1)$$

or equivalently

$$\int \frac{d^{n\times k}C\; \delta^{2\times k}(C\cdot\tilde\lambda)\, \delta^{2\times(n-k)}(C_\perp\cdot\lambda)}{GL(K)\, M_1 M_2 \cdots M_n.}\, \delta^{4K}(C\cdot\eta) \qquad (A.2)$$

To establish the connection, one needs to clarify a few points

1. Tree amplitudes are generically rational functions of external kinematic data. How do we get a rational function out of an integral?

   At 3 pts we see that the bosonic delta functions would completely localize the integral. Indeed from (A.2)

$K=1 \quad n=3$

~~$1\times 3\ (C_s) - 1\ (GL(1)) - 2 -$~~

$$1\times 3 - 2 - 4 - 1 + 4 = 0$$

$(C_s) \quad (\delta(C\cdot\tilde\lambda)) \quad (\delta(C\lambda) \quad GL(1) \quad \delta^4(p)$

Not all delta functions can be used on C.

$K=2 \quad n=3$

$$2\times 3 - 4 - 2 - 4 + 4 = 0$$

$(C_s) \quad (\delta(C\tilde\lambda)) \quad (\delta(C\lambda)) \quad GL(2) \quad \delta^4(p)$

However for general n, K, we have

$$K\times n - 2K - 2(n-K) - K^2 + 4 = (K-2)(n-K-2) \qquad (A.3)$$

So we see that in general there aren't sufficient delta functions to localize the integrals

Indeed for $n=6$ $k=3$ we have 1 integral left. (note that A.3 is invariant under $k \leftrightarrow n-k$ as indicated by parity) In general, we will have more integration variables than delta functions, how can we get rational things?

2. What does the n-pt amplitude actually look like? are they uniquely determined?

3. What exactly is K?

The last question is the easiest. Recall that K also appears in the fermionic part of (A.1) (A.2) comes with multiples of K. The grassmann odd variable η_i^I has SU(4) R-symmetry indices $I=1,2,3,4$ and.

$$\delta^{4K}(C\cdot\eta^I) = \cancel{\epsilon_{I_1I_2I_3I_4}\,\delta^K(C\cdot\eta^{I_1})}$$

$$\frac{1}{4!}\epsilon_{I_1I_2I_3I_4}\,\delta^K(C\cdot\eta^{I_1})\,\delta^K(C\cdot\eta^{I_2})\,\delta(C\cdot\eta^{I_3})\,\delta(C\cdot\eta^{I_4})$$

where $\epsilon_{I_1I_2I_3I_4}$ is the SU(4) invariant Levi-cevita tensor.

So our amplitude is categrorized interms of the number of such SU(4) invariants. K means that we have K of them. In terms of component amplitude this K is related to the pure glouon amplitude have K negative helicities. This can be seen from the on-shell superfield having the expansion

$$\Phi(\eta) = A^{+1} + \eta^I\psi_I^{+\frac{1}{2}} + \frac{1}{2}\eta^I\eta^J\phi^0_{IJ} + \frac{1}{3!}\eta^I\eta^J\eta^K\epsilon_{IJKL}\,\psi^{-\frac{1}{2}L}$$

$$+\frac{1}{4!}\eta^I\eta^J\eta^K\eta^L\epsilon_{IJKL}\,A^{-1}$$

§ n-pt recurssions and the grassmannian §.

The BCFW recurssion relations:

Consider a deformation of our amplitude via a complex parameter z.

$$A_n \Big|_{\substack{\vec{P}_1 \to \vec{P}_1 + z\vec{q} \\ \vec{P}_n \to \vec{P}_n - z\vec{q}}} = A_n(z)$$

Here, we perform the shift on legs 1 and n although shifting any two legs works

where the shift vector $\vec{q}$ satisfies

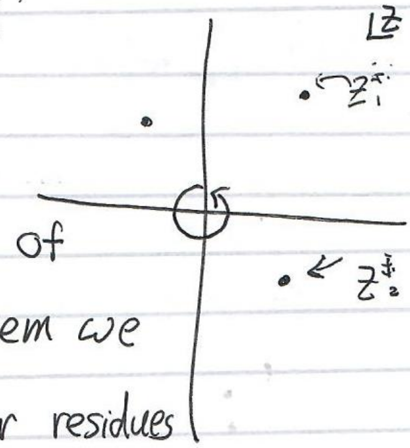
$$\vec{q}\cdot\vec{P}_1 = \vec{q}\cdot\vec{P}_n = \vec{q}^{\,2} = 0$$

Since P_1, P_2 are massless there are two solutions

$$q^{\alpha\dot\alpha} = \lambda_1^{\alpha}\tilde{\lambda}_n^{\dot\alpha} \quad \text{or} \quad \lambda_n^{\alpha}\tilde{\lambda}_1^{\dot\alpha}$$

We now proceed with a trivial identity:

$$A_n = A_n(0) = \frac{1}{2\pi i}\oint \frac{dz}{z} A_n(z)$$

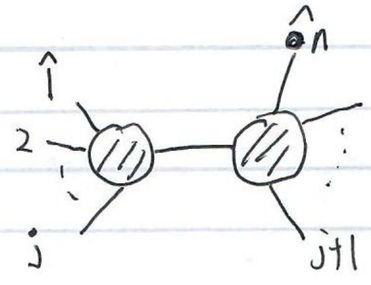


where the contour encircles the origin of the complex z plane. From Cauchy's theorem we know that this is equal to the sum over residues in z

$$A_n = \sum_{z^*} \operatorname{Res}_{z=z^*}\left[\frac{A_n(z)}{z}\right]$$

The reason we know that there are poles on the complex z plane is because tree-level amplitudes have factorization pole.

For example



has a singularity $\dfrac{1}{(\hat{P}_1+P_2+\cdots P_j)^2}$

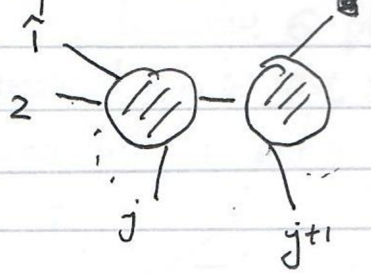
where $\hat{P}_1 = P_1 + z q$.

Note that $\dfrac{1}{(p_1+zq+p_2+\cdots p_j)^2} = \dfrac{1}{(p_1+p_2+\cdots p_j)^2 + z\, q\cdot(p_2+p_3+\cdots p_j)^2} = \dfrac{1}{P_I^2 + 2z\, q\cdot P_I}$

where $P_I = (p_1+p_2+\cdots+p_j)$ is the momenta flowing through the propagator. Thus we see that factorization singularity introduces a simple pole at

$$z = z^{*} = -\frac{P_I^2}{2q\cdot P_I}$$

The residue of this pole corresponds to a product of two on-shell amplitudes



$$\Bigg|_{z \Rightarrow -\frac{P_I^2}{2q\cdot P_I} = z^{*}} = \frac{A_L(z^{*}) \otimes A_R(z^{*})}{2q\cdot P_I\left(z + \frac{P_I^2}{2q\cdot P_I}\right)}$$

where $A_L(z^{*}) = A_L(\hat{p}_1, p_2, \cdots, p_j, -\hat{P}_I)\big|_{z=z^{*}}$ where $\hat{P}_I = \hat{p}_1 + p_2 + \cdots p_j$

$A_R(z^{*}) = A_R(p_{j+1}, \cdots, \hat{p}_n, \hat{P}_I)\big|_{z=z^{*}}$

Thus the n-pt amplitude can be computed via the recursion relation

$$A_n = \sum_{z=z^{*}} \mathrm{Res}\left[\frac{A_n(z)}{z}\right] = \sum_{z=z^{*}} \frac{A_L(z^{*}) \otimes A_R(z^{*})}{P_I^2}$$

where the sum sums over all possible factorization channels with the shifted legs on opposite side of the factorization channels.

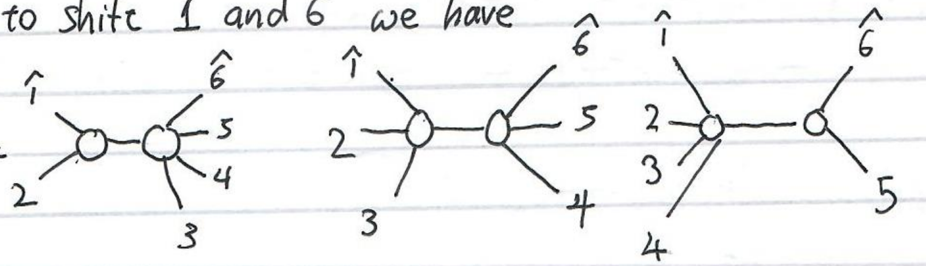
Comment 1. In principle we need to worry about poles at infinity. For our purpose, it suffice to say that for YM and gravity ~~no~~ one can ~~organize~~ arrange the deformation such that no poles appear at infinity.

Comment 2: One might be suspicious as it would seem that factorization channels associated with the two shifted legs on the same side appears to be missing. No worries, they are hidden in the non-local nature of $z^{I} = \frac{-P_I^2}{2q\cdot P_I}$

Comment 3: Different choice of shifted momenta, actually gives different but equivalent representation of the same result.

Example: consider the 6-pt amplitude for YM theory. The tree-level amplitude can be cast into trace basis. Then the partial amplitude is ordered. Choosing to shift 1 and 6 we have

~~tree-level~~

$A_6 =$ 

$$= \frac{A_3(z)\,A_5(z)}{(P_1+P_2)^2}\Bigg|_{z=\frac{P_{12}^2}{-2q\cdot P_{12}}} + \frac{A_4(z)\,A_4(z)}{P_{123}^2}\Bigg|_{z=\frac{P_{123}^2}{-2q\cdot P_{123}}} + \cancel{\frac{A_5(z)\,A_3(z)}{P_{45}^2}}\;\; z=\frac{P_{45}^2}{-2q\cdot P_{45}}$$

Fig 1.

Let us choose $q=\lambda_1\tilde{\lambda}_6$. This gives a "particular" representation of the amplitude. I say particular, because we could have choosen $q=\lambda_6\tilde{\lambda}_1$ or to shift two different legs entirely. each modification will yield a different representation of the amplitude, where the answer is split up into different pieces. ~~This begs the question~~ Said in another way, the amplitude seems to be some invariant object where the bcfw terms are the building blocks for this object.

What is this object?

Let's take a small detour and consider our favorite object:

$$C = \{C_{ai}\} = \begin{Bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{k1} & & \cdots & C_{kn} \end{Bmatrix}$$

If we view this as a collection of row vectors, we have k-vectors in n-dimensions. Recall that we are interested in such objects, mod GL(K) transformations. This is precisely the "space" of k-planes in n-dimmensions. We denote this as Gr(k,n)

We will be interested in a subclass of grassmaniann termed "positive grassmannian" where all ordered k×k minors are non-negative, i.e.

$$\det[C_{i_1} C_{i_2} \cdots C_{i_k}] = \frac{1}{k!} \epsilon_{a_1 a_2 \cdots a_k} C_{a_1 i_1} C_{a_2 i_2} \cdots C_{a_k i_k} \geq 0$$

$$\forall\ i_1 < i_2 < \cdots < i_k$$

For such grassmannian, the space can be conveniently stratified in terms of "cells", which is like characterizing the boundaries of a polytope. Due to positivity constraints the boundary occurs when consecutive columns become colinear, and hence the minors can vanish. The "Top Cell" is then the space where all ordered minors are strictly positive

The codimension-1 cells are those which "one" of the consecutive minors vanish, i.e $\{M_i = 0\}$. Co-dimension-2 cells are those where 2 ~~co-dimension~~ consecutive minors are zero

$$\{M_i = 0,\ M_j = 0\}$$

and so forth.

Example:

We are now ready to put the two stories together

Firstly, we see that at six point, a BCFW representation will involve 3 terms, and different deformations lead to different sets of 3 terms that ~~same~~ add to the same amplitude. Is there a deeper understanding of this equivalence?

Second, we see that the grassmaniann integral formula at $n=6$ $k=3$ is a one dimensional integral, i.e.

$$\int \frac{dC^{3\times 6}}{GL(3)\, M_1 M_2 \cdots M_6} \delta^{3\times 2}(C_\perp \tilde{\lambda})\, \delta^{3\times 2}(C_\perp \cdot \lambda)\, \delta^{3\times 4}(C \cdot \eta)$$

$$= \delta^4(p) \int \frac{d\tau \cdot \delta^{3\times 4}(C(\tau)\cdot \eta)}{M_1(\tau) M_2(\tau) \cdots M_6(\tau)} \qquad (A.3)$$

where in the second line, we used the fact that the bosonic delta functions are sufficient to solve C down to 1 d.o.f. which we denote as τ and the solution $C(\tau)$.

~~It turns out~~, To get a rational function, it is natural to consider (A.3) as a contour integral which localizes on the zeros of $M_1(\tau) \cdots M_6(\tau)$. So we will get 6 rational functions, one for each residue! We denote as

$\{M_1=0\}$ $\{M_2=0\}$ $\{M_3=0\}$ $\{M_4=0\}$ $\{M_5=0\}$ $\{M_6=0\}$

Remarkably, these are 1 to 1 correspond~~ing~~ing to the BCFW terms! For example, the 3 terms in fig 1 with $q = \lambda_1 \tilde{\lambda}_6$ are precisely $\{M_1=0\}$ $\{M_3=0\}$ $\{M_5=0\}$

while for $q = \lambda_6 \tilde{\lambda}_1$ are precisely the negative of $\{M_2=0\}$ $\{M_4=0\}$ $\{M_6=0\}$

Thus the equivalence of two different BCFW deformations can be simply represented as.

$$\{M_1=0\} + \{M_3=0\} + \{M_5=0\} = -\{M_2=0\} - \{M_4=0\} - \{M_6=0\}$$

This is nothing but Cauchy identity! Further more recall that the condition of $\{M_i=0\}$ is equivalent to ~~that~~ the ~~co~~ codimension 1 cell of $Gr(3,6)$.

Thus we see that each BCFW term is identify with a lower dimensional cell of $Gr^{\geq 0}(k,n)$, these cells have dimension

$$\underset{\substack{\uparrow \\ \text{dim of top cell}}}{(n\times k-k^2)} - \underset{\substack{\uparrow \\ \text{number of} \\ \text{integrals to localize.}}}{(k-2)(n-k-2)} = \underline{2n-4}$$

This fascinating connection leads to the following question.

What are these cells building? or What is the amplitude?