Stochastic quantization of Liouville conformal field theory

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June 24, 2022

with

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Harmonic Analysis, Stochastics and PDEs

in Honour of the 80th Birthday of Nicolai Krylov

Liouville conformal field theory (LCFT)

Consider the Liouville action $S_{\mathcal{L}}$ is defined on paths $u : \mathcal{M} \to \mathbb{R}$ by

$$\text{(LCFT)} \qquad S_{\mathcal{L}}(u; \mathrm{g}) \stackrel{\text{def}}{=} \frac{1}{4\pi} \int_{\mathcal{M}} \left\{ |\nabla_{\mathrm{g}} u|^2 + Q\mathcal{R}_{\mathrm{g}} u + 4\pi\nu e^{\beta u} \right\} dV_{\mathrm{g}},$$

- $(\mathcal{M}, \mathrm{g})$ is a two-dimensional connected, closed (compact, boundaryless), orientable Riemannian manifold.
- $\mathcal{R}_{\mathrm{g}}$ is the Ricci scalar curvature.
- $\nu > 0$, $\beta > 0$ and the charge $Q = \frac{2}{\beta} + \frac{\beta}{2}$.

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Answer: No.

- $e^{\beta u}$ does not make sense for rough u.
- Uniform disribution at the zero mode.

LCFT - punctures and renormalization

Consider

$$d\rho_{\{a_\ell,x_\ell\},g}(u) = \overbrace{\prod_{\ell=1}^{L} e^{a_\ell u(x_\ell)}}^{\text{punctures}}\, e^{-\frac{1}{4\pi}\int_{\mathcal{M}}\{|\nabla_g u|^2 + Q\mathcal{R}_g u + 4\pi\nu \overbrace{:e^{\beta u}:}^{\text{renormalization}}\}dV_g} Du.$$

where $a_\ell \in \mathbb{R}\backslash\{0\}$ and $x_\ell \in \mathcal{M}$.

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- Sub-critical regime: $0 < \beta < 2$;
- First Seiberg bound: $\chi(\mathcal{M})Q < \sum_{\ell=1}^{L} a_\ell$;
- Second Seiberg bound: $\max_{1\le\ell\le L} a_\ell < Q = \frac{2}{\beta} + \frac{\beta}{2}$.

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Remark: The above implies $\chi(\mathcal{M}) < L$. Therefore $\chi(\mathbb{S}^2) = 2$ implies $L \geq 3$; and $\chi(\mathbb{T}^2) = 0$ implies $L \geq 1$.

Stochastic Liouville equations (SL): Consider

$$\text{(SL)} \qquad \partial_t u - \frac{1}{4\pi}\Delta_{\mathrm{g}} u + \frac{Q}{8\pi}\mathcal{R}_{\mathrm{g}} + \frac{1}{2}\nu\beta : e^{\beta u} := \frac{1}{2}\sum_{\ell=1}^{L} a_\alpha \delta_{x_\ell} + \xi_{\mathrm{g}},$$

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Main Goal

(i) Local and global dynamics on $\mathcal{M}$.
(ii) Invariance of the measure $\rho_{\{a_\ell, x_\ell\},\mathrm{g}}$ under the resulting flow.

Main result - stochastic quantisation

Main Theorem (Oh-Robert-Tzvetkov-W., 20)

Let $a_{\ell_{\max}} = \max\{a_\ell\}$ and assume

- Sub-critical regime: $0 < \beta < \sqrt{2}$;
- First Seiberg bound: $\chi(\mathcal{M})Q < \sum_{\ell=1}^{L} a_\ell$;
- Integrable insertions: $a_{\ell_{\max}} < \frac{2}{\beta}$.

Measure construction independent of the approximation procedure.

- Extra condition: $0 < \beta < \sqrt{a_{\ell_{\max}}^2 + 4} - a_{\ell_{\max}}$.

Globally well-posed and the invariance of the measure.

Remark: The conditions are not optimal.

Related results - I: measure construction

For measure construction:

- $\mathcal{R}_g > 0$, David-Kupiainen-Rhodes-Vargas (2016),
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Remark: (i) Oh-Robert-Tzvetkov-W. dealt with all cases at once, but with a smaller range.

(ii) They used the circle average process $X_\varepsilon(z) = \int_0^{2\pi} X(z + \varepsilon e^{i\theta}) \frac{d\theta}{2\pi}$.

Related results - II: stochastic Ricci flow

Fix a metric g_0, consider

$$g = e^{u} g_0,$$

where the function u is the conformal factor. Consider

$$\partial_t u = e^{-2u} \Delta_{g_0} u + \nu e^{-u} \xi_{g_0} - \lambda$$

- Dubédat-Shen '19 constructed weak solutions.
- The measure is not normalizable.

Related results - III: Hoegh-Krohn model

A related model

$$\partial_t u + mu - \frac{1}{4\pi}\Delta_g u + \frac{1}{2}\nu\beta : e^{\beta u} := \xi_g$$

with $m > 0$ and $\mathcal{M} = \mathbb{T}^2$.

- For $\beta < \sqrt{2}$, Hoshino-Kawabi-Kusuoka, Oh-Robert-W. '19.
- For $\beta < 2$, Hoshino-Kawabi-Kusuoka '20.
- Hyperbolic case with $\beta < \sqrt{0.43}$, Oh-Robert-W. '19.

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Remark: (i) Albeverio-Röckner '91 constructed weak solutions on $\mathbb{R}^2$.
(ii) The mass terms with $m > 0$ destructs the conformal invariance of the Gibbs measure.

Known results - IV Garban (2019)

For the dynamical problem:

Garban (2019) considered when $\mathcal{M} = \mathbb{S}^2$ or $\mathbb{T}^2$, the last condition replaced by

$$\frac{\beta^2}{2} - 2\sqrt{2}\beta + \min\left(0, \frac{\beta}{2\sqrt{2}} - a_{\ell_{\max}}\beta\right) > -2.$$

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For instance, if $\mathcal{M} = \mathbb{T}^2$, $L = 1$, and $a_1 = \beta$,

- (Garban) $0 < \beta < \frac{\sqrt{2}}{2} \approx 0.707$;
- (ORTW) $0 < \beta < \sqrt{\frac{4}{3}} \approx 1.15$.

Main tools - Green's function

We set $\{\varphi_n\}_{n\geq 0} \subset C^\infty(\mathcal{M})$ to be a basis of $L^2(\mathcal{M}, \mathrm{g})$ consisting of eigenfunctions of Δ_{g} associated with the eigenvalue $-\lambda_n^2$, assumed to be arranged in increasing order: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ....$ In particular $\varphi_0 \equiv V_{\mathrm{g}}(\mathcal{M})^{-\frac{1}{2}}$ is constant.

- The Green's function

$$G_{\mathrm{g}}(x, y) = \sum_{n\geq 1} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^2} \sim -\frac{1}{2\pi} \log\left(\mathbf{d}_{\mathrm{g}}(x, y)\right) + f(x, y).$$

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Lemma (Oh-Robert-Tzvetkov-W., 20)

Let $\psi \in \mathcal{S}(\mathbb{R})$ *such that* $\psi(0) = 1$. *Then,*

$$\Big|(\psi \otimes \psi)\big(-N^{-2}\Delta_\mathrm{g}\big)G_\mathrm{g}(x, y) + \frac{1}{2\pi}\log\big(\mathbf{d}_\mathrm{g}(x, y) + N^{-1}\big)\Big| \lesssim 1$$

Remark: We may choose different ψ for approximations, they are all equivalent.

Stochastic convolution:

$$\Psi(t) = e^{\frac{t}{4\pi}\Delta_g} X_g + \int_0^t e^{\frac{t-t'}{4\pi}\Delta_g} dW_g(t').$$

- Mass-less Gaussian free field,

$$X_g(\omega) = \sum_{n\geq 1} \frac{\sqrt{2\pi}h_n}{\lambda_n}\phi_n.$$

- Wiener process

$$W_g(t) = \sum_{n\geq 1} \langle \xi_g, \mathbf{1}_{[0,t]}\phi_n\rangle_{t,g}\phi_n.$$

- $\Psi \in C(\mathbb{R}_+ : H_0^s(\mathcal{M}))$ for any $s < 0$.

Ideals - "punctured" Gaussian multiplicative chaos

Define the truncated "punctured" Gaussian multiplicative chaos Θ_N:

$$\Theta_N(t,x) = e^{\pi\beta^2 C_{\mathbf{P}}} N^{-\frac{\beta^2}{2}} \exp\left(\beta \mathbf{P}_N \Psi(t,x) + 2\pi\beta \sum_{\ell=1}^{L} a_\ell (\mathbf{P}_N \otimes \mathbf{P}_N) G_{\mathrm{g}}(x_\ell, x)\right)$$

where $C_{\mathbf{P}}$ is a constant depends on $\{\mathbf{P}_N\}$, $\mathbf{P}_N$ is a regularization operators.

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Lemma (ORTW 2020)

$$\int_{H_0^s(\mathcal{M},\mathrm{g})} \mathbb{E}\left[|Q_M \Theta_N(t,x)|^p\right] d\mu_{\mathrm{g}} \lesssim M^{p(\alpha-\varepsilon)} \left(f_{\alpha-\varepsilon,\{x_\ell\}}(x)\right)^{\frac{p}{2}}$$

Here Q_M is a projection, $a_\ell^+ = \max(a_\ell, 0)$, and

$$f_{\alpha-\varepsilon,\{x_\ell\}}(x) = \sum_{\substack{\ell_1,\ell_2=1 \\ \ell_1\neq\ell_2}}^{L} \left(1 + \mathbf{d}_{\mathrm{g}}(x_{\ell_1}, x)^{\alpha-\varepsilon-\beta a_{\ell_1}^+}\right)\left(1 + \mathbf{d}_{\mathrm{g}}(x_{\ell_2}, x)^{\alpha-\varepsilon-\beta a_{\ell_2}^+}\right)$$

$$+ \sum_{\ell=1}^{L} \mathbf{d}_{\mathrm{g}}(x_\ell, x)^{2\alpha-2\varepsilon-(p-1)\beta^2-2\beta a_\ell^+}$$

Ideas - sign-definite structure

After using the Da Prato-Debussche trick, (SL) is reduced to

$$\partial_t U - \frac{1}{4\pi}\Delta_g U + \frac{1}{2}\nu\beta e^{\beta(z+U)}\Theta = 0,$$

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Sign-definite structure: the above can be written as,

$$U(t,x) = -\frac{\nu\beta}{2}\int_0^t \int_{\mathcal{M}} P_{\mathrm{g}}(t-t', x, y)[e^{\beta(z+U)}\Theta](t', y)dV_{\mathrm{g}}(y)dt'$$

where $\nu, \beta > 0$ and P_{g} is the heat kernel on $(\mathcal{M}, \mathrm{g})$.

Thanks

Thank you for your attention!