Two-dimensional topological order and operator algebras

Yasu Kawahigashi

Edinburgh, October 2023

Two-dimensional topological order, fusion categories, and subfactors

Anyons are certain types of quasi-particles in two-dimensional topological order which are expected to be useful for realizing topological quantum computations. A mathematical framework to study anyons is modular tensor category.

Such tensor categories can be studied with certain 4-tensors and matrix product operators, where everything is finite dimensional. They are the same as bi-unitary connections studied in subfactor theory of Jones. We understand certain results in two-dimensional topological order in condensed matter physics in terms of subfactor theory.

Outline of the talk

We give the following presentations. We start with an abstract setting of fusion and modular tensor categories, and then present methods of studying such tensor categories using 4-tensors. Then we relate them to subfactor theory.

1. Anyons and modular tensor categories
2. Fusion categories and Drinfel′d centers
3. 4-tensors and matrix product operators
4. Subfactors and bi-unitary connections
5. Bi-unitary connections and 4-tensors
6. Hilbert spaces for gapped Hamiltonians
7. α-induction for bi-unitary connections

Anyons and topological quantum computing

A certain quasi-particle in dimension 2 is called an anyon and a set of finitely many anyons can display braid group statistics.

A modular tensor category gives a mathematical description of such a system of anyons and is expected to be useful for topological quantum computing, where an irreducible object of such a category corresponds to an anyon.

Traces of n anyons on a plane gives a braid of n strands in the 3-dimensional space. We are interested in a certain unitary representation of such a braid group arising from the modular tensor category and a braid plays a role of a program for topological quantum computation.

A modular tensor category and the Drinfel′d center

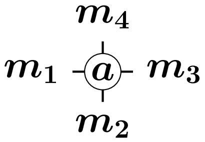
A braiding in a fusion category naturally comes in a pair — overcrossing and undercrossing. It is more interesting if these two are really different. If this is the case, the fusion category is called a modular tensor category.

For a given fusion category, the Drinfel′d center construction naturally gives a modular tensor category. An object of the Drinfel′d center is a pair of an object in the original fusion category and its half-braiding, which gives certain commutativity of the tensor products with all the other objects.

It is understood in terms of Ocneanu's tube algebra, a certain finite dimensional C^*-algebra arising from the fusion category.

Tensor networks

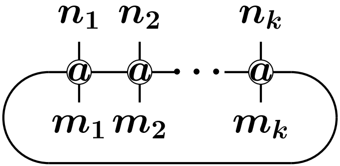
Suppose we have a family $\{a_{m_1,m_2,m_3,m_4}\}$ of finitely many complex numbers depending on 4 indices m_1, m_2, m_3, m_4. We draw a picture for this as follows.



We call this a 4-tensor since the value depends on 4 indices. Note that an ordinary vector is a 1-tensor and an ordinary matrix is a 2-tensor. A pictorial combination of tensors is called a tensor network. All fusion and modular tensor categories are described with tensor networks using 4-tensors.

Matrix product operators

We define a matrix product operator (MPO) O_a^k as follows, where a is a label for a 4-tensor arising from a subfactor and k is a positive integer. (We have a finite family of 4-tensors labeled with a.)

$\sum$ 

$| n_1 \cdots n_k \rangle \langle m_1 \cdots m_k |$

We show that the range of a weighted sum of O_a^k, a Hilbert space for their gapped Hamiltonian, has a nice interpretation in operator algebras.

Subfactors

We are interested in an algebra M of bounded linear operators acting on a fixed Hilbert space of states. We require that M is closed in the $*$-operation and an appropriate topology. If M does not decompose into a direct sum of two such algebras, then we say M is a factor. A finite dimensional factor is a matrix algebra $M_n(\mathbb{C})$. We are typically interested in so-called type II_1 factors.

When one factor N is contained in another factor M, we say $N \subset M$ is a subfactor. We can measure the size of M relative to N as the Jones index $[M : N]$. This is a positive real number ≥ 1 or ∞, not necessarily an integer.

The Jones theory

Jones developed a Galois type theory to describe the symmetry of $N \subset M$. From today's viewpoint, a natural algebraic structure for this type of symmetry is a fusion category.

By "reflecting" the inclusion $N \subset M$, we obtain a larger inclusion $M \subset M_1$. This is called the basic construction of Jones. We can repeat this to obtain the Jones tower.

$$N \subset M \subset M_1 \subset M_2 \subset \cdots.$$

The higher relative commutants $N' \cap M_k$ are finite dimensional C^*-algebras and contain important information about the subfactor.

A commuting square

Consider $\begin{array}{ccc} A & \subset & B \\ \cap & & \cap \\ C & \subset & D \end{array}$ where A, B, C, D are finite dimensional C^*-algebras with a trace on D. We say this is a commuting square if the restriction to C of the conditional expectation E_B from D to B is equal to the conditional expectation E_A from C to A.

In order to avoid some not-so-interesting examples, we require that BC, the span of the products bc with $b \in B$ and $c \in C$, is equal to D. Such a commuting square is said to be nondegenerate. In this talk, a commuting square means a finite dimensional nondegenerate commuting square.

Basic construction and subfactors

We start with a commuting square and repeat basic constructions horizontally.

$$\begin{array}{ccccccccc} A_{00} & \subset & A_{01} & \subset & A_{02} & \subset & A_{03} & \subset & \cdots \\ \cap & & \cap & & \cap & & \cap & & \\ A_{10} & \subset & A_{11} & \subset & A_{12} & \subset & A_{13} & \subset & \cdots \end{array}$$

This gives a sequence of commuting squares. The GNS-completions of $\bigcup_{n=1}^\infty A_{0n} \subset \bigcup_{n=1}^\infty A_{1n}$ with respect to trace give a type II_1 subfactor $A_{0,\infty} \subset A_{1,\infty}$ of finite Jones index. The vertical basic constructions give finite dimensional C^*-algebras A_{kn} with trace and we have the Jones tower:

$$A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset A_{3,\infty} \subset \cdots.$$

A bi-unitary connection

For a choice of one edge each from the four Bratteli diagrams of a commuting square, the connection W gives a complex number to each such square with the following, which is bi-unitarity.

$$\sum_{z,\xi_1,\xi_2} \begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array} \; \overline{\begin{array}{ccc} x & \xi_4' & y' \\ \xi_1 & \boxed{W} & \xi_3' \\ z & \xi_2 & w \end{array}} = \delta_{\xi_3,\xi_3'}\delta_{\xi_4,\xi_4'}$$

$$\begin{array}{ccc} y & \tilde{\xi}_4 & x \\ \xi_3 & \boxed{W'} & \xi_1 \\ w & \tilde{\xi}_2 & z \end{array} = \sqrt{\frac{\mu_x\mu_w}{\mu_y\mu_z}} \; \overline{\begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array}}$$

Bi-unitary connections on the Dynkin diagrams

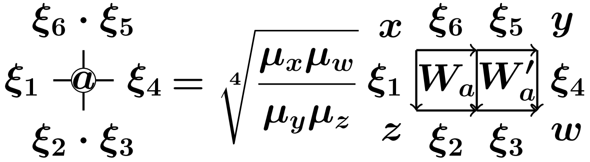
We give an example of a bi-unitary connection as follows. Fix one of the $\boldsymbol{A}$-$\boldsymbol{D}$-$\boldsymbol{E}$ Dynkin diagrams and use it for the four Bratteli diagrams. Let n be its Coxeter number and set $\varepsilon = \sqrt{-1}\exp\frac{\pi\sqrt{-1}}{2(n+1)}$. We write μ_x for the Perron-Frobenius eigenvector entry for a vertex x. Then our bi-unitary connection is given as follows.

$$\begin{array}{ccc} j & & k \\ & \boxed{W} & \\ l & & m \end{array} = \delta_{kl}\varepsilon + \sqrt{\frac{\mu_k\mu_l}{\mu_j\mu_m}}\delta_{jm}\bar{\varepsilon}$$

This is similar to a Boltzmann weight for a lattice model.

A 4-tensor and a bi-unitary connection

Suppose we have a bi-unitary connection W_a. We then define a 4-tensor a as follows.



Here W'_a stands for the horizontal reflection of W_a. We also use the vertical reflection so that we can concatenate 4-tensors as usual. The reflection corresponds to basic construction and the vertical concatenation of 4-tensors corresponds to the product of bi-unitary connections.

The range of a projector matrix product operator

The above matrix product operator O_a^k based on 4-tensors is studied in the context of 2-dimensional topological order by Bultinck et al. In this situation, we have finitely many anyons and each of them corresponds to a 4-tensor. Their certain weighted sum is a finite dimensional projection and its range has physical significance related to gapped Hamiltonians.

We started with a commuting square producing a subfactor with a certain finiteness condition. We can construct another subfactor $A_{\infty,0} \subset A_{\infty,1}$ by repeating basic constructions vertically. We have proved that the range of the above projection is equal to the higher relative commutant of $A_{\infty,0} \subset A_{\infty,1}$ (K 2021).

Possible 4-tensors for describing a given fusion category

Different 4-tensors can give the same subfactor and the same fusion category through the matrix product operators. When this happens is described in terms of Morita equivalence. Our 4-tensors are quantum $6j$-symbols arising from a fusion 2-category (K 2022).

It is easy to see that horizontal self-concatenation of 4-tensors produces isomorphic matrix product operators and isomorphic subfactors. There is another method, called reduction, which also produces the isomorphic subfactors and matrix product operators. We can show that except for these rather trivial operations, we have only finitely many 4-tensors that realize a given fusion category.

Subfactors in conformal field theory

A 2-dimensional conformal field theory is a quantum field theory with conformal symmetry. It splits into two chiral halves and each lives on S^1, a compactified light ray. In algebraic quantum field theory, we consider a conformal net $\{A(I)\}_{I\subset S^1}$ where I is an interval in the circle. Each $A(I)$ is a factor generated by observables in I.

Unitary representations of a conformal net give a braided category of Doplicher-Haag-Roberts superselection sectors. If we have only finitely many irreducible representations, we get a modular tensor category (K-Longo-Müger). Each object is realized as an endomorphism of $A(I)$ and the image of this endomorphism gives a subfactor of $A(I)$.

α-induction in conformal field theory

Let $\{A(I)\}_{I\subset S^1}$ be a conformal net. Suppose it has only finitely many irreducible representations (rationality). Its representation category is a modular tensor category. Commutative Frobenius algebras in it are in a bijective correspondence to conformal nets extending $\{A(I)\}_{I\subset S^1}$.

Fix an extension $\{B(I)\}_{I\subset S^1}$ of $\{A(I)\}_{I\subset S^1}$. We have an induction procedure for representations, called α-induction, but we now have positive induction and negative induction. The irreducible objects that simultaneously arise from both positive and negative α-inductions exactly correspond to irreducible representations of $\{B(I)\}_{I\subset S^1}$.

α-induction for endomorphisms

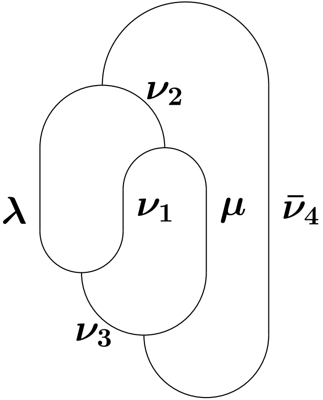
If we have a Frobenius algebra in a braided fusion category of endomorphisms of N corresponding to a subfactor $N \subset M$, then each endomorphism λ in the fusion category has an extension $\alpha_\lambda^\pm$ to M depending on the choice of a braiding as follows.

$$\alpha_\lambda^\pm = \bar{\iota}^{-1} \cdot \mathrm{Ad}(\varepsilon^\pm(\lambda, \theta)) \cdot \lambda \cdot \bar{\iota}$$

Here ι is the inclusion map of N into M, $\theta = \bar{\iota}\iota$, and we have $M = Nv$ with a nice isometry v. We have $\alpha_\lambda^\pm(x) = \lambda(x)$ for $x \in N$ and $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda, \theta)^* v$. This was first defined by Longo-Rehren and studied by Xu, Böckenhauer-Evans, and Böckenhauer-Evans-K in detail.

A fusion category and a bi-unitary connection

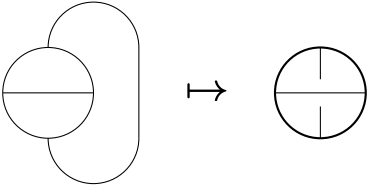
Suppose we have a fusion category and $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}, \boldsymbol{\nu_3}, \boldsymbol{\nu_4}$ are irreducible objects. For fixed $\boldsymbol{\lambda}, \boldsymbol{\mu}$, the following diagram gives a bi-unitary connection.



(quantum $\boldsymbol{6j}$-symbols)

α-induction for bi-unitary connections

We now have α-induction, a new induction machinery, for bi-unitary connections (K 2023).



The diagrams represent complex numbers. The ones represented by the left diagram are quantum $6j$-symbols and they contain all information about the original fusion category. Those represented by the right diagram are α-induced bi-unitary connections and they contain all information about the α-induced fusion category.

α-induced bi-unitary connections and flatness

The bi-unitary connections on the Dynkin diagrams are classified into two classes.

Flat ones: A_n, D_{2n}, E_6 and E_8.
Non-flat ones: D_{2n+1} and E_7.

Here flatness represents an extra symmetry. The original bi-unitary connections on A_n arise from the quantum $6j$-symbols of the Wess-Zumino-Witten model $SU(2)_{n-1}$ or the quantum groups $U_q(sl_2)$ at roots of unity.

The difference between the two classes is understood from a viewpoint of commutativity of the Frobenius algebra in this framework of α-induction.

Summary

Behaviors of anyons are described with a modular tensor category. Fusion and modular tensor categories are described with matrix product operators arising from 4-tensors. Then everything is finite dimensional. These 4-tensors are exactly the same as bi-unitary connections appearing in subfactor theory.

Which kind of bi-unitary connections appear in this context is determined within the framework of subfactor theory and they are quantum $6j$-symbols arising from a fusion 2-category. The range of a certain projector matrix product operator is identified with the higher relative commutant of a subfactor. A new bi-unitary connection arising from the α-induction is determined.