INFINITE-DIMENSIONAL LIE ALGEBRAS AND THEIR MULTIVARIABLE GENERALIZATIONS

LECTURES BY BRIAN R. WILLIAMS

Here is a brief summary of the lectures:

- **Lecture 1**. We begin by introducing two important examples of infinite-dimensional Lie algebras, the Kac–Moody and Virasoro Lie algebras. We will provide a brief survey of the importance of these objects in contexts of representation theory, the moduli of Riemann surfaces, and conformal field theory. Along the way, we will provide background on some fundamental concepts in Lie theory including central extensions, Lie algebra (co)homology, and universal enveloping algebras. We will define a class of modules called *vacuum* modules and highlight an important construction known as the "free field realization".
- **Lecture 2**. In this lecture, we introduce an algebraic structure present in the modules covered in the first lecture. In some sense, this structure is comparable to that of an associative algebra, and we will provide a geometric interpretation that unifies the two. The key technical tool in formulating this structure is through a certain *colored operad* that is built from holomorphic disks inside of a Riemann surface. Algebras over this colored operad, so-called "prefactorization algebras", will be the central objects of study in the remainder of the lecture series. In part, the data of a prefactorization algebra on certain Riemann surfaces encodes all of the infinite-dimensional Lie algebras in Lecture 1, together with the structure of a "vertex algebra" on the corresponding vacuum modules.
- **Lecture 3**. We begin this lecture with the following question: what are the "multivariable" generalizations of the infinite dimensional Lie algebras introduced in Lecture 1? We will see that prefactorization algebras on higher dimensional complex manifolds provide a very natural family of such enhancements, including generalizations of the associated vacuum modules. A necessary structure to understand is the central extensions present in the multivariable case, where we will find it is necessary to work in a *derived* context. We will give a purely algebraic definition of the higher dimensional Kac–Moody and Virasoro algebras and end with an eye towards their theory of representations, where we provide a

free field realization that can be understood in terms of a very simple higher dimensional algebra.

1. Lecture 1: Kac–Moody and Virasoro algebras and their representations

This lecture series introduces a family of infinite dimensional Lie algebras that form the bedrock for many modern topics in representation theory. Much like their finite dimensional counterparts, infinite dimensional Lie algebras appear naturally as the symmetries of certain systems.

We begin with an example from "gauge theory". Consider a vector bundle E over a smooth manifold M. The space of "infinitesimal gauge symmetries" is the space of sections of the endomorphism bundle $\Gamma(M, \mathrm{End}(E))$. The space of gauge symmetries forms a Lie algebra using the commutator of endomorphisms. Given a section $s \in \Gamma(M, E)$, and a gauge symmetry $\phi \in \Gamma(M, \mathrm{End}(E))$, we obtain a new section $\phi(s)$ of E. In this way, infinitesimal gauge symmetries act on sections of E.

Exercise 1.1. This Lie algebra of infinitesimal gauge symmetries can be thought of as the Lie algebra of some (infinite-dimensional) Lie group. Describe it.

Many important structures associated to a vector bundle are preserved under a gauge symmetry. For instance, suppose E is equipped with a connection $\nabla : \Gamma(M, E) \to \Omega^1(M, E)$ whose curvature will be denoted $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$. If ϕ is a gauge symmetry, we obtain a new connection ∇_ϕ. Moreover, if F_{∇_ϕ} is the curvature of this new connection, the difference $F_\nabla - F_{\nabla_\phi}$ is d_{dR}-exact:

$$F_\nabla - F_{\nabla_\phi} = \mathrm{d}\omega_\phi$$

for some $\omega_\phi \in \Omega^1(M, \mathrm{End}(E))$. Thus, the *cohomology class* of the curvature is left unchanged upon performing a gauge transformation.

The Kac–Moody algebra is rooted in the study of holomorphic gauge symmetries on Riemann surfaces. It is built from the loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$, consisting of Laurent polynomials valued in a Lie algebra $\mathfrak{g}$. admits a non-trivial central extension $\widehat{\mathfrak{g}}$ for each choice of invariant pairing on $\mathfrak{g}$.

Our other main example, the Virasoro algebra, is built from the Lie algebra of vector fields on a Riemann surface. This describes the symmetry of any *natural bundle* that can be built on a Riemann surface, and so has fundamental relationship the moduli space of Riemann surfaces.

1.1. The Virasoro algebra.

Let D denote the algebraic disk, whose algebra of functions is the algebra of polynomials in a single variable

$$\mathcal{O}(\widehat{D}) = \mathbb{C}[z].$$

In other words, $D = \operatorname{Spec}\mathbb{C}[z]$. Let $D^\times$ denote the formal punctured disk, whose algebra of functions is the algebra of Laurent polynomials

$$\mathcal{O}(D^\times) = \mathbb{C}[z, z^{-1}]$$

In other words, $D^\times = \operatorname{Spec}\mathbb{C}[z, z^{-1}]$.

Definition 1.2. The *Witt algebra* $\mathfrak{vir}_0$ is the Lie algebra of derivations of the commutative algebra $\mathbb{C}[z, z^{-1}]$. In other words, the Witt algebra is the Lie algebra of vector fields on the punctured disk $D^\times$.

Remark 1.3. There are "formal" versions of each of the objects above. This amounts to replacing polynomials by power series.

Remark 1.4. Geometrically, the Witt algebra plays an essential role in studying the moduli space of Riemann surfaces. There is a way to "describe" a Riemann surface which one obtains by first fixing a disk $D \subset \Sigma$. The Riemann surface Σ can be glued from the disk D and its complement $\Sigma \setminus D$ along the punctured disk $D^\times$. The possible gluing data is described, then, but automorphisms of the punctured disk. The infinitesimal, or Lie algebraic, version is precisely the Witt algebra.

A basis for the Witt algebra is provided by the symbols

$$L_n = z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z} \quad , \quad n \in \mathbb{Z}$$

which satisfy the commutation relations

$$[L_n, L_m] = (m - n)L_{n+m}.$$

We are interested in *central extensions* of the Witt algebra.

Definition 1.5. A *central extension* of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{c}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ which sits in an exact sequence of Lie algebras

$$\mathfrak{c} \to \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

with the property that $\mathfrak{c}$ lies in the center of $\widetilde{\mathfrak{g}}$.

Exercise 1.6. Define an equivalence relation on the set of central extensions.

Before classifying central extensions, we remark on a fundamental aspect of Lie theory. Given a Lie algebra $\mathfrak{g}$, we let $U\mathfrak{g}$ denote its universal enveloping algebra which is equipped with a canonical map $\mathfrak{g} \to U\mathfrak{g}$. The algebra is universal in the sense that if A is any other algebra which admits a Lie algebra map $f : \mathfrak{g} \to A$, there is a unique map $\widetilde{f} : U\mathfrak{g} \to A$, for which

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "f"] \arrow[dr] & & A \\ & U\mathfrak{g} \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

is commutative. Explicitly, one obtains $U\mathfrak{g}$ as a quotient of the tensor algebra $T(\mathfrak{g}) = \oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$. The famous Poincaré-Birkhoff-Witt theorem identifies $U\mathfrak{g}$ with $\mathrm{Sym}(\mathfrak{g})$ as vector spaces (but not algebras!).

There is a classification of central extensions in terms of *Lie algebra cohomology.* Given a Lie algebra $\mathfrak{g}$ and a module M, the Lie algebra cohomology is the derived functor

$$H^n(\mathfrak{g}; M) = \mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{C}, M).$$

Here, we view $\mathbb{C}$ as a trivial $\mathfrak{g}$-representation.

We will use a particular model for Lie algebra cohomology.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-representation. The Chevalley–Eilenberg cochain complex $\mathrm{C}^\bullet_{\mathrm{Lie}}(\mathfrak{g}; M)$ computing Lie algebra cohomology is the cochain complex whose underlying graded vector space is

$$\mathrm{Hom}\left(\mathrm{Sym}\left(\mathfrak{g}[1]\right), M\right) = \mathrm{Hom}\left(\oplus_{k\geq 0}(\wedge^k\mathfrak{g})[k]), M\right).$$

The differential is defined as follows. Given a k-cochain $\varphi : \wedge^k\mathfrak{g} \to M$, the $(k+1)$-cochain $\mathrm{d}_{\mathrm{CE}}(\varphi)$ is defined by

$$\begin{aligned} \mathrm{d}_{\mathrm{CE}}(\varphi)(x_1, \ldots, x_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}x_i \cdot \varphi(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1}) \\ &+ \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\varphi([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{k+1}). \end{aligned}$$

It is a tedious, but straightforward exercise to verify that $\mathrm{d}_{\mathrm{CE}} \circ \mathrm{d}_{\mathrm{CE}} = 0$, hence this defines a cochain complex. The Chevalley–Eilenberg cochain complex computes the Lie algebra cohomology $H^\bullet(\mathfrak{g}; M) = H^\bullet(\mathrm{C}^\bullet_{\mathrm{Lie}}(\mathfrak{g}; M))$.

Remark 1.8. There is a linear dual cochain complex $\mathrm{C}_\bullet^{\mathrm{Lie}}(\mathfrak{g}; M)$ which computes Lie algebra *homology*. As a graded vector space $\mathrm{C}_\bullet^{\mathrm{Lie}}(\mathfrak{g}; M)$ is

$$\mathrm{Sym}(\mathfrak{g}[1]) \otimes_{\mathbb{C}} M = \left(\oplus_{k\geq 0}(\wedge^k \mathfrak{g})[k]\right) \otimes_{\mathbb{C}} M.$$

We leave it as an exercise to write down the differential of this cochain complex which is linear dual to the one above.

Lemma 1.9. *Central extensions of $\mathfrak{g}$ by a $\mathfrak{a}$ are in one-to-one correspondence with the second cohomology group of $\mathfrak{g}$ with values in $\mathfrak{a}$*

$$H^2(\mathfrak{g}\ ;\ \mathfrak{a}).$$

Proof. We will use our model for Lie algebra cohomology. Suppose φ is a CE 2-cocycle representing a class in $H^2(\mathfrak{g}; \mathfrak{a})$. Then, define the Lie algebra $\widetilde{\mathfrak{g}}$, which as a vector space is $(x, a) \in \mathfrak{g} \oplus \mathfrak{a}$ with Lie brackets

$$\begin{aligned} [x, x'] &= [x, x']_{\mathfrak{g}} + \varphi(x, x') \\ [x, a] &= 0 \\ [a, a] &= 0. \end{aligned}$$

Here $[\cdot, \cdot]_{\mathfrak{g}}$ denotes the original Lie bracket on $\mathfrak{g}$.

Conversely, suppose $\widetilde{\mathfrak{g}}$ is such a central extension with Lie bracket $[\cdot, \cdot]$. Then, define the 2-cochain

$$\varphi(x, x') = [x, x'] - [x, x']_{\mathfrak{g}}$$

It is immediate to check that φ is a 2-cocycle.

We leave it as an exercise to show that equivalent central extensions determine cohomologous 2-cocycles. $\square$

The following result is well-known. See [1] for more details.

Proposition 1.10. *The space of central extensions of the Witt algebra $\mathfrak{vir}_0$ is one-dimensional and is spanned by the class of the 2-cocycle $\varphi_{\mathrm{Vir}} \in \mathrm{C}^2_{\mathrm{Lie}}(\mathfrak{vir}_0; \mathbb{C})$ defined by*

$$\varphi_{\mathrm{Vir}}(L_m, L_n) = \frac{1}{12}\delta_{m,-n}(m^3 - m).$$

Remark 1.11. The Lie algebra $\mathfrak{vir}_0$ is infinite dimensional, so one must use caution when defining the CE cochain complex. When we write Hom in the definition of Lie algebra cohomology, we mean continuous linear homomorphisms. With this convention, the CE complex is the one studied by Gelfand and Fuks [1].

Remark 1.12. Let $\mathrm{Res} : \mathbb{C}((z))\mathrm{d}z \to \mathbb{C}$ be the formal residue map, which sends a Laurent 1-form $\sum a_n z^n \mathrm{d}z$ to a_{-1}. In terms of vector fields on the formal punctured disk, one can rewrite this cocycle as

$$\varphi_{\mathrm{Vir}}\left(f(z)\frac{\mathrm{d}}{\mathrm{d}z}, g(z)\frac{\mathrm{d}}{\mathrm{d}z}\right) = \frac{1}{12}\mathrm{Res}_z\left(f'(z)\mathrm{d}g'(z)\right).$$

Here, d is the formal de Rham differential $\mathrm{d}(h(z)) = h'(z)\mathrm{d}z$. The factor of $\frac{1}{12}$ is conventional and can be traced back to bosonic string theory.

Definition 1.13. The *Virasoro algebra* $\mathfrak{vir}$ is the Lie algebra central extension of the Witt algebra $\mathfrak{vir}_0$ defined by the 2-cocycle φ_{Vir}.

Remark 1.14. Geometrically, we remarked that the Witt algebra played an essential role in the description of the moduli space of Riemann surfaces. The Virasoro algebra naturally acts on sections of a certain *determinant line bundle* over the moduli space of Riemann surfaces. This line bundle is uniquely characterized by the central charge.

1.2. **Oscillator algebras.** We move on to our first example of a representation for the Virasoro algebra. This is an example of an algebra obtained from a "free field" in conformal field theory.

Introduce the following Lie algebra $\mathfrak{s}$ spanned by elements $\{a_n K \mid n \in \mathbb{Z}\}$ with commutation relations

$$[a_n, a_m] = m\delta_{m,-n}\hbar$$
$$[a_n, K] = 0.$$

Exercise 1.15. Show that $\mathfrak{s}$ is isomorphic to the central extension of the *abelian* Lie algebra $\mathbb{C}((z))$ defined by the 2-cocycle

$$f(z), g(z) \mapsto \mathrm{Res}_z(f\mathrm{d}g).$$

There is an abelian sub Lie algebra

$$\mathbb{C}[[z]] \oplus \mathbb{C} \cdot K \subset \mathfrak{s}$$

which is spanned by the elements $\{a_n, K \mid n \geq 0\}$. In other words, this is the algebra of "non-negative" modes. Denote by $\mathbb{C}_k$ the one-dimensional module for the abelian Lie algebra $\mathbb{C}[[z]]\oplus\mathbb{C}\cdot K$ whereby $\mathbb{C}[[z]]$ acts trivially and K acts by $k \in \mathbb{C}$.

Definition 1.16. The *vacuum module of* $\mathfrak{s}$ *at level* $k \in \mathbb{C}$ is the induced module

$$\begin{aligned}\mathrm{Vac}_k &= \mathrm{Ind}^{\mathfrak{s}}_{\mathbb{C}[[z]]\oplus\mathbb{C}} \mathbb{C}_k \\ &= U(\mathfrak{s}) \otimes_{U(\mathbb{C}[[z]])[K]} \mathbb{C}_k.\end{aligned}$$

We briefly explain the reason why this is called the "vacuum module". Notice that as a vector space, there is an identification of the vacuum module with the vector space

$$\mathrm{Sym}(z^{-1}\mathbb{C}[z^{-1}]) = \mathbb{C}[z^{-1}, z^{-2}, \ldots]$$

Typically in physics, this is called the "Fock module". The element $|0\rangle = 1$ is a "vacuum vector" since it is annihilated by $a_k \in \mathfrak{s}$ for $k \geq 0$. Furthermore, the vacuum module is spanned by elements of the form

$$a_{-k_n} \cdots a_{-k_1}|0\rangle$$

for $k_1, \ldots, k_n \geq 0$. For this reason, the "negative modes" $\{a_{-k} | k \geq 0\}$ are called *creation operators.*

Exercise 1.17. For $\mu \in \mathbb{C}$, we can modify the module structure to $a_0 \cdot |0\rangle = \mu|0\rangle$ and we obtain what is known as a "Verma module" of $\mathfrak{s}$ weight μ and level k. Show that this results in a new module for $\mathfrak{s}$.

Proposition 1.18. *For* $k \neq 0$*, define the operators acting on* Vac_k*:*

$$L_n^{\mathrm{Fock}} \overset{\mathrm{def}}{=} \frac{1}{2k} \sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}.$$ [1]

Then, the $\{L_n^{\mathrm{Fock}}\}_{n\in\mathbb{Z}}$ *define an action of the Virasoro algebra on* Vac_k *of central charge* $c =??$.

Remark 1.19. We view this result as the simplest "free field realization" of the Virasoro algebra. Roughly, we have exhibited the commutation relations of a very nontrivial Lie algebra $\mathfrak{vir}$ in terms of some algebra of operators on the Fock module associated to the oscillator algebra. In physics, this Fock module describes a "free quantum theory", which are among the simplest to study.

An important invariant of a Virasoro module is its *q-character*, or *q-dimension.* Suppose M is such a module where $L_0 \in \mathfrak{vir}$ acts diagonally. The *conformal weight* of a $v \in M$ is the number $|v| \in \mathbb{C}$ such that $L_0(v) = |v|v$. For M such a module, one writes the q-character as the formal q-expansion

$$\chi_q(M) = \sum_n q^n \cdot \dim M^{(n)}$$

where $M^{(n)} \subset M$ is the subspace of elements of conformal weight n.

[1] We always assume "normal ordering", so $(a_ia_i)b = a_i(a_jb)$ if $i \leq j$ and $(a_ia_j)b = a_j(a_ib)$ for $i > j$.

Exercise 1.20. Show that $\chi_q(\mathrm{Vac}_k(M)) = \eta(q)^{-1}$ where $\eta(q)$ is the Dedekind η-function

$$\eta(q) = \prod_{n\geq 1}(1-q^n).$$

1.3. **The Kac–Moody algebra.** Suppose that $\mathfrak{g}$ is a Lie algebra. For applications in representation theory, one often restricts to the case that $\mathfrak{g}$ is semi-simple, but for now it will make no difference.

We can tensor the commutative algebra $\mathcal{O}(\widehat{D}^\times) = \mathbb{C}((z))$ with the Lie algebra $\mathfrak{g}$ to obtain a new Lie algebra

$$\mathbb{C}((z)) \otimes \mathfrak{g}.$$

We refer to this as the *current algebra* associated to $\mathfrak{g}$. Equivalently, if we think of $\mathfrak{g}$ as an affine variety, this Lie algebra is the same as maps from the formal punctured disk $\widehat{D}^\times$ to $\mathfrak{g}$. We will write elements of this Lie algebra as $f(z) \otimes x$. Explicitly, the Lie bracket is defined by

$$[f(z)\otimes x, g(z)\otimes y] = (f\cdot g)(z) \otimes [x,y].$$

As in the previous section, we will be interested in a certain central extension of this Lie algebra. To define it, we introduce the following terminology. The algebra of polynomials on a vector space V is

$$\mathbb{C}[V] = \mathrm{Sym}(V^*).$$

When $V = \mathfrak{g}$ we note that $\mathfrak{g}$ acts on its polynomials $\mathbb{C}[\mathfrak{g}]$ by the adjoint representation, which we will denote by ad_x.

Definition 1.21. An *invariant polynomial* of $\mathfrak{g}$ is a polynomial P on $\mathfrak{g}$ such that $\mathrm{ad}_x(P) = 0$ for all $x \in \mathfrak{g}$.

Definition 1.22. Suppose κ is an invariant quadratic polynomial of $\mathfrak{g}$. Define the 2-cochain $\varphi_\kappa \in \mathrm{C}^2_{\mathrm{Lie}}(\mathbb{C}((z))\otimes\mathfrak{g};\mathbb{C})$ by the formula

$$\varphi_\kappa(f(z)\otimes x, g(z)\otimes y) = \mathrm{Res}_z(f\mathrm{d}g)\kappa(x,y).$$

It is immediate to verify that φ_κ is a cocycle and hence defines the following central extension.

Definition 1.23. The *Kac–Moody algebra*, or *affine algebra*, $\widehat{\mathfrak{g}}_\kappa$ associated to the invariant quadratic polynomial κ is the central extension of $\mathbb{C}((z))\otimes\mathfrak{g}$ defined by the 2-cocycle φ_κ.

Remark 1.24. Notice that when $\mathfrak{g}$ is the one-dimensional Lie algebra, the Kac-Moody algebra is the oscillator algebra of the previous section.

Example 1.25. There is a natural quadratic invariant polynomial associated to any Lie algebra. Note that $\mathfrak{g}$ acts on itself by the adjoint. The *Killing form* is the invariant polynomial

$$\kappa_{\mathrm{Kill}}(x,y) = \mathrm{Tr}\left(\mathrm{ad}_x \circ \mathrm{ad}_y\right)$$

where the trace on the right-hand side is in the adjoint representation. When $\mathfrak{g}$ is simple, this is the unique invariant quadratic polynomial up to scale. Furthermore, $\mathfrak{g}$ is semisimple if and only if κ_{Kill} is nondegenerate.

We end this lecture by giving a definition of a Fock or vacuum module associated to any Kac–Moody algebra. For this definition, we observe that there is a subalgebra $\mathfrak{g}[[z]] \oplus \mathbb{C} \subset \widehat{\mathfrak{g}}_\kappa$, where the Lie bracket is just the Lie bracket induced from $\mathfrak{g}$.

The reader should notice the similarities with Definition 1.16

Definition 1.26. Let $\mathfrak{g}$ be a Lie algebra and κ a non-degenerate quadratic invariant polynomial. The *Kac–Moody vacuum module at level* $k \in \mathbb{C}$ is the induced $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathrm{Vac}_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[z]]\oplus\mathbb{C}}^{\widehat{\mathfrak{g}}} \mathbb{C}_k = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[z]])[K]} \mathbb{C}_k.$$

Remark 1.27. In the previous section, we saw how to represent the Virasoro algebra on a the Fock module for the oscillator algebra. There is a "nonabelian" version of this which represents the Virasoro algebra acting on the vacuum module $\mathrm{Vac}_k(\mathfrak{g})$ provided that k does not equal the *critical level* $\kappa_c = -h^\vee$, where $h^\vee$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$. This is known as the Sugawara construction, but we will not spend any more time on it here.

Remark 1.28. The q-character of the Kac–Moody vacuum module can be computed using the so-called Weyl–Kac denominator formula.

Exercise 1.29. Introduce the analogous "vacuum module" for the Virasoro algebra. Hint: consider the subalgebra of formal vector fields on the (unpunctured) disk.

The affine Lie algebra $\widehat{\mathfrak{g}}_\kappa$ and its cousin, the Kac-Moody vertex algebra, which we will meet in the next lecture, are foundational objects in representation theory and conformal field theory. A natural question then arises: do there exists multivariable, or higher dimensional, generalizations of the affine Lie algebra and Kac-Moody vertex algebra?

2. Lecture 2: Holomorphic (pre)factorization algebras and higher dimensional algebras

In this section, we provide a unifying geometric interpretation of the affine and Virasoro algebras of the previous section together with their vacuum modules. In fact, the vacuum modules in the previous section have an extra algebraic structure known as a *vertex algebra*. The language of vertex algebras provides a very efficient landscape for the study of affine algebras and their representations. In their own right, additionally, vertex algebras describe the physics of "local operators" in conformal field theory on Riemann surfaces.

By a standard procedure, there is a way of enhancing the affine algebra to a vertex algebra. The so-called Kac-Moody or Virasoro vertex algebras, as developed in [2]–[4]. Roughly, a vertex algebra is a vector space equipped V with some algebraic structure. The fundamental piece of data is called a *vertex operator*, which is a map of the form

$$Y(-;z) : V \to \mathrm{End}(V)[z, z^{-1}].$$

One should think of this as a z-dependent *family* of multiplication maps of the form $V \otimes V \to V$ where $z \in D^{\times}$, the punctured disk.

The rigorous definition of a vertex algebra is quite complicated at first glance. For this reason, we will choose to work with a more geometric description using the language of *prefactorization algebras*.

2.1. Prefactorization algebras.

The geometric description of a vertex algebra will be given in terms of an algebra over some (colored) operad of disks. This is a holomorphic analog of an important structure historically studied in topology and homotopy theory.

We begin with some abstract definitions.

Definition 2.1. A *colored operad* $\mathsf{D}^{\boxtimes}$ consists of a collection of objects together with:

(i) for every finite nonempty set I a Σ_I-equivariant set of "multi-homs"

$$\mathsf{D}^{\boxtimes}\left(\boxtimes_{i\in I} d_i, e\right)$$

where $\{d_i\}_{i\in I}$ is a I-indexed collection of objects and B is a single object;

(ii) for every surjection of finite sets $\pi : J \twoheadrightarrow I$ a $\Sigma_I \times (\bigtimes_{i\in I} \Sigma_{\pi^{-1}(i)})$-equivariant composition rule

$$\circ : \mathsf{D}^{\boxtimes}\left(\boxtimes_{i\in I} d_i, e\right) \times \left(\bigtimes_{i\in I} \mathsf{D}^{\boxtimes}\left(\boxtimes_{j\in\pi^{-1}(i)} f_j, d_i\right)\right) \to \mathsf{D}^{\boxtimes}\left(\boxtimes_{j\in J} f_j, e\right).$$

The composition maps must obey some natural associativity laws which we do not spell out here.

By a *map* of colored operads, we mean a map on objects and multihoms which intertwines the composition laws.

Remark 2.2. A colored operad with one object is called a (symmetric) operad. Examples include the commutative operad, associative operad, Lie operad, etc..

A monoidal category is a category $\mathcal{C}^{\otimes}$ equipped with a "tensor product"

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which obeys associativity and unit axioms. A *symmetric* monoidal category is a monoidal category with a natural equivalence $A \otimes B \to B \otimes A$. Examples of symmetric monoidal categories include $\mathrm{Set}^{\times}$, $\mathrm{Vect}_k^{\otimes_k}$, $\mathrm{Top}^{\times}$, $\mathrm{Top}^{\sqcup}$, $\mathrm{Rep}_{\mathfrak{g}}$.

Exercise 2.3. Suppose $\mathcal{C}^{\otimes}$ is a symmetric monoidal category. Construct a colored operad whose objects are the same as the objects of $\mathcal{C}^{\otimes}$.

The collection of open sets on a manifold M form a poset. In fact, this can be enhanced to the structure of a colored operad as follows.

Definition 2.4. Let M be a manifold. The colored operad $\mathrm{Open}_M^{\sqcup}$ has objects given by the open sets of M, and mutli-homs

$$\mathrm{Open}_M^{\sqcup}\left(\boxtimes_{i\in I} U_i, V\right)$$

defined by:

- the singleton set $\{\star\}$ if the collection of open sets $\{U_i\}$ is mutually disjoint and each U_i is contained in V;
- the empty set, otherwise.

Exercise 2.5. Define the composition law and formulate the associativity axiom for $\mathrm{Open}_M^{\sqcup}$.

Let $\mathsf{D}^{\boxtimes}$ be a colored operad and suppose $\mathcal{C}^{\otimes}$ is a symmetric monoidal category. A $\mathsf{D}^{\boxtimes}$-*algebra with values in* $\mathcal{C}^{\otimes}$ is a map of colored operads

$$A : \mathsf{D}^{\boxtimes} \to \mathcal{C}^{\otimes}.$$

Spelling this data out, an algebra consists of an object $A(d) \in \mathcal{C}^{\otimes}$ for each object $d \in \mathsf{D}^{\boxtimes}$ together with a "multiplication rule"

$$m_F : \otimes_{i\in I} A(d_i) \to A(e)$$

for every element F of $\mathsf{D}^{\boxtimes}(\boxtimes_{i\in I} d_i, e)$.

Definition 2.6. A *prefactorization algebra* on M with values in a symmetric monoidal category $\mathcal{C}^{\otimes}$ is an $\mathrm{Open}_M^{\sqcup}$-algebra with values in $\mathcal{C}^{\otimes}$. Unpacking, this is an assignment

$$A : U \subset M \mapsto A(U)$$

together with a "multiplication rule"

$$m_{\{U_i\},V} : \otimes_{i\in I} A(U_i) \to A(V)$$

whenever the open sets $\{U_i\}$ are mutually disjoint and contained in V. These multiplications are required to satisfy some natural associativity axiom.

A prefactorization algebra is called *locally constant* if for every embedding of open balls $B \hookrightarrow B'$ in M that the induced map

$$A(B) \xrightarrow{\simeq} A(B')$$

is an isomorphism.

Theorem 2.7 (Lurie)**.** *There is an equivalence of categories between locally constant prefactorization algebras on* $\mathbb{R}$ *and associative algebras.*

2.1.1. *Working in the differentially graded setting.* From hereon, we will mostly be working in the setting where "vector spaces" are replaced by dg vector spaces, or cochain complexes. In particular, unless we say otherwise all of the prefactorization algebras we will mention will take values in some category of cochain complexes. These provide an efficient resolution, as we will see, of the affine algebras of the previous section.

A cochain complex is a $\mathbb{Z}$-graded vector space $V^\bullet = \oplus_k V^k[-k]$ equipped with a linear operator of degree $+1$, $\mathrm{d} : V^\bullet \to V^{\bullet+1}$. We will often write this as a pair $(V^\bullet, \mathrm{d})$.

A *dg Lie algebra* is a triple $(\mathfrak{g}^\bullet, \mathrm{d}, [\cdot,\cdot])$ where $(\mathfrak{g}, \mathrm{d})$ is a cochain complex and

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is a bilinear map of degree zero which satisfies three conditions:

(i) graded skew symmetry $[a,b] = (-1)^{|a||b|+1}[b,a]$ for all $a, b \in \mathfrak{g}$;

(ii) graded Jacobi identity

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]].$$

(iii) graded derivation

$$\mathrm{d}[a,b] = [\mathrm{d}a, b] + (-1)^{|a|}[a, \mathrm{d}b]$$

Of course, one can just remember the underlying cochain complex of a dg Lie algebra. Its cohomology has the structure of a graded Lie algebra.

Finally, a *commutative dg algebra* is a triple $(A, \mathrm{d}, \cdot)$ such that (A, d) is a cochain complex, $\cdot$ is a graded commutative product on A, and together they satisfy the graded derivation rule

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot \mathrm{d}b.$$

We saw that given a commutative algebra and a Lie algebra we can tensor them together to obtain a new Lie algebra. Similarly, if A is a commutative dg algebra and $\mathfrak{g}$ is a dg Lie algebra, then $A \otimes \mathfrak{g}$ can also be given the structure of a dg Lie algebra. This is one of the most common ways dg Lie algebras appear for us.

2.1.2. *Examples on Riemann surfaces.* Recall, given a Lie algebra $\mathfrak{g}$, its CE cochain complex $\mathrm{C}^{\mathrm{Lie}}_\bullet(\mathfrak{g}; M) = (\mathrm{Sym}(\mathfrak{g}[1]) \otimes M, \mathrm{d}_{\mathrm{CE}})$ computing Lie algebra *homology* is defined. In fact, this CE cochain complex is defined for any *dg* Lie algebra $(\mathfrak{g}, \mathrm{d}, [\cdot,\cdot])$. One simply takes the definition from before but uses the new differential

$$\mathrm{d}_{\mathrm{CE}} + \mathrm{d}$$

which acts on $\mathrm{Sym}(\mathfrak{g}[1]) \otimes M$ in the obvious way way.

Example 2.8. *The level zero Kac–Moody prefactorization algebra.* Let Σ be a Riemann surface, and consider the cosheaf of commutative dg algebras $\Omega^{0,\bullet}_{\Sigma,c}$ of compactly supported Dolbeault forms. On each open set, the differential is given by the $\overline{\partial}$-operator and the product is wedge product of differential forms.

By tensoring with a Lie algebra $\mathfrak{g}$, we obtain a precosheaf[2] of dg Lie algebras $\Omega^{0,\bullet}_{\Sigma,c} \otimes \mathfrak{g}$. The differential is $\overline{\partial} \otimes \mathrm{id}_{\mathfrak{g}}$ and the bracket uses the wedge product of forms together with the Lie bracket of $\mathfrak{g}$.

For each open set $U \subset \Sigma$, we can consider the Chevalley–Eilenberg cochain complex computing Lie algebra homology:

$$\mathrm{C}^{\mathrm{Lie}}_\bullet(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}).$$

As a graded vector space, this cochain complex is

$$\mathrm{Sym}\left(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}[1]\right) = \oplus_{k \geq 0} \mathrm{Sym}^k \left(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}[1]\right).$$

[2]The direct sum is not a categorical coproduct in the category of Lie algebras.

The differential splits as a sum of two terms $\overline{\partial} + \mathrm{d}_{\mathfrak{g}}$, where $\overline{\partial}$ is the usual $\overline{\partial}$ differential and $\mathrm{d}_{\mathfrak{g}}$ encodes the Lie bracket of $\mathfrak{g}$.

Letting U vary, the assignment $U \mapsto \mathrm{C}^{\mathrm{Lie}}_{\bullet}\left(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}\right)$ has the structure of a prefactorization algebra.

The structure maps for the configuration $i : U \sqcup V \hookrightarrow W$ are defined by the following composite

$$\begin{tikzcd}
\mathrm{Sym}\left(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}[1]\right) \otimes \mathrm{Sym}\left(\Omega^{0,\bullet}_c(V) \otimes \mathfrak{g}[1]\right) \arrow[r, "\cong"] \arrow[dr, "m_{U,V;W}"'] & \mathrm{Sym}\left(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}[1] \oplus \Omega^{0,\bullet}_c(V) \otimes \mathfrak{g}[1]\right) \arrow[d, "i_*"] \\
& \mathrm{Sym}\left(\Omega^{0,\bullet}_c(W) \otimes \mathfrak{g}[1]\right).
\end{tikzcd}$$

We denote this prefactorization by $\mathrm{KM}_0(\mathfrak{g})$.

Example 2.9. *The central charge zero Virasoro prefactorization algebra.* Consider now the holomorphic tangent bundle $T\Sigma$ of a Riemann surface Σ. The compactly supported Dolbeault complex $\Omega^{0,\bullet}(\Sigma, T\Sigma)$ carries the structure of a precosheaf of dg Lie algebras. The differential is the $\overline{\partial}$-operator and the Lie bracket is induced from the ordinary Lie bracket of vector fields together with the wedge product of Dolbeault forms.

One can show similarly to above that $U \subset \Sigma \mapsto \mathrm{C}^{\mathrm{Lie}}_{\bullet}(\Omega^{0,\bullet}(U, TU)$ has the structure of a prefactorization algebra. This is the charge zero Virasoro prefactorization algebra and will be denoted Vir_0.

2.1.3. *Holomorphic translation invariance.* The theory of vertex algebras is a "holomorphic" analog of the theory of associative algebras. We will see this analogy at the level of prefactorization algebras. Associative algebras, we have seen, correspond to locally constant prefactorization algebras on $\mathbb{R}$. In the holomorphic setting one asks for a weaker, though still important condition that we not introduce.

Definition 2.10. A ((n) infinitesimal) *translation invariant* prefactorization algebra on $\mathbb{R}^n$ is a factorization algebra equipped with an action by the n-dimensional abelian Lie algebra $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$.

In complex analysis, one way to formulate holomorphicity is to impose the requirement that a certain object be annihilated by the $\overline{\partial}$-operator. At the level of prefactorization algebras, it will be useful for us to work with this condition in a homotopical way.

Definition 2.11. A *holomorphically translation invariant* prefactorization algebra on $\mathbb{C}^n$ is a translation invariant factorization algebra whereby the anti-holomorphic translations $\{\partial_{\overline{z}_i}\}$ act homotopically trivially.

Consider a prefactorization algebra built from the cosheaf of compactly supported Dolbeault forms $\Omega_c^{0,\bullet}(\Sigma)$. When $\Sigma = \mathbb{C}$, there is a natural action of the Lie algebra of translations $\{\partial_z, \partial_{\overline{z}}\}$. Moreover, Cartan's magic formula provides a canonical trivialization of the anti-holomorphic translations:

$$\left[\overline{\partial}, \iota_{\partial_{\overline{z}}}\right] = \partial_{\overline{z}}.$$

This is how we endow the structure of a holomorphically translation invariant factorization algebra for the examples of the Kac–Moody and Virasoro prefactorization algebras

Exercise 2.12. Show that on $\Sigma = \mathbb{C}$, the prefactorization algebra $\mathrm{C}_\bullet^{\mathrm{Lie}}(\Omega_c^{0,\bullet} \otimes \mathfrak{g})$ carries the structure of a holomorphically translation invariant prefactorization algebra.

2.2. **A functorial relationship.** We now turn to the holomorphic analog of Theorem 2.7.

Theorem 2.13 ([5]). *Suppose $\mathcal{F}$ is a holomorphic prefactorization algebra on $\mathbb{C}$. Then, for any disk D centered at zero, the graded vector space*

$$\mathbb{V}\mathrm{ert}(\mathcal{F}) \stackrel{\text{def}}{=} \oplus_{n\in\mathbb{Z}} H^\bullet(\mathcal{F}^{(n)}(D))$$

has the structure of a $\mathbb{Z}$-graded vertex algebra. The assignment $\mathcal{F} \mapsto \mathbb{V}\mathrm{ert}(\mathcal{F})$ determines a functor from the category of holomorphically translation invariant prefactorization algebras to $\mathbb{Z}$-graded vertex algebras.

Remark 2.14. By a *holomorphic prefactorization algebra* we mean a holomorphically translation invariant prefactorization algebra on $\Sigma = \mathbb{C}$ which satisfies some additional technical conditions. For details see [5].

Sketch of the construction. Since we haven't given the formal definition of a vertex algebra, we will only sketch the idea of how one can recover a "z-dependent family" of multiplications.

We can restrict the factorization algebra to the submanifold $\mathbb{C}^\times \subset \mathbb{C}$. Then, we obtain a prefactorization algebra $r_*\mathcal{F}$ on $\mathbb{R}_{>0}$ where $r : \mathbb{C}^\times \to \mathbb{R}_{>0}$ is radial projection. The subfactorization algebra

$$\mathcal{A}_{\mathrm{rad}} \stackrel{\text{def}}{=} r_*\left(\oplus_{n\in\mathbb{Z}}\mathcal{F}^{(n)}\right) = \oplus_n r_*\mathcal{F}^{(n)} \tag{2.1}$$

is *locally constant* hence can be identified with an associative algebra.

The factorization product of an annulus on a disk gives $\oplus_{n\in\mathbb{Z}} H^\bullet(\mathcal{F}^{(n)}(D))$ the structure of a module for $\mathcal{A}_{\mathrm{rad}}$. In fact, this is the "vacuum module" in all of the examples we have introduced in the first lecture. $\square$

2.3. **Local cocycles.** We now want to consider how to "centrally extend" the holomorphic prefactorization algebras we introduced earlier in the lecture. The key idea is that at the level of prefactorization algebras, these extensions appear as *deformations* of the differential. We will focus on the Kac–Moody example first.

Consider the cosheaf of Lie algebras $\Omega_c^{0,\bullet} \otimes \mathfrak{g}$ that we used to built the prefactorization algebra of Example 2.8. Given an invariant quadratic polynomial κ, we notice that on this cosheaf we have a very similar "2-cocycle" to the one φ_κ introduced in the first lecture. It is defined by the formula

$$\psi_\kappa(\alpha, \alpha') = \int_\Sigma \kappa(\alpha \partial \alpha').$$

Notice that instead of taking the residue, as in the definition of φ_κ, we are integrating along Σ. Because of this, we see that ψ_κ is actually of *cohomological degree* $+1$.

Using ψ_k, we can perform the following deformation of the factorization algebra introduced in Example 2.8. To an open set $U \subset \Sigma$ consider the cochain complex

$$\mathrm{KM}_\kappa(U) = \left(\mathrm{Sym}\left(\Omega_c^{0,\bullet}(U) \otimes \mathfrak{g}[1]\right)[K] \; , \; \overline{\partial} + \mathrm{d}_\mathfrak{g} + K\psi_\kappa \right).$$

Notice that the first two terms in the differential are identical to the differential in $\mathrm{C}_\bullet^{\mathrm{Lie}}(\Omega_c^{0,\bullet}(U) \otimes \mathfrak{g})$. We have also introduced a polynomial variable K, which we will see plays the role of the central term. This polynomial variable can be specialized at any complex number $K = k \in \mathbb{C}$.

Theorem 2.15. *The assignment $U \mapsto \mathrm{KM}_\kappa(U)$ is a holomorphic prefactorization algebra with values in dg $\mathbb{C}[K]$-modules. On $\Sigma = \mathbb{C}$, the resulting vertex algebra*

$$\mathrm{Vert}\left(\mathrm{KM}_\kappa|_{K=k}\right)$$

is isomorphic to the Kac–Moody vertex algebra at level $k \in \mathbb{C}$. In particular the value of the prefactorization algebra on a disk is precisely the level k vacuum module.

A key step in the proof of this fact is to identify the "radial algebra" of Equation 2.1 $\mathcal{A}_{\mathrm{rad}}$ associated to the prefactorization algebra KM_κ with $U(\widehat{\mathfrak{g}}_\kappa)$.

There is a totally analogous result for the Virasoro algebra. Here, the local cocycle one deforms by can be written as

$$\alpha(z, \overline{z})\partial_z, \beta(z, \overline{z})\partial_z \mapsto \frac{1}{12} \int_\Sigma (\partial_z \alpha) \partial (\partial_z \beta).$$

3. Lecture 3: Higher dimensional algebras

We turn to the driving question of this lecture series. To what extent are there multivariable generalizations of Kac–Moody and Virasoro algebras? Furthermore, what do their category of representations look like?

In the second lecture we have seen how to recover affine algebras and their modules using the notion of a prefactorization alegbra. Our approach for finding multivariable enhancements is geometric and a direct higher dimensional generalization of the constructions in the previous lecture.

3.1. A naive generalization.

Before turning to our geometric approach, we briefly point out some subtleties in defining multivariable versions of affine algebras.

One fundamental issue is the following. In defining central extensions, we have extensively used the fact that there is a residue pairing on the algebra of functions on the punctured one-dimensional disk $\mathcal{O}(\widehat{D}^\times)$. Furthermore, the modules we have built have all essentially been induced along the map of algebras $\mathcal{O}(D) \to \mathcal{O}(D^\times)$.

If D^d is a d-dimensional disk, with $d > 1$, then the obvious map of algebras $\mathcal{O}(D^d) \to \mathcal{O}((D^d)^\times)$ is an *isomorphism*. This is a classical fact from algebraic geometry known as "Hartog's theorem". In particular, the naive generalization leads to a trivial theory of algebras and their modules in higher dimensions.

Notice that in the case of prefactorization algebras on Riemann surfaces, we used the Dolbeault complex to provide a "free resolution" of holomorphic sections. If X is a complex n-manifold we still have the Dolbeault complex $\Omega^{0,\bullet}(X)$, which is a resolution for the sheaf of holomorphic functions on X. When $X = \mathbb{C}^d \setminus 0$, $d > 1$, Hartog's theorem implies that $H^{0,\bullet}(X)$ is the same as $\mathcal{O}^{hol}(X)$. However, there is higher cohomology!

For now, let's return to the algebraic category. Let $(D^d)^\times \subset D^d$ denote the punctured algebraic disk. Then

$$H^i((D^d)^\times, \mathcal{O}) = \begin{cases} 0, & i \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & i = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & i = d-1 \end{cases} .$$

(For instance, use the cover by the affine opens of the form $D^d \setminus \{z_i = 0\}$.) When $d = 1$, this computation recovers the Laurent polynomials, so we should view the cohomology in degree $d-1$ as providing the derived replacement of the polar part of the Laurent polynomials.

There is a convenient model for the derived global sections of the structure sheaf of the punctured d-disk which we summarize below.

Proposition 3.1. *There is a dg algebra* $(A_d^\bullet, \overline{\partial})$ *satisfying the following properties:*

- *the complex* $A_d^\bullet$ *is concentrated in degrees* $0, \ldots, d-1$.
- *there is an isomorphism of graded algebras* $H^\bullet(A_d) \cong H^\bullet((D^d)^\times, \mathcal{O})$.
- *there is a map of dg algebras*

$$j : (A_d, \overline{\partial}) \hookrightarrow (\Omega^{0,\bullet}(\mathbb{C}^d), \overline{\partial})$$

  which is dense at the level of cohomology.
- *there is a "residue map"*

$$\mathrm{Res} : A_d^\bullet \to \mathbb{C}[-d+1]$$

  satisfying

$$\mathrm{Res}(\omega) = \oint_{S^{2d-1}} j(\omega) \mathrm{d}^d z$$

  for any $\omega \in A^{d-1}$.

This particular dg algebra A_d has appeared in [6] in their definition of the higher dimensional Kac–Moody algebra, which we will arrive at below.

3.2. **Higher dimensional holomorphic prefactorization algebras.** Let X be a complex manifold of complex dimension d. The construction of the prefactorization algebras on Riemann surfaces of the previous lectures carry over with no more difficulty to define the following two factorization algebras on X:

- the *level zero Kac–Moody* prefactorization algebra on X, $\mathrm{KM}_{d,0}$, which assigns to an open set $U \subset X$ the cochain complex

$$\mathrm{C}^{\mathrm{Lie}}_\bullet(\Omega^{0,\bullet}_c(U) \otimes \mathfrak{g}).$$

- the *charge zero Virasoro* prefactorization algebra on X, $\mathrm{Vir}_{d,0}$, which assigns to an open set $U \subset X$ the cochain complex

$$\mathrm{C}^{\mathrm{Lie}}_\bullet(\Omega^{0,\bullet}_c(U, TU)).$$

3.2.1. *Classification of central extensions.* A natural question to ask is what data we can use to "twist" these higher dimensional prefactorization algebras. Actually, the situation is very close to the one-dimensional case.

As in the one-dimensional case, we obtain such twists by deforming the differential on $C_\bullet^{\mathrm{Lie}}(\Omega_c^{0,\bullet}(X) \otimes \mathfrak{g})$ and $C_\bullet^{\mathrm{Lie}}(\Omega_c^{0,\bullet}(X, TX))$. Consequently, we are interested in degree $+1$ cocycles.

Theorem 3.2. *Let X be a complex d-dimensional manifold.*

- *There is a linear embedding*

$$\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow H^1_{\mathrm{Lie}}(\Omega_c^{0,\bullet}(X) \otimes \mathfrak{g}).$$

  To an invariant polynomial θ, we assign the cocycle

$$\psi_\theta(\alpha_1, \ldots, \alpha_{d+1}) = \int_X \theta(\alpha_1 \partial\alpha_2 \cdots \partial\alpha_d).$$

- *There is a linear embedding*

$$H^{2d+1}(\mathfrak{w}_d) \hookrightarrow H^1_{\mathrm{Lie}}(\Omega_c^{0,\bullet}(X, TX)).$$

As in the one-dimensional case, we find that the cocycles on $\Omega_c^{0,\bullet}(X) \otimes \mathfrak{g}$ and $\Omega_c^{0,\bullet}(X, TX)$ are built from natural polydifferential operators. In fact, when restricted functionals of this type, the cocycles described in the previous theorem classify *all* such cocycles up to equivalence, see [7].

Remark 3.3. The precise formulas for the Lie algebra cocycles on holomorphic vector fields are more difficult to describe than in the Kac–Moody case. In dimension one, the fact that $H^2(\mathfrak{vir}_{1,0}) = H^3(\mathfrak{w}_1)$ is one-dimensional means that there is essentially a one notion of "central charge".

Equip

$$\mathfrak{vir}_{d,0} \overset{\mathrm{def}}{=} A_d \otimes T_0 = A_d \{\partial_{z_1}, \ldots, \partial_{z_d}\}$$

with the Lie bracket of vector fields. Let $\mathfrak{w}_d$ be the Lie algebra of vector fields on the unpunctured disk. Then, there is natural map of Lie algebras $\mathfrak{w}_d \to \mathfrak{vir}_{d,0}$ which factors through H^0 of $\mathfrak{vir}_{d,0}$.

In higher dimensions, it turns out that $H^2(\mathfrak{vir}_{d,0}) = H^{2d+1}(\mathfrak{w}_d)$ is isomorphic to $H^{2d+2}(B\mathrm{U}(d))$. For example, in dimension two, there are *two* central charge parameters.

There is an isomorphism of vector spaces

$$\varphi_{\mathrm{Vir}} : H^{2d+1}(\mathfrak{w}_d) \to H^2(\mathfrak{vir}_{d,0})$$

defined as follows. Suppose $\varphi : (\mathfrak{w}_d)^{\otimes(2d+1)} \to \mathbb{C}$ is a cochain. We can restrict this to a cochain $\varphi : (\mathfrak{vir}_0)^{\otimes(2d+1)} \to \mathbb{C}$. Consider the translation invariant vector fields $\partial_{z_1}, \ldots, \partial_{z_d}$. Then, define

$$\overline{\varphi}(X_1, \ldots, X_{d+1}) = \varphi(\partial_{z_1}, \ldots, \partial_{z_d}, X_1, \ldots, X_{d+1}) + \text{symmetrize}.$$

We then obtain a cochain of $\mathfrak{vir}_0$ by the formula

$$\widetilde{\varphi}(X_1, \ldots, X_{d+1}) = \text{Res}\left(\overline{\varphi}(X_1, \ldots, X_{d+1})\mathrm{d}^d z\right).$$

The advantage of working with our model of prefactorization algebra is that the derived nature of the problem is naturally built into the formalism via the Dolbeault resolutions.

3.3. **Multivariable Lie algebras.** In the previous lecture, we pointed out how to extract an important associative algebra from the data of a prefactorization algebra on $\Sigma = \mathbb{C}$. The idea is to restrict to the punctured line $\mathbb{C}^\times \subset \mathbb{C}$ and to pushforward along the radial projection $r : \mathbb{C}^\times \to \mathbb{R}_{>0}$. The algebra $\mathcal{A}_{\text{rad}}$ was defined as a certain sub factorization algebra of this one-dimensional prefactorization algebra that had the property that it was locally constant (its value did not depend on the size of the interval).

In the case of the Kac–Moody prefactorization algebra on $\Sigma = \mathbb{C}$, the radial algebra $\mathcal{A}_{\text{rad}}$ is isomorphic to the enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$. Similarly, for the one-dimensional Virasoro prefactorization algebra the radial algebra is $U(\mathfrak{vir})$.

For the higher dimensional Kac–Moody and Virasoro algebras on $\mathbb{C}^n$, we can perform and analogous construction. Like the one-dimensional case, the content of this result is to interpret the geometrically defined radial algebra in terms of some algebra obtained from a central extension of the multivariable versions of $\mathfrak{g}$-currents and vector fields.

Proposition 3.4. *The radial algebra of both the higher Kac–Moody and Virasoro algebra is isomorphic the enveloping algebra of a central extension of some dg Lie algebra:*

- *The radial algebra $\mathcal{A}_{\text{rad}}$ of the prefactorization algebra $\mathrm{KM}_{\theta,k}$ on $\mathbb{C}^d$ is quasi-isomorphic to an algebra of the form $U(\widehat{\mathfrak{g}}_{d,\theta})$.*
- *The radial algebra $\mathcal{A}_{\text{rad}}$ of the prefactorization algebra $\mathrm{Vir}_{\mathbf{c}}$ on $\mathbb{C}^d$ is quasi-isomorphic to an algebra of the form $U(\mathfrak{vir}_{d,c})$.*

We proceed to describe $\widehat{\mathfrak{g}}_{d,\theta}$ and $\mathfrak{vir}_{d,c}$ in algebraic terms.

3.3.1. *An "algebraic" description.* Recall that we have introduced the dg model A_d for the derived global sections of the punctured d-disk. A_d is a commutative dg algebra whose cohomology is isomorphic to the cohomology of the punctured d-disk.

Definition 3.5. Define the following two dg Lie algebras.

- For a Lie algebra $\mathfrak{g}$, the *current algebra in complex dimension d* is the dg Lie algebra $A_d \otimes \mathfrak{g}$. We denote it by $\mathfrak{g}_d^\bullet$.
- The d-dimensional Witt algebra is the dg Lie algebra $\mathfrak{vir}_{d,0}$ which is spanned by elements of the form

$$\alpha(z,\overline{z})\frac{\partial}{\partial z_i}$$

  where $\alpha \in A_d$ and $i = 1, \dots, d$. The Lie bracket is the natural extension of the Lie bracket of vector fields.

There are natural central extensions of this dg current algebra as prescribed by the following cocycles.

Definition 3.6. Define the *multivariable Kac–Moody* and *Virasoro* dg Lie algebras as follows.

- For any invariant polynomial of degree d

$$\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \subset \mathbb{C}[\mathfrak{g}],$$

  the *d-dimensional Kac–Moody algebra* $\widehat{\mathfrak{g}}_{d,\theta}^\bullet$ is the central extension of the dg current algebra $A_d \otimes \mathfrak{g}$ determined by the degree $+2$ cocycle

$$\begin{array}{rccc} \varphi_\theta : & (A_d \otimes \mathfrak{g})^{\otimes(d+1)} & \to & \mathbb{C} \\ & a_0 \otimes \cdots \otimes a_d & \mapsto & \mathrm{Res}_{z=0}\,\theta(a_0 \partial a_1 \cdots \partial a_d) \end{array}.$$

- For any $c \in H^{2n+1}(\mathfrak{w}_d) \cong^{\varphi_{\mathrm{Vir}}} H^2(\mathfrak{vir}_{d,0})$, the d-dimensional Virasoro algebra of charge c, $\mathfrak{vir}_{d,c}$, is the central extension of the dg Witt algebra $\mathfrak{vir}_{d,0}$.

The multivariable Kac–Moody algebra has first appeared in the literature [6]. It's relationship to quantum field theory and factorization algebras, which we follow here, can be found in [7]. The multivariable Virasoro has appeared in the works [8] and in [9].

3.3.2. *L_∞-models.* We'd like to point out a subtle point in the definition above. We have shown in Lecture 1 that degree two cohomology classes of an ordinary Lie algebra are in bijective correspondence with equivalences classes of central extensions by $\mathbb{C}$. The same proof carries over with no more difficulty to the *dg* Lie algebra case. So, abstractly, we know that the cohomology classes in φ_θ and $\varphi_{\rm Vir}(c)$ correspond bijectively to central extensions of the dg Lie algebras $A_d \otimes \mathfrak{g}$ and $\mathfrak{vir}_{d,0}$ respectively.

To construct an explicit model, however, it is convenient to use a slightly weaker notion of a dg Lie algebra. An *L_∞-algebra* $\mathfrak{h}$ is a like a dg Lie algebra except the Jacobi identity only holds up to a (prescribed) homotopy. This is encoded by a sequence of maps $\{\ell_k\}_{k\geq 1}$ where ℓ_k is a graded alternating map

$$\ell_k : \mathfrak{h}^{\otimes k} \to \mathfrak{h}[2-k]$$

which satisfies a list of relations. The first few relations read

$$\begin{aligned} \ell_1^2 &= 0 \\ \ell_1 \circ \ell_2 + \ell_2 \circ (\ell_1 \otimes 1 + 1 \otimes \ell_1) &= 0 \\ \ell_1 \circ \ell_3 + \ell_3(\ell_1 \otimes 1 \otimes 1 + 1 \otimes \ell_1 \otimes 1 + 1 \otimes 1 \otimes \ell_1) &= \ell_2 \circ (1 \otimes \ell_2) + \ell_2 \circ (\ell_2 \otimes 1) \circ (1 \otimes \tau) \\ \vdots \quad & \quad \vdots \end{aligned}$$

The first equation implies that ℓ_1 is a differential and the second equation implies that it is a derivation for the "bracket" ℓ_2. The third equation states that ℓ_2 satisfies the ordinary Jacobi identity up the a homotopy defined by ℓ_3.

There are L_∞ models for the higher Kac–Moody and Virasoro algebras that we consider which are fairly easy to comprehend. For example, the d-dimensional Kac–Moody Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$ has $\ell_1 = \overline{\partial}$, $\ell_2 = [-,-]$ and $\ell_{d+1} = \theta$. Notice that here the Jacobi identity strictly holds since $\ell_{d+1} = \theta$ is closed for $\ell_1 = \overline{\partial}$.

3.4. **dg vacuum modules.** Interpreting the multivariable algebras geometrically in terms of prefactorization algebras leads one to the very natural notion of "vacuum modules".

A key step in passing from prefactorization algebras to vertex algebras was in the identification of a subspace of the value of the factorization algebra on a disk

$$\mathcal{V}_{\rm disk} \subset \mathcal{F}(D)$$

as a module for the radial algebra $\mathcal{A}_{\rm rad}$.

For higher dimensional Kac–Moody and Virasoro algebras, we have an analogous construction. Let $\mathcal{F}$ denote either prefactorization algebra on $\mathbb{C}^d$ (with or without central term introduced). Consider the action by dilations $\mathbb{G}_m$ on $\Omega^{0,\bullet}(\mathbb{C}^n)$. Denote by $\mathcal{V}_{\text{disk}}$ the direct sum over all weights

$$\mathcal{V}_{\text{disk}} = \oplus_{n \geq 0} \mathcal{F}^{(n)}(\mathbb{C}^d).$$

Then, it is immediate to check that the factorization structure maps of disks and spheres gives $\mathcal{V}_{\text{disk}}$ the structure of a $\mathcal{A}_{\text{rad}}$-module.

Definition 3.7. The $\mathcal{A}_{\text{rad}}$-module $\mathcal{V}_{\text{disk}}$ associated to the Kac–Moody or Virasoro prefactorization algebra on $\mathbb{C}^d$ is called the *dg disk module* associated to $\mathcal{A}_{\text{rad}} = U(\widehat{\mathfrak{g}}_{d,\theta})$ or $\mathcal{A}_{\text{rad}} = U(\mathfrak{vir}_{d,\theta})$ respectively.

It turns out that we can recast this definition in completely algebraic terms mimicking the notion of a "vacuum module" in complex dimension one.

Definition 3.8. Let $d > 1$, and consider the dg algebra A_d. Define the A_d-module of *positive modes*

$$A_{d,+} = H^{d-1}(A_d).$$

Note that there is a natural map of dg A_d-modules $A_d \to A_{d,+}[-d+1]$. Define the dg ideal of *negative modes*

$$A_{d,-} = \ker\left(A_d \to A_{d,+}[-d+1]\right).$$

Remark 3.9. We are modeling our terminology on the usual definition of positive and negative modes for Laurent polynomials in one-variable $A_1 = \mathbb{C}[z, z^{-1}]$ via

$$A_{1,+} = \mathbb{C}[z] \subset \mathbb{C}[z, z^{-1}] \quad \text{and} \quad A_{1,-} = z^{-1}\mathbb{C}[z^{-1}] \subset \mathbb{C}[z, z^{-1}]$$

respectively.

Definition 3.10. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and let $k \in \mathbb{C}$. The *vacuum module* $\mathrm{Vac}_{(\theta,k)}$ associated to the pair (θ, k) is the induced $\widehat{\mathfrak{g}}_{d,\theta}$-module

$$\mathrm{Ind}_{U(A_{d,+}\otimes\mathfrak{g})[K]}^{U(\mathfrak{g}_{d,\theta})}(\mathbb{C}_K) = U(\mathfrak{g}_{d,\theta}) \otimes_{U(A_{d,+}\otimes\mathfrak{g})[K]} \mathbb{C}_k.$$

When θ is understood, we refer to this as the *level k* vacuum module.

Remark 3.11. There is a variant of this definition that makes sense for a fixed θ and no specification of k. It is defined by

$$\mathrm{Vac}_\theta = U(\widehat{\mathfrak{g}}_{d,\theta}) \otimes_{U(A_{d,+}\otimes\mathfrak{g})[K]} \mathbb{C}[K].$$

This is a $U(\widehat{\mathfrak{g}}_{d,\theta})$-module in the category of $\mathbb{C}[K]$-modules.

Proposition 3.12. *Let $\mathcal{V}_{\mathrm{disk}}$ be the disk module of $U(\widehat{\mathfrak{g}}_{d,\theta})$. There is an isomorphism of dg $U(\widehat{\mathfrak{g}}_{d,\theta})$-modules*

$$\mathcal{V}_{\mathrm{disk}} \cong \mathrm{Vac}_{(\theta,k)}.$$

Exercise 3.13. Define the notion of a dg Virasoro module in the style of the above definition for the Kac–Moody.

3.5. **Higher vertex structure.** A result of [5], which we summarized in Theorem 2.13, states how the holomorphic factorization algebra associated to a Lie algebra recovers the Kac–Moody vertex algebra. The key point is that the operator product expansion is encoded by the factorization product between disks embedded in $\mathbb{C}$. Our proposed factorization algebra, then, provides a higher dimensional enhancement of this vertex algebra through the factorization product of balls or polydisks in $\mathbb{C}^d$. This structure can be thought of as a holomorphic analog of an algebra over the operad of little d-disks. Writing down the precise axioms of a "higher dimensional vertex algebra" is currently work in progress.

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