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DUALITY THEOREMS FOR SHIFT-INVARIANT SYSTEMS AND GREEN'S IMPRIMITIVITY THEOREM

Franz Luef
(joint work Ulrik Enstad)
Applied Matrix Positivity II
ICMS, Bayes Centre, Edinburgh

Theorem

The following are equivalent:

(i) $\{\pi(\alpha k, \beta l)g : k, l \in \mathbb{Z}\}$ is a Gabor frame for $L^2(\mathbb{R})$.

(ii) There exist $A, B > 0$ such that the spectrum of the Ron-Shen matrix $G(x) = (\sum_{j\in\mathbb{Z}} g(x + \alpha j - \frac{k}{\beta})\overline{g}(x + \alpha j - \frac{l}{\beta}))$ is contained in $[A, B]$.

(iii) There exist $A, B > 0$ such that

$$A\|c\|_2^2 \le \sum_j |\sum_k g(x + \alpha j - \tfrac{k}{\beta})c_k|^2 \le B\|c\|_2^2, \qquad \text{a.a. } x \in \mathbb{R},\ c \in \ell^2(\mathbb{Z}).$$

Remark

(iii) says that $x + \alpha\mathbb{Z}$ is a set of sampling for the shift-invariant space $V_{\frac{1}{\beta}} = \{f \in L^2(\mathbb{R}) : f = \sum_{k\in\mathbb{Z}} c_k g(\cdot - \frac{k}{\beta})\}$ with uniform constants for all $x \in \mathbb{R}$.

Main goal

Developing a better understanding of the characterization of Gabor frames in terms of Ron-Shen matrices.

Periodization trick

$$\int_{\mathbb{R}} f(x)\, dx = \int_0^{\alpha} \sum_{k\in\mathbb{Z}^d} f(x+\alpha k)\, dx$$

Use the partition of $\mathbb{R}$ into translates of the interval $[0,\alpha)$, i.e. $\cup_{k\in\mathbb{Z}^d}(\alpha k + [0,\alpha))$ and $(\alpha k + [0,\alpha)) \cap (\alpha l + [0,\alpha)) = \emptyset$ for $k \neq l$.

Consequence: **Poisson summation formula:**

$$\sum_{k\in\mathbb{Z}} f(x+\alpha k) = \alpha^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}\left(\tfrac{k}{\alpha}\right) e^{2\pi i k x/\alpha},$$

for "nice functions", e.g. in Feichtinger’s algebra or Schwartz class.

Setting

- second-countable locally compact group G with identity element e
- Γ lattice in G, i.e., a discrete subgroup such that there exists a finite G-invariant Borel probability measure μ on the left G-space G/Γ of left cosets of Γ in G.

Weil's formula

$$\int_G f(x)dx = \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x\gamma)d\mu(x\Gamma), \qquad f \in C_c(G).$$

Key observation

Weil's formula implies the direct integral decompositions of $L^2(G)$:

$$L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma)d\mu(x\Gamma)$$

The identification of $L^2(G)$ with the direct integral is given by mapping $f \in L^2(G)$ to the section of $(\ell^2(x\Gamma))_{x\Gamma \in G/\Gamma}$ given by the family of restrictions $(f|_{x\Gamma})_{x\Gamma \in G/\Gamma}$.

Field of Hilbert spaces

A *field* of Hilbert spaces over X is a collection $(\mathcal{H}_x)_{x\in X}$ of Hilbert spaces indexed by X. We write $\langle\cdot,\cdot\rangle_x$ and $\|\cdot\|_x$ for the inner product and norm of $\mathcal{H}_x$, respectively.

An element f of the product $\prod_{x\in X}\mathcal{H}_x$ is called a section and we denote the projection of f onto $\mathcal{H}_x$ by f_x.

Measurable fields

Let X be a measurable space. The field $(\mathcal{H}_x)_x$ is called *measurable* when it comes equipped with a linear subspace V of $\prod_{x\in X}\mathcal{H}_x$ such that the following hold: There exists a countable family $(\eta^i)_{i=1}^{\infty}$ in V such that

1. $\{\eta^i_x : i \in \mathbb{N}\}$ is dense in $\mathcal{H}_x$ for every $x \in X$, and
2. an element $f \in \prod_{x\in X}\mathcal{H}_x$ is in V if and only if $x \mapsto \langle f_x, \eta^i_x\rangle_x$ is measurable for every $i \in \mathbb{N}$.

We call elements of V *measurable sections*.

Direct integral

Let μ be a measure on X. A measurable section f of $(\mathcal{H}_x)_{x \in X}$ is called *integrable* with respect to μ if $\int_X \|f_x\|_x^2 d\mu(x) < \infty$.

The *direct integral* of $(\mathcal{H}_x)_{x \in X}$ with respect to μ, denoted by $\int_X \mathcal{H}_x d\mu(x)$, is the set of equivalence classes of integrable sections under equality μ-almost everywhere on X. It is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_X \langle f_x, g_x \rangle_x d\mu(x), \qquad f, g \in \int_X \mathcal{H}_x d\mu(x).$$

We set $\mathcal{H} = \int_X \mathcal{H}_x d\mu(x)$.

Operators for direct integrals

Let $(\mathcal{H}_x)_{x\in X}$ and $(\mathcal{K}_x)_{x\in X}$ be two measurable fields of Hilbert spaces over X.

- A collection $T=(T_x)_{x\in X}$ of bounded linear maps $T_x\colon \mathcal{H}_x\to\mathcal{K}_x$ defines a map $T\colon \prod_{x\in X}\mathcal{H}_x\to\prod_{x\in X}\mathcal{K}_x$ given by $(Tf)(x)=T(f(x))$.
- We call $(T_x)_x$ a *measurable field* if the associated map T maps measurable sections of $(\mathcal{H}_x)_x$ to measurable sections of $(\mathcal{K}_x)_x$.
- If μ is a measure on X, then $(T_x)_x$ is called *μ-essentially bounded* if ${}_{x\in X}\|T_x\|_{\mathcal{H}_x\to\mathcal{K}_x}<\infty$.

  In that case T defines a bounded linear map from $\int_X \mathcal{H}_x d\mu(x)$ to $\int_X \mathcal{K}_x d\mu(x)$. Bounded linear operators between direct integrals of this form are called *decomposable*.
- Furthermore, a decomposable operator $T\in\mathcal{B}(\mathcal{H})$ is positive if and only $T=S^*S$ for some operator $S\in\mathcal{B}(\mathcal{H})$ and thus S is decomposable too. Hence $T_x=S_x^*S_x$ for μ-almost every $x\in X$, which means that μ-almost every T_x is positive.

Lemma

Let $T \in \mathcal{B}(\mathcal{H})$ be a decomposable operator. Then the following hold:

1. T is self-adjoint if and only if T_x is self-adjoint for μ-almost every $x \in X$.
2. T is positive if and only if T_x is positive for μ-almost every $x \in X$.

Let us develop the basic notions of frame theory for direct integrals:

Fibered frames and fibered Riesz bases

A sequence $(g^j)_{j\in J} \in \mathcal{H} \cong \int_X \mathcal{H}_x d\mu(x)$ is called a *fibered frame* (resp. Riesz sequence) with bounds $c, C > 0$ if $(g^j_x)_{j\in J}$ is a frame (resp. Riesz sequence) for $\mathcal{H}_x$ with bounds $c, C > 0$ for μ-almost every $x \in X$.

If an upper frame bound in the definition of a fibered frame exists but not necessarily a lower frame bound, we call $(g^j)_{j\in J}$ a *fibered Bessel sequence*.

Let d denote the counting measure on J. We denote by $\mathcal{H}' = \int_X \ell^2(J) d\mu(x)$ the direct integral of the constant field $(\ell^2(J))_{x \in X}$ with respect to μ, which is isomorphic to $L^2(J \times X, d \times \mu)$, and denote its elements by $a = (a_x^j)_{x \in X, j \in J}$.

Let $(g^j)_{j \in J}$ be a fibered Bessel sequence with upper Bessel bound $C > 0$.

Fibered analysis and synthesis operator

- The *analysis operator* $\mathcal{C}_x \colon \mathcal{H}_x \to \ell^2(J)$ is given by

$$\mathcal{C}_x f = (\langle f, g_x^j \rangle)_{j \in J}, \quad f \in \mathcal{H}_x,$$

  for μ-almost every $x \in X$. These define a μ-essentially bounded field of operators from $(\mathcal{C}_x)_{x \in X}$ to the constant field $(\ell^2(J))_{x \in X}$. The corresponding decomposable operator $\mathcal{C} \colon \mathcal{H} \to \mathcal{H}'$, which we call the *fibered analysis operator*, satisfies $\|\mathcal{C}\|^2 \leq C$.
- The synthesis operators $\mathcal{D}_x = \mathcal{C}_x^*$ given by

$$\mathcal{D}_x a = \sum_{j \in J} a^j g_x^j, \quad a \in \ell^2(J),$$

  for μ-almost every $x \in X$ also define a μ-essentially bounded field of operators with associated *fibered synthesis operator* $\mathcal{D} \colon \mathcal{H}' \to \mathcal{H}$.

Fibered frame / Gramian operator

The *fibered frame operator* (resp. *fibered Gramian operator*) of a sequence $(g^j)_{j\in J}$ in $\mathcal{H}$ is given by $\mathcal{S} = \mathcal{C}^*\mathcal{C} \in \mathcal{B}(\mathcal{H})$ (resp. $\mathcal{G} = \mathcal{D}^*\mathcal{D} \in \mathcal{B}(\mathcal{H}')$) where $\mathcal{C}$ and $\mathcal{D}$ denote the associated analysis and synthesis operators, respectively.

Basic observation

Let $(g^j)_j$ be a sequence in $\mathcal{H}$. Then the following hold:

1. $(g^j)_{j\in J}$ is a fibered frame for $\mathcal{H}$ with bounds $c, C > 0$ if and only if $_{\mathcal{B}(\mathcal{H})}(\mathcal{S}) \subseteq [c, C]$, that is,

$$c\|f\|^2 \leq \int_X \sum_{j\in J} |\langle f_x, g_x^j\rangle|^2 d\mu(x) \leq C\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

2. $(g^j)_{j\in J}$ is a fibered Riesz sequence for $\mathcal{H}$ with bounds $c, C > 0$ if and only if the associated fibered Gramian operator satisfies $_{\mathcal{B}(\mathcal{H})}(\mathcal{G}) \subseteq [c, C]$, that is,

$$c\|a\|_2^2 \leq \int_X \Big\| \sum_{j\in J} a_x^j g_x^j \Big\|^2 d\mu(x) \leq C\|a\|_2^2 \quad \text{for all } a \in L^2(J \times X, c \times \mu). \tag{2}$$

Like for ordinary frames, it can also be verified that a sequence $(g^j)_{j\in J}$ in $\mathcal{H}$ is a fibered frame for $\mathcal{H} = \int_X \mathcal{H}_x d\mu(x)$ if and only if there exists another sequence $(h^j)_{j\in J}$ in $\mathcal{H}$ such that for μ-almost every $x \in X$ we have that

$$\langle f, f' \rangle = \sum_{j\in J} \langle f, g_x^j \rangle \overline{\langle f', h_x^j \rangle}, \qquad f, f' \in \mathcal{H}. \tag{3}$$

We call $(h^j)_j$ a *fibered dual frame* to $(e^j)_j$. Similarly, $(g^j)_j$ is a Riesz sequence if and only if it admits a *fibered biorthogonal sequence*, that is, a sequence $(h^j)_j$ that satisfies

$$\langle g_x^j, h_x^{j'} \rangle = \delta_{j,j'} \qquad \text{for } \mu\text{-almost every } x \in X. \tag{4}$$

Proposition

Let $g \in L^2(G)$. Then the following hold:

1. The family $(L_\lambda g)_{\lambda \in \Lambda}$ is a fibered frame for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma) d\mu(x\Gamma)$ with bounds $c, C > 0$ if
$$c\|a\|^2 \leq \sum_{\lambda \in \Lambda} \left| \sum_{\gamma \in \Gamma} a^\gamma \overline{g(\lambda^{-1} x\gamma)} \right|^2 \leq C\|a\|^2 \quad \text{for all } a = (a^\gamma)_\gamma \in \ell^2(\Gamma)$$
holds for μ-almost every $x\Gamma \in G/\Gamma$.
2. The family $(L_\lambda g)_{\lambda \in \Lambda}$ is a fibered Riesz sequence for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma) d\mu(x\Gamma)$ with bounds $c, C > 0$ if
$$c\|a\|^2 \leq \sum_{\gamma \in \Gamma} \left| \sum_{\lambda \in \Lambda} a^\gamma g(\lambda^{-1} x\gamma) \right|^2 \leq C\|a\|^2 \quad \text{for all } a = (a^\gamma)_\gamma \in \ell^2(\Gamma)$$
holds for μ-almost every $x\Gamma \in G/\Gamma$.

Proposition – continued

1. The family $(R_\gamma g)_{\gamma\in\Gamma}$ is a fibered frame for $L^2(G) \cong \int_{\Lambda\backslash G} \ell^2(\Lambda x) d\nu(x\Lambda)$ with bounds $c, C > 0$ if

$$c\|b\|^2 \leq \sum_{\gamma\in\Gamma} \left| \sum_{\lambda\in\Lambda} b^\lambda \overline{g(\lambda x \gamma)} \right|^2 \leq C\|b\|^2 \quad \text{for all } b \in \ell^2(\Lambda)$$

   holds for ν-almost every $\Lambda x \in \Lambda\backslash G$.

2. The sequence $(R_\gamma g)_{\gamma\in\Gamma}$ is a fibered Riesz sequence for $L^2(G) \cong \int_{\Lambda\backslash G} \ell^2(\Lambda x) d\nu(x\Lambda)$ with bounds $c, C > 0$ if and only if

$$c\|b\|^2 \leq \sum_{\lambda\in\Lambda} \left| \sum_{\gamma\in\Gamma} b^\lambda g(\lambda x \gamma) \right|^2 \leq C\|b\|^2 \quad \text{for all } b \in \ell^2(\Lambda)$$

   holds for ν-almost every $\Lambda x \in \Lambda\backslash G$.

We denote the left translation operator by $(L_\lambda f)(x) = f(\lambda^{-1}x)$ and the right translation operator $(R_\gamma f)(x) = f(x\gamma)$ for $\lambda \in \Lambda$ and $\gamma \in \Gamma$.

Duality theorem

Let $g \in L^2(G)$. The following are equivalent:

1. The family $(L_\lambda g)_{\lambda \in \Lambda}$ is a fibered frame for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma) d\mu(x\Gamma)$ with bounds $c, C > 0$, that is,

$$c\|a\|^2 \leq \sum_{\lambda \in \Lambda} \Big| \sum_{\gamma \in \Gamma} a^\gamma \overline{g(\lambda^{-1} x \gamma)} \Big|^2 \leq C\|a\|^2 \quad \text{for all } a = (a^\gamma)_\gamma \in \ell^2(\Gamma)$$

2. The family $(R_\gamma g)_{\gamma \in \Gamma}$ is a fibered Riesz sequence for $L^2(G) \cong \int_{\Lambda \backslash G} \ell^2(\Lambda x) d\nu(\Lambda x)$ with bounds $c, C > 0$, that is,

$$c\|b\|^2 \leq \sum_{\lambda \in \Lambda} \Big| \sum_{\gamma \in \Gamma} b^\lambda g(\lambda x \gamma) \Big|^2 \leq C\|b\|^2 \quad \text{for all } b \in \ell^2(\Lambda).$$

Theorem

Let $g, h \in L^2(G)$. Then the following are equivalent:

1. $(L_\lambda g)_{\lambda\in\Lambda}$ and $(L_\lambda h)_{\lambda\in\Lambda}$ are fibered dual frames for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma) d\mu(x\Gamma)$, that is,

$$\langle f, f' \rangle = \int_{G/\Lambda} \sum_{\lambda\in\Lambda} \sum_{\gamma,\gamma'\in\Gamma} f(x\gamma)\overline{g(\lambda^{-1}x\gamma) f(x\gamma')} h(\lambda^{-1}x\gamma') d\mu(x\Gamma), \qquad f, f' \in L^2(G).$$

2. $(R_\gamma g)_{\gamma\in\Gamma}$ and $(R_\gamma h)_{\gamma\in\Gamma}$ are fibered biorthogonal systems for $L^2(G) \cong \int_{\Lambda\backslash G} \ell^2(\Lambda x) d\nu(\Lambda x)$, that is,

$$\sum_{\lambda\in\Lambda} \overline{f(\lambda x)} g(\lambda x\gamma) = \delta_{\gamma,e}, \qquad \text{for all } \gamma \in \Gamma \text{ and } \mu\text{-almost every } x \in X.$$

Suitable class of Lie groups

Denote by $\mathcal{R}$ the class of Lie groups for which their radical (i.e., largest, connected, normal, solvable subgroup) R such that G/R contains no nontrivial, connected, compact, normal subgroups. We write $G \in \mathcal{R}$ to indicate that G belongs to this class.

Density theorem

For $G \in \mathcal{R}$ we have:

- If $(L_\lambda g)_{\lambda \in \Lambda}$ is a fibered frame for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma) d\mu(x\Gamma)$, then

$$\mathrm{covol}(\Gamma) \leq \mathrm{covol}(\Lambda).$$

- If $(R_\gamma g)_{\gamma \in \Gamma}$ is a fibered Riesz basis for $L^2(G) \cong \int_{\Lambda \backslash G} \ell^2(\Lambda x) d\nu(\Lambda x)$, then

$$\mathrm{covol}(\Lambda) \leq \mathrm{covol}(\Gamma).$$

Rieffel computed in 1981 the center-valued von Neumann dimensions and his results imply these density theorems for frames and Riesz bases for our direct integrals.

Recall that for $L^2(\Lambda \times G/\Gamma)$ the space $\Lambda \times G/\Gamma$ is equipped with the product measure $d \times \mu$ where d denotes the counting measure on Λ, and similarly for $L^2(\Gamma \times \Lambda\backslash G)$.
Note that on $C_c(\Lambda \times G/\Gamma)$ (resp. $C_c(\Gamma \times \Lambda\backslash G)$ of $L^2(\Gamma \times \Lambda\backslash G)$) we may define the representation π (resp. ρ) induced by these two actions as follows:

$$(\pi(a)f)(x) = \sum_{\lambda \in \Lambda} a(\lambda, x\Gamma) f(\lambda^{-1}x), \tag{5}$$

$$(\rho(b)f)(x) = \sum_{\gamma \in \Gamma} b(\gamma, \Lambda x) f(x\gamma), \tag{6}$$

for $a \in \ell^1(\Lambda, G/\Gamma)$ and $b \in \ell^1(\Gamma, \Lambda\backslash G)$.

Observation

The fibered synthesis operator $\mathcal{D}_g$ of $(L_\lambda g)_{\lambda \in \Lambda}$ is given by

$$(D_g a)(x) = \sum_{\lambda \in \Lambda} a(\lambda, x\Gamma) g(\lambda^{-1}x), \qquad a \in C_c(\Lambda \times G/\Gamma).$$

These two actions allow the definition of noncommutative measure spaces:

Crossed product von Neumann algebras

We briefly define the crossed product algebras associated to the group actions:

(i) Left action of the lattice Λ on the space G/Γ by left translations, i.e., $\lambda \cdot (x\Gamma) = \lambda x \Gamma$ for $\lambda \in \Lambda$ and $x \in G$.

(ii) Right action of Γ on the space $\Lambda \backslash G$ by right translations, i.e. $(\Lambda x) \cdot \gamma = \Lambda x \gamma$ for $x \in G$ and $\gamma \in \Gamma$.

(iii) For $\lambda \in \Lambda$ and $g \in L^\infty(G/\Gamma)$ we define unitary operators u_λ and m_g on $L^2(\Lambda \times G/\Gamma)$ by

$$(u_\lambda \xi)(\lambda', x\Gamma) = \xi(\lambda^{-1}\lambda', x\Gamma), \qquad (m_g \xi)(\lambda, x\Gamma) = g(\lambda x \Gamma)\xi(\lambda, x\Gamma), \quad \xi \in L^2(\Lambda \times G/\Gamma).$$

Crossed products–continued

(iv) We define unitary operators v_γ for $\gamma \in \Gamma$ and n_h for $h \in L^\infty(G/\Gamma)$ on $L^2(\Gamma \times \Lambda\backslash G)$ by

$$(v_\gamma\xi)(\gamma', \Lambda x) = \xi(\gamma'\gamma, x\Gamma), \qquad (n_h\xi)(\gamma, \Lambda x) = h(\Lambda x\gamma)\xi(\gamma, \Lambda x), \quad \xi \in L^2(\Gamma \times \Lambda\backslash G).$$

(v) The crossed product $M = L^\infty(G/\Gamma) \rtimes \Lambda$ is defined to be the von Neumann algebra on $L^2(\Lambda \times G/\Gamma)$ generated by the operators U_λ and m_g for $\lambda \in \Lambda$ and $g \in L^\infty(G/\Gamma)$. Similarly, the crossed product $N = L^\infty(\Lambda\backslash G) \rtimes \Gamma$ is the von Neumann algebra generated by the operators v_γ and n_h for $\gamma \in \Gamma$ and $h \in L^\infty(\Lambda\backslash G)$.

Both of these von Neumann algebras come equipped with faithful normal traces $\tau\colon M \to \mathbb{C}$ and $\kappa\colon N \to \mathbb{C}$ determined by μ and ν, respectively.

$$\tau(u_\lambda m_g) = \delta_{\lambda,e} \int_{G/\Gamma} g d\mu, \quad \lambda \in \Lambda,\ g \in L^\infty(G/\Gamma),$$
$$\kappa(v_\gamma n_h) = \delta_{\gamma,e} \int_{\Lambda\backslash G} h d\nu, \quad \gamma \in \Gamma,\ h \in L^\infty(\Lambda\backslash G).$$

In short, these are “nice" noncommutative measure spaces.

Partial Fourier transform

Let G be a locally compact abelian group. Hence, we may associate to the lattice Γ the dual lattice in $\widehat{G}$ given by

$$\Gamma^{\perp} = \{\omega \in \widehat{G} : \omega|_{\Gamma} = 1\}.$$

There is a natural isomorphism $\widehat{G/\Gamma} \cong \Gamma^{\perp}$, so by performing a partial Fourier transform to elements of $\ell^{1}(\Lambda \times \Gamma^{\perp})$ in the first argument we obtain elements of $\ell^{1}(\Lambda, G/\Gamma)$.

Denote this map by $\mathcal{F} \colon \ell^{1}(\Lambda \times \Gamma^{\perp}) \to \ell^{1}(\Lambda, G/\Gamma)$, which we define using the convention

$$\mathcal{F}(a)(\lambda, x\Gamma) = \sum_{\tau \in \Gamma^{\perp}} a(\lambda, \tau)\tau(x), \qquad a \in \ell^{1}(\Lambda, \Gamma^{\perp}).$$

Ron-Shen matrix characterization

Let $g \in L^2(G)$. Then the following are equivalent:

1. $(\pi(z)g)_{z\in\Lambda\times\Gamma}$ is a Gabor frame with bounds $c, C > 0$.
2. The family $(T_\lambda g)_{\lambda\in\Lambda}$ is a fibered frame for $L^2(G) \cong \int_{G/\Gamma} \ell^2(x\Gamma)d\mu(x\Gamma)$ with bounds $c, C > 0$, that is,

$$c\|a\|^2 \leq \sum_{\lambda\in\Lambda} \Big| \sum_{\gamma\in\Gamma} a^\gamma \overline{g(x+\lambda^{-1}+\gamma)} \Big|^2 \leq C\|a\|^2 \quad \text{for all } a = (a^\gamma)_\gamma \in \ell^2(\Gamma)$$

Shift-invariant spaces

We associate with our left and right actions by the lattices Λ and Γ the shift-invariant spaces V_Λ and V_Γ:

$$V_\Lambda = \{f \in L^2(G) : \ f = \sum_{\lambda \in \Lambda} c_\lambda L_\lambda g, \quad c \in \ell^2(\Lambda)\}$$

$$V_\Gamma = \{f \in L^2(G) : \ f = \sum_{\gamma \in \Gamma} c_\gamma R_\gamma g, \quad c \in \ell^2(\Gamma)\}$$

Then the duality result may be rephrased as follows: $x + \Gamma$ is a sampling set for V_Λ if and only if $x + \Lambda$ is a set of interpolation for V_Γ.

Thank you for your attention

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