COMBINATORIAL INDICES AND MATRIX POSITIVITY

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The talk is based on a series of works with
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Positive and Non-negative matrices

Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix with the real entries.
A is positive if all its entries are positive, $a_{ij} > 0$,
A is non-negative, if all $a_{ij} \geq 0$.

Combinatorial matrix theory is an efficient approach to investigate non-negative matrices. Here

$$\text{matrix properties} \longrightarrow \text{graph theory constructions}$$

Graphs

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- A closed walk is a $u \to v$ walk where $u = v$.
- A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$.
- The length of a shortest cycle in G is called the girth of G.

Correspondence between matrices and digraphs

Let $A = (a_{ij}) \in M_n(\mathbf{B})$. A corresponds to a digraph $G = G(A)$ of order n as follows. The vertex set is the set $V = \{1, \ldots, n\}$. There is an edge (i, j) from i to j iff $a_{ij} \neq 0$. A is adjacency matrix of G.

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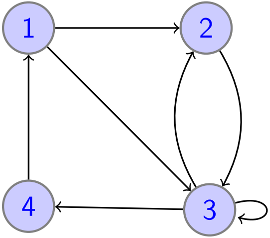
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \longleftrightarrow$$



$G(A)$

Definition

Non-negative $A \in M_n$, $A \geq 0$, $n \geq 2$, is called decomposable if $\exists$ permutation matrix $P \in M_n$ such that

$$A = P \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} P^t,$$

where B, D are square matrices and C is possibly a rectangular matrix. If A is not decomposable, then it is called indecomposable.

Definition

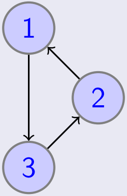
G is strongly connected iff for any $u, v \in V(G)$ there is an oriented path from u to v.

Theorem

Let $A \in M_n$, $A \geq 0$. TFAE

- A *is indecomposable,*
- $G(A)$ *is strongly connected,*
- $(I + A)^{n-1} > 0$,
- $\forall\ i, j,\ i \neq j,\ \exists\ k$: (i, j)*-th element of* A^k *is positive.*

Example



$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$(I + A)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Primitive digraphs

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Then $A^{k+1} = A^k \cdot A > 0$.

Theorem

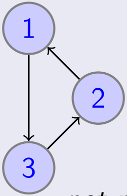
Let G be an digraph. THAE

- *G is primitive,*
- *G is strongly connected and the GCD of all cycle lengths in G is 1,*
- *$A(G)$ is primitive.*

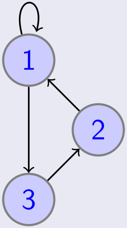
Corollary

Let G be a primitive digraph. Then $\exp(G) = \exp(A(G))$.

Example



$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *is indecomposable and is not primitive:* $A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $A^3 = I$, $A^4 = A$, *etc.*



$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *is primitive:* $A^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

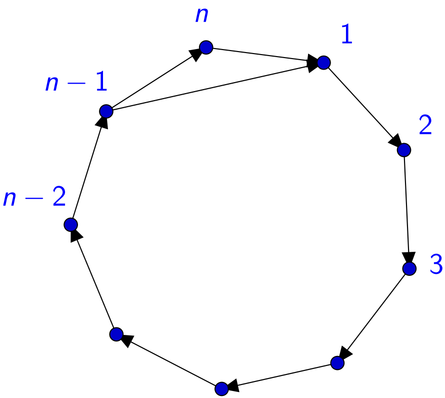
The Wielandt matrix is

$$W_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Theorem (Wielandt)

Let $A \in M_n$, $A \geq 0$. *Then* $\exp(A) \leq \exp(W_n) = (n-1)^2 + 1$.

Classical example



W_n

W_n is called a Wielandt digraph. It is the digraph with the maximal possible exponent, $(n-1)^2+1$.

Akelbek and Kirkland, 2009

Definition

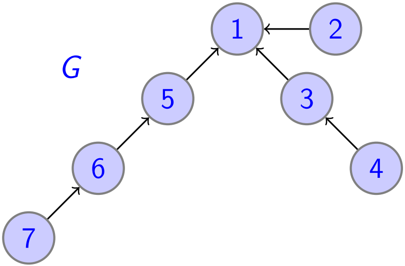
The scrambling index of a digraph G is the smallest positive integer k such that for every pair $u, v \in V(G)$, exists $w \in V(G)$ such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in G.

The scrambling index of G is denoted by $k(G)$. If such w does not exist, let $k(G) = 0$.

$$k(W_n) = \left\lceil \frac{(n-1)^2 + 1}{2} \right\rceil < (n-1)^2 + 1 = \exp(W_n)$$

How to compute the scrambling index?

Simple structure of digraph



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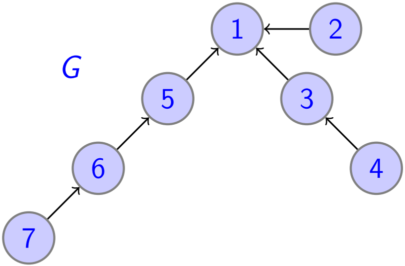

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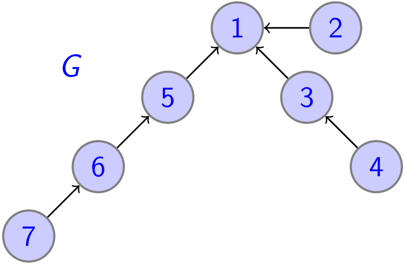

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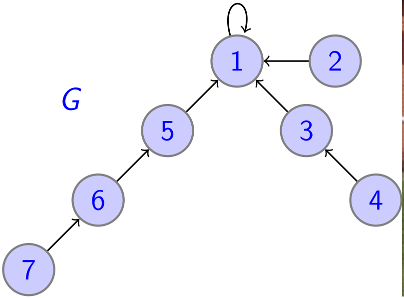
Applications



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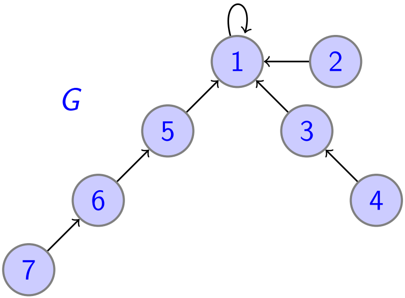

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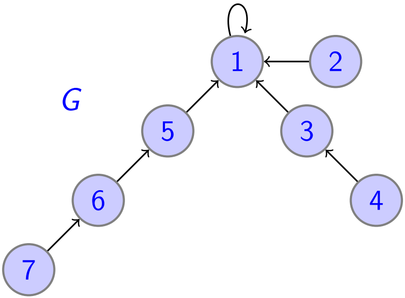

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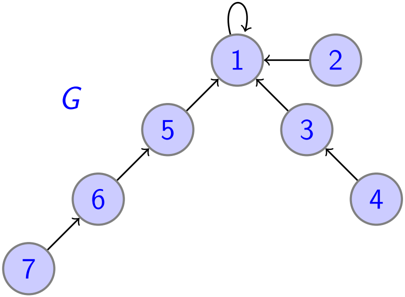

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$$\tau(P) = \frac{1}{2}\max_{i,j}\sum_{l=1}^{n}|p_{il} - p_{jl}|$$

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Theorem (Akelbek, Kirkland)

Let $P = (p_{ij})$ be an $n \times n$ primitive stochastic matrix with $k(P) = k$ and suppose that λ is a non-unit eigenvalue of P. Then $\tau(P^k) < 1$ and $|\lambda| \leq (\tau(P^k))^{1/k}$.

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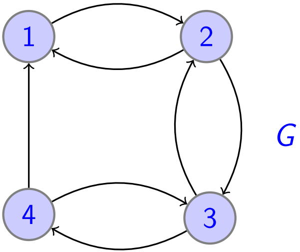
- The system continues in this way.
- For some digraphs after certain time there exists a vertex that knows both bits of the information, independently on the choice of the initial two vertices. When and what digraphs?

How to compute the scrambling index?

Theorem (Lewin)

G is primitive iff G is strongly connected and $k(G) \neq 0$.

What is the value of $k(G)$?



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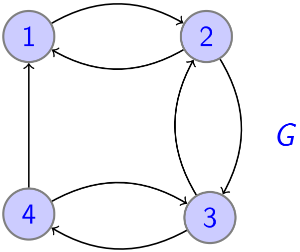

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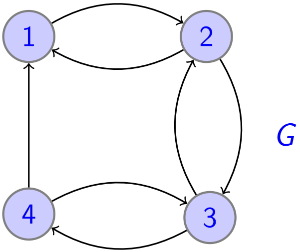

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$\Longrightarrow$ $k(G) = 0$.

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Let G be symmetric, i.e. for any vertices u and v, (u,v) is an edge ***iff*** *(v,u) is an edge, and G be primitive. Then $k(G) = \left\lceil \frac{exp(G)}{2} \right\rceil$.*

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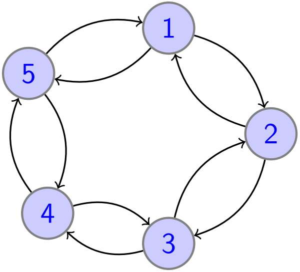

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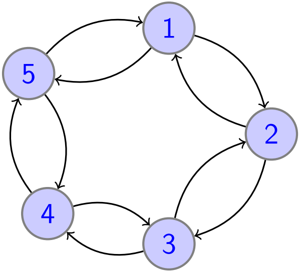

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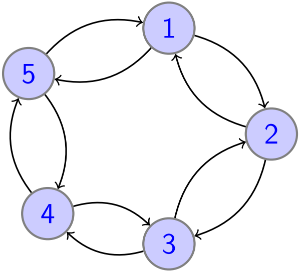

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Scrambling index in terms of the matrix theory

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Matrix $A \in M_n(\mathbf{B})$ *is named scrambling matrix if no two rows of it are orthogonal. Equivalently, if any two rows have at least one non-zero element in coincident position.*

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The scrambling index of a matrix $A \in M_n(\mathbf{B})$ is the smallest positive integer k such that A^k is the scrambling matrix.

The scrambling index of A is denoted by $k(A)$. If such k does not exist, let $k(A) = 0$.

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Let G be a primitive digraph. Then $k(G) = k(A(G))$.

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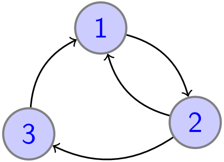

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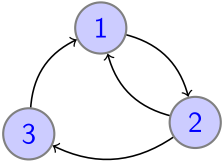

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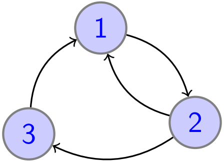

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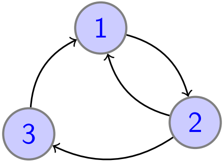

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$u = 2, v = 3$. Then $w = 3$ and the shortest paths are $2 \to 1 \to 2 \to 3$ and $3 \to 1 \to 2 \to 3$.

Some known bounds for the scrambling index

Theorem (Huang, Liu)

Let G de a primitive digraph of order $n \geq 2$ with d loops. Then

$$k(G) \leq n - \left\lceil \frac{d}{2} \right\rceil.$$

Denote

$$K(n,s) = n - s + \begin{cases} \left(\dfrac{s-1}{2}\right) n, & \text{when } s \text{ is odd,} \\ \left(\dfrac{n-1}{2}\right) s, & \text{when } s \text{ is even.} \end{cases}$$

Theorem (Akelbek, Kirkland)

Let G be a primitive digraph with n vertices and girth s. Then $k(G) \leq K(n,s)$.

Some known bounds for the scrambling index

Theorem (Akelbek, Kirkland)

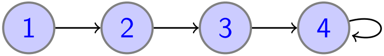
Let G be a primitive digraph of order $n \geq 3$. Then

$$k(G) \leq \left\lceil \frac{(n-1)^2+1}{2} \right\rceil.$$

Equality holds iff $G \cong W_n$.

Actually, we do not need to require primitivity...

G:



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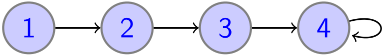

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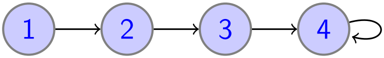

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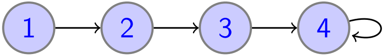

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- G is not strongly connected.
- $k(G) = 3 \neq 0$.

Characterization of digraphs with $k(G) \neq 0$

Theorem (GM, 2019)

For an arbitrary digraph G the following conditions are equivalent:

1. $k(G) \neq 0$.
2. *There exists a primitive subgraph G' of G s.t. $\forall\ v \in V(G)\ \exists\ w \in V(G')$ for which $\exists$ a directed walk from v to w in G.*

$(G_1 \to G_2)$-partition

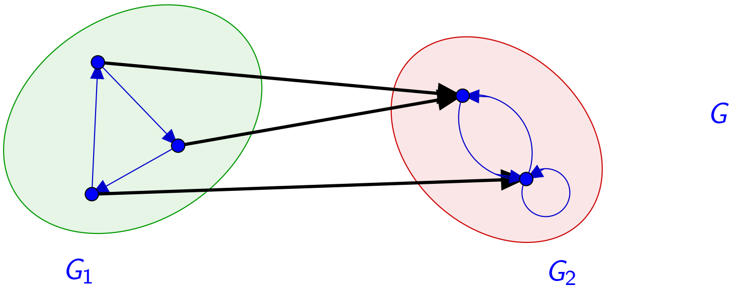
Definition

Let G be a directed graph. G has a $(G_1 \to G_2)$-partition if G_1 and G_2 are non-empty subgraphs of the digraph G such that:

1. $V(G) = V(G_1) \sqcup V(G_2)$;
2. *for each directed edge $e = (v_1, v_2) \in E(G)$, either $e \in E(G_1)$, or $e \in E(G_2)$, or $v_1 \in V(G_1), v_2 \in V(G_2)$.*

Illustration

For a not strongly connected digraph G let us consider a $(G_1 \to G_2)$-partition:



Remark

Geometrically this means that $V(G)$ is partitioned into two non-intersecting components $V(G_1)$ and $V(G_2)$ that are connected only by edges from G_1 to G_2.

New upper bounds

Let G is not strongly connected digraph of order n with $k(G) \neq 0$ and G_1, G_2 be its $(G_1 \to G_2)$-partition.

Theorem (GM, 2019)

Let s be a girth of G_2. Then

$$k(G) \leqslant 1 + K(n-1, s).$$

Here,

$$K(n,s) = n - s + \begin{cases} \left(\dfrac{s-1}{2}\right) n, & \text{when } s \text{ is odd,} \\ \left(\dfrac{n-1}{2}\right) s, & \text{when } s \text{ is even.} \end{cases}$$

New upper bounds

Let G is not strongly connected digraph of order n with $k(G) \neq 0$ and G_1, G_2 be its $(G_1 \to G_2)$-partition.

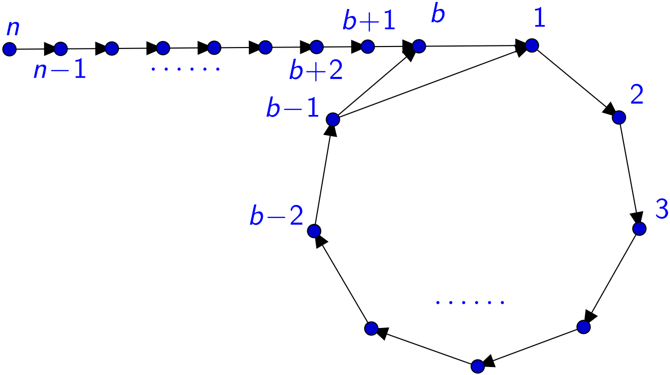
Theorem (GM, 2019)

Assume that $|G_2| = b \leqslant n - 1$. *Then*

$$k(G) \leqslant n - b + \left\lceil \frac{(b-1)^2 + 1}{2} \right\rceil.$$

Sharpness of the upper bound

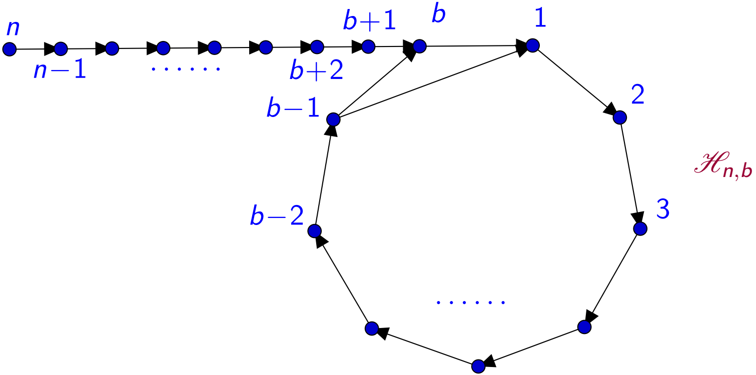
Let $n \geq 3,\ b \leq n-1$. Define a digraph $\mathscr{H}_{n,b}$:



$\mathscr{H}_{n,b}$

Sharpness of the upper bound

Let $n \geq 3,\ b \leq n-1$. Define a digraph $\mathscr{H}_{n,b}$:



If $b > 1$, then $k(\mathscr{H}_{n,b}) = n - b + \left\lceil \frac{(b-1)^2+1}{2} \right\rceil$.

New upper bounds

Let G is not strongly connected digraph of order n with $k(G) \neq 0$ and G_1, G_2 be its $(G_1 \to G_2)$-partition.

Theorem (GM, 2019)

Assume that $|G_2| = b \leqslant n-1$. *Then*

$$k(G) \leqslant n - b + \left\lceil \frac{(b-1)^2+1}{2} \right\rceil.$$

If $4 \leqslant n < 2b$, *then equality holds if and only if* $G \cong \mathscr{H}_{n,b}$.

Corollaries

Theorem (GM, 2019)

Let G be an arbitrary digraph of order $n \geqslant 3$. Then

$$k(G) \leqslant \left\lceil \frac{(n-1)^2+1}{2} \right\rceil .$$

The equality holds if and only if $G \cong W_n$.

Corollaries

Theorem (GM, 2019)

Let G be an arbitrary digraph of order $n \geqslant 3$. Then

$$k(G) \leqslant \left\lceil \frac{(n-1)^2+1}{2} \right\rceil .$$

The equality holds if and only if $G \cong W_n$.

Theorem (GM, 2019)

Let G be a not strongly connected digraph of order $n \geqslant 3$. Then

$$k(G) \leqslant 1 + \left\lceil \frac{(n-2)^2+1}{2} \right\rceil .$$

When $n \geqslant 4$, the equality holds if and only if $G \cong \mathscr{H}_{n,n-1}$.

Chain rank

Definition

Recall that rows i and j of the matrix A are intersecting if they have positive elements in a certain common column.

The scrambling matrix is such that all its rows are intersecting.

Definition

We say that indices i and j are in solidarity relation in the matrix A (A-solidarity relation), if there exists a sequence of indices $i = i_1, i_2, \ldots, i_s = j$ such that rows with the indices i_k, i_{k+1} are intersecting for $k = 1, \ldots, s-1$.

Definition

The matrix is chainable if all its rows are in the same solidarity class.

A-solidarity relation is indeed an equivalence relation on $\mathbf{n}$. The number of equivalence classes by this relation is called chain rank of A and is denoted by $\mathrm{crk}\,(A)$.

Chainable matrices

Definition

A is called a chainable matrix, if one of the following equivalent conditions is satisfied:
1. $\mathrm{crk}\,(A) = 1$.

2. $A = (a_{ik})$ is a chainable matrix iff $\forall$ couple of its positive entries a_{ik}, a_{pq} $\exists$ a sequence of positive entries $a_{i_1 k_1}, a_{i_2 k_2}, \ldots, a_{i_n k_n}$ satisfying following conditions:
a) $i_1 = i, k_1 = k$,
b) $i_n = p, k_n = q$,
c) $\forall$ $l \in \{1, 2, ..., n-1\}$ it is true that $i_l = i_{l+1}$ or $k_l = k_{l+1}$.

Consider every entry as a square of a chessboard, where the rook is allowed to stay only on positive entries. Matrix is chainable if the rook can reach any positive entry from any other positive entry.

Theorem

A is a scrambling matrix $\Longrightarrow$ A is a chainable matrix.

Reminder: A is a scrambling matrix, iff $\forall\ i, p\ \exists\ q$: $a_{iq} \neq 0 \& a_{pq} \neq 0$.

But the converse does not hold:

Examples

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$ is chainable, since

row 1 intersect row 3, row 3 intersect row 4, row 4 intersect row 2, but is not a scrambling matrix, since row 1 does not intersect row 4.

Properties of the chain rank

$\mathbb{P}$ is the set of non-negative matrices without zero rows&columns

Theorem (Al'pin, Bashkin, 2020)

For any $A \in \mathbb{P}$ *it holds that* $1 \leq \operatorname{crk}(A) \leq n$ *and*

$$\operatorname{crk}(A^t) = \operatorname{crk}(A)$$

If $A, B \in \mathbb{P}$ *and the product* AB *exists then*

$$\operatorname{crk}(AB) \leq \min\{\operatorname{crk}(A), \operatorname{crk}(B)\},$$

$$\operatorname{crk}(AA^t) = \operatorname{crk}(A) = \operatorname{crk}(A^t A)$$

Theorem (Guterman, Shafeev, 2024)

$$\operatorname{crk}(ABC) \leq \operatorname{crk}(AB) + \operatorname{crk}(BC) - \operatorname{crk}(B)$$

Properties of the chain rank

Corollary

$\forall$ $A, B \in \mathbb{P}$ *such that the product* AB *exists*

$$\operatorname{crk}(A) + \operatorname{crk}(B) - n \leq \operatorname{crk}(AB) \leq \min\{\operatorname{crk}(A), \operatorname{crk}(B)\}$$

Corollary

$\forall$ *square* $A \in \mathbb{P}$

$$\operatorname{crk}(A^k) - \operatorname{crk}(A^{k+1}) \leq \operatorname{crk}(A^{k-1}) - \operatorname{crk}(A^k)$$

Potentially chainable and primitive matrices

Definition

If a certain power of a matrix A is a chainable matrix then A is called potentially chainable matrix.

The notion of potentially chainable matrix is an analog of the notion of a primitive matrix. There a certain degree is a positive matrix, and here it is a chainable matrix.

If A is primitive $\overset{\Longrightarrow}{\not\Longleftarrow}$ A is potentially chainable.

Theorem

Let A be indecomposable. Then A is primitive iff A is potentially chainable.

However, there are potentially chainable decomposable matrices.

Example

$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $M_2^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. M_2 is potentially chainable, since M_2^2 is chainable.

Perron – Frobenius Theory

Theorem

- *Any primitive matrix* $A \in M_n$ *has a positive eigenvalue* ρ *which is a simple root of* $\chi_A(x)$.
- $\forall\ \lambda,\ \lambda \neq \rho$ *being eigenvalue of* $A \Rightarrow |\lambda| < |\rho|$.
- *Maximal eigenvalue corresponds to the eigenvector* z *with all positive coordinates.*

ρ *is called Perron–Frobenius eigenvalue of* A.

If A is not primitive but indecomposable, the following generalization is true:

Frobenius Normal Form

Theorem (Frobenius)

Let $A \in M_n$ be indecomposable. Then either A is primitive or by the permutation similarity A can be reduced to the block form

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{h-1,h} \\ A_{h1} & 0 & 0 & \dots & 0 \end{pmatrix}$$

where all the blocks are primitive matrices.
In this case Perron–Frobenius eigenvalues of all blocks are different but have the same absolute value.

Definition

h is the imprimitivity index of A.

Further generalizations

[Protasov and Voynov, 2012]: matrix semigroups.

Definition 1

Matrix semigroup $\mathcal{S}$ is called indecomposable if for any indices i and j there exists $A \in \mathcal{S}$ such that $a_{ij} > 0$.

To compare: A is indecomposable, if $\forall$ i, j, $i \neq j$, $\exists$ k: (i, j)-th element of A^k is positive.

If $\exists$ indecomposable $A \in S$, then S is an indecomposable semigroup.

$\exists$ indecomposable semigroups without indecomposable matrices!

Example

Let $\mathcal{S}_\infty$, be a semigroup generated by

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$A_1 + A_2A_1 + A_3A_1 > 0 \Rightarrow S_1$ indecomposable.

Definition 2

Matrix semigroup $\mathcal{S}$ is called primitive if $\exists$ a positive $A \in \mathcal{S}$.

To compare: A is primitive, if $\exists$ k: A^k is positive.

In particular, A is a primitive matrix iff $\langle A \rangle$ is a primitive semigroup.

Imprimitivity index

Let $v \in \mathbb{R}^n, v \geq 0$. $\operatorname{supp}(v)$ is the set of positive coordinates of v.

Definition

$\mathcal{S} \subseteq M_n$ is a semigroup. Imprimitivity index of a semigroup $\gamma(\mathcal{S})$ is the biggest $\gamma \in \mathbb{N}$, s.t. $\exists\ e_{i_1}, \ldots, e_{i_\gamma} \in \mathbb{R}^n$: $\forall\ A \in \mathcal{S}$ $\operatorname{supp}(Ae_{i_1}), \ldots, \operatorname{supp}(Ae_{i_\gamma})$ are pairwise non-intersecting.

Lemma. If $A \in M_n$ is indecomposable matrix, then $h(A) = \gamma(\langle A \rangle)$.

Example (Illustrating example)

$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in M_n$ is indecomp., $h(B) = 2 = h(B^2) = \gamma(\langle B \rangle)$, indeed, $B^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Example (For decomposable matrices the equality does not hold.)

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then $\operatorname{supp}(Ae_1) = \{1, 2\}$, $\operatorname{supp}(Ae_2) = \{3\} = \operatorname{supp}(Ae_3)$. Hence $h(A) = 2$. But $h(A^2) = 1$, since A is decomp.

Protasov–Voynov Theorem, 2012

Definition

$\alpha = (\alpha_1, \ldots, \alpha_t)$ is a certain partition of the set $\mathbf{n}$. Matrix $A \in \mathbb{P}_n$ is acting on α as a permutation if, for any set α_i $\exists$ a unique set α_j, such that $\alpha_i A = \alpha_j$, i.e., all vectors of the standard basis with the numbers from α_i are mapped by A to the linear combinations of the vectors with the indices from α_j.

Theorem

Let $\mathcal{S} \subseteq M_n$ be a matrix semigroup satisfying:
1. matrices in S are without zero rows or columns,
2. S is irreducible semigroup.
Then TFAE:

1. *$\mathcal{S}$ does not contain a positive matrix,*
2. *imprimitivity index $\gamma(\mathcal{S}) > 1$.*
3. *There exists a partition $\mathbf{n} = \bigsqcup_{k=1}^{m} \alpha_k$, $m \geq 2$, on which all matrices from $\mathcal{S}$ act like permutations.*

Protasov–Voynov Theorem, 2012

Corollary (1)

If one of the conditions of the Theorem hold then:

1. $\exists$ *a partition of the set* $\mathbf{n}$ *onto* $m = \gamma(\mathcal{S})$ *sets, on which matrices from* $\mathcal{S}$ *act like permutations.*
2. *All matrices from* $\mathcal{S}$ *can be reduced by one permutation similarity to the block-monomial form with* $\gamma(\mathcal{S})$ *blocks.*
3. *The semigroup* $\mathcal{S}$ *contains a matrix with strictly positive blocks.*

Corollary (2)

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ *be a family of matrices of size* n*. Then there exists an algorithm which requires* $O(2mn^3)$ *operations and determines if there exists a product of matrices in* $\mathcal{A}$*, which is positive.*

To do this it is necessary to determine the partition α from Protasov–Voynov Theorem. Then a positive product does exist iff $m = \gamma(\mathcal{S}) = 1$.

Definition

Indices $i, j \in \mathbf{n}$ are intersecting in the semigroup $\mathcal{S} \subseteq \mathbb{P}_n$, if $\exists$ $A \in \mathcal{S}$: rows i, j are intersecting in A.

Definition

Indices i, j are in solidarity relation in the semigroup $\mathcal{S} \subseteq \mathbb{P}_n$, if there exists a sequence of indices $i = i_1, i_2, \ldots, i_s = j$, such that neighbor indices (i_k, i_{k+1}) are intersecting in $\mathcal{S}$, i.e., $\forall$ $k = 1, \ldots, s-1$ $\exists$ $A_k \in \mathcal{S}$ such that (i_k, i_{k+1}) are intersecting in A_k.

Reminder: indices i and j are in solidarity relation in the matrix A, if $\exists$ a sequence of indices $i = i_1, i_2, \ldots, i_s = j$ such that rows with the indices (i_k, i_{k+1}) are intersecting for $k = 1, \ldots, s-1$.

Lemma

Solidarity relation in $\mathcal{S}$ is an equivalence relation on $\boldsymbol{n}$.

Lemma

Let $A \in \mathbb{P}_n$. Then $i, j \in \boldsymbol{n}$ are in solidarity relation in $\langle A \rangle$ iff i, j are in solidarity relation in A^{n-1}.

Chainable properties of semigroups

Definition

Insolidarity index of $\mathcal{S} \subseteq \mathbb{P}_n$ is the number of $\mathcal{S}$-solidarity classes, denote $i(\mathcal{S})$.

Theorem (Al'pin, Guterman, Shafeev, 2024)

Let $\mathcal{S} \subseteq \mathbb{P}_n$ is a semigroup, $i(\mathcal{S}) = r$. Then
1. If $r = 1$, then $\exists$ a potentially chainable matrix in $\mathcal{S}$.
2. If $r \geq 2$, then $\forall\ A \in \mathcal{S}$ acts on solidarity classes as permutation. I.e., $\exists$ a permutation P such that $\forall\ A \in \mathcal{S}$ the matrix PAP^t is block-monomial with r blocks and $\exists\ X \in \mathcal{S}$ such that all non-zero blocks of PXP^T are potentially chainable.

Generalization of Frobenius Theorem

Corollary (Al'pin, Guterman, Shafeev, 2024)

Let $A \in \mathbb{P}_n$.
1. If $i(A) = 1$, then A is potentially chainable, i.e., A^k are chainable $\forall\ k \geq n-1$.
2. If $r = i(A) \geq 2$, then $\exists$ a permutation P such that PAP^t is block-monomial with r blocks and $\forall\ k \geq n-1$ all non-zero blocks of A^k are chainable.
3. If $\mathcal{S} \subseteq \mathbb{P}_n$ is an indecomposable semigroup, then the partition to solidarity classes coincide with the partition to equivalence classes under intersection relation.

Remarks

- In Theorem we show, what is possible to save from Protasov-Voynov theorem if we change the condition of indecomposability to a not so strong condition of the absence of zero rows and columns.
- Corollary is a generalization of Frobenius theorem. But absence of indecomposability leads to change of primitive blocks with potentially chainable blocks.

Maps preserving scrambling index

Definition

- *We say that T is a map preserving the scrambling index, if for all $A \in M_n(\mathbf{B})$ we have that $k(T(A)) = k(A)$.*
- *We say that T is a map preserving the non-zero scrambling index, if for all $A \in M_n(\mathbf{B})$, for which $k(A) \neq 0$, we have that $k(T(A)) = k(A)$.*
- *We say that T is a map preserving the scrambling index on the set of primitive matrices if $\forall$ primitive $A \in M_n(\mathbf{B})$ we have that $k(T(A)) = k(A)$.*

Theorem (Frobenius, 1896)

$T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ — *linear, bijective,*

$$\det(T(A)) = \det A \qquad \forall A \in M_n(\mathbb{C})$$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{C}), \det(PQ) = 1 :$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{C})$$

or

$$T(A) = PA^t Q \quad \forall A \in M_n(\mathbb{C})$$

Definition

$T : M_{m\,n}(\mathbb{F}) \to M_{m\,n}(\mathbb{F})$ is standard iff
$\exists P \in GL_m(\mathbb{F}),\ Q \in GL_n(\mathbb{F})$:

$$T(A) = PAQ \quad \forall A \in M_{m,n}(\mathbb{F})$$

or $m = n$ and

$$T(A) = PA^tQ \quad \forall A \in M_{m,n}(\mathbb{F})$$

Let $X \in M_{m,n}(\mathbb{C})$. Then $C_r(X) \in M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C})$ consists from r-minors of X ordered lexicographically by rows and columns.

Theorem

[Schur, 1925] Let $T : M_{m\,n}\mathbb{C}) \to M_{m\,n}(\mathbb{C})$ be bijective and linear, r, $2 \leq r \leq \min\{m, n\}$, be given. $\exists$ bijective linear $S : M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C}) \to M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C})$ s.t.

$$C_r(T(X)) = S(C_r(X)) \ \ \forall \ \in M_{m,n}(\mathbb{C})$$

iff T is standard.

Theorem (Dieudonné, 1949)

$\Omega_n(\mathbb{F})$ *is the set of singular matrices*
$T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ — *linear, bijective,* $T(\Omega_n(\mathbb{F})) \subseteq \Omega_n(\mathbb{F})$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{F})$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{F})$$

or

$$T(A) = PA^t Q \quad \forall A \in M_n(\mathbb{F})$$

E.B. Dynkin, Maximal subgroups of classical groups // The Proceedings of the Moscow Mathematical Society, **1** (1952) 39-166.

$$St_n(\mathbb{F}) \subseteq Fix(S) \subseteq GL_{n^2}(\mathbb{F})$$

The quantity of Linear Preservers for a given matrix invariant is a measure of its complexity. Indeed, to compute the invariant for a given matrix, we reduce it to a certain good form, where computations are easy.

$$\det(A) = \sum_{\sigma \in S_n} (-1)^n a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

• Computations of det require $\sim O(n^3)$ operations

$$\mathrm{per}\,(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

• Computations of per require
$\sim (n-1) \cdot (2^n - 1)$ multiplicative operations (Raiser formula).

The explanation

There are just few linear preservers of permanent in comparison with the determinant. Indeed,

Theorem (Marcus, May)

Linear transformation T is permanent preserver iff
$T(A) = P_1 D_1 A D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$, *or*
$T(A) = P_1 D_1 A^t D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$
where D_i are invertible diagonal matrices, $i = 1, 2, \det(D_1 D_2) = 1$
P_i are permutation matrices, $i = 1, 2$

- Group theory

Question Is it possible that two non-isomorphic finite groups have the same group determinant?

Theorem (E. Formanek, D. Sibley)

A group determinant determines the group up to an automorphism

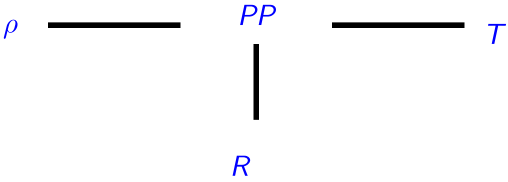
Proof is based on an extension of Dieudonne singularity preserver theorem to the direct products of matrix algebras.

Preserve Problems

$\rho : M_n(R) \to S$ is a certain matrix invariant
$T : M_n(R) \to M_n(R)$

$$\rho(T(A)) = \rho(A) \quad \forall A \in M_n(R)$$

$$T = ?$$



Let $\mathbb{F}$ be a field

$\emptyset \neq S \subseteq M_n(\mathbb{F})$	$T(S) \subseteq S$
$\rho : M_n(\mathbb{F}) \to \mathbb{F}$ $\forall A \in M_n(\mathbb{F})$	$\rho(T(A)) = \rho(A)$
$\sim : M_n(\mathbb{F})^2 \to \{0,1\}$	$A \sim B \Rightarrow T(A) \sim T(B)$ $\forall A, B \in M_n(\mathbb{F})$
P – property in $M_n(\mathbb{F})$	$A \in P \Rightarrow T(A) \in P$

$T = ?$

The standard solution in linear case

There are $P, Q \in GL_n(\mathbb{F})$:

$$T(X) = PXQ \quad \forall X \in M_n(\mathbb{F})$$

or

$$T(X) = PXQ \quad \forall X^t \in M_n(\mathbb{F})$$

Basic methods to investigate PPs

1. Matrix theory
2. Theory of classical groups
3. Projective geometry
4. Algebraic geometry
5. Differential geometry
6. Dualisations
7. Tensor calculus
8. Functional identities
9. Model theory

Maps preserving scrambling index

Theorem

Let $n \geq 3$ *and* $T: M_n(\mathbf{B}) \to M_n(\mathbf{B})$ *be an arbitrary mapping. Then* T *is a bijective additive operator which preserves non-zero scrambling index*

$$\Updownarrow$$

$\exists$ *permutation matrix* P *such that* $T(A) = P^T A P$, $\forall A \in M_n(\mathbf{B})$.

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Let $n \geq 3$ and $T: M_n(\mathbf{B}) \to M_n(\mathbf{B})$ be an arbitrary mapping. Then T is a bijective additive operator which preserves non-zero scrambling index

$$\Updownarrow$$

$\exists$ permutation matrix P such that $T(A) = P^T A P$, $\forall A \in M_n(\mathbf{B})$.

For $A \in M_n(\mathbf{B})$ let us use the notation:

$$A_{id} = \sum_{k:\ A(k,k)=1} E_{kk}; \quad A_{od} = \sum_{i \neq j:\ A(i,j)=1} E_{ij}.$$

Maps preserving distinct values of the scrambling index

Theorem

Let $n \geqslant 3$ *and* $T : M_n(\mathbf{B}) \to M_n(\mathbf{B})$ *be an additive bijective map.*

- T *preserves* $k = 1$ *iff* $\exists$ *permutation matrices* P, Q *s.t.*

$$T(A) \ = \ PA\,Q.$$

- T *preserves* $k = 0$ *iff* $\exists$ *a permutation matrix* P, *s.t.*

$$T(A) \ = \ P^T A\,P.$$

- T *preserves* $k = \max$ *iff* $\exists$ *permutation matrices* P, Q *s.t.*

$$T(A) = P^T A_{od}\,P + Q^T A_{id}\,Q \quad \textit{for all } A \in M_n(\mathbf{B})$$

$$T(A) = P^T A_{od}^T\,P + Q^T A_{id}\,Q \quad \textit{for all } A \in M_n(\mathbf{B})$$

Maps preserving scrambling index

Theorem

Let $n \geqslant 3$ and $T\colon M_n(\mathbf{B}) \to M_n(\mathbf{B})$ be the additive map preserves the scrambling index. Then T is a bijection.

Steps of the proof

1. Let $A, B \in M_n$. If A is primitive, then $A + B$ is primitive.
2. Let $A, B \in M_n$. If $k(A) \neq 0$, then $k(A+B) \neq 0$ and $k(A+B) \leqslant k(A)$.
3. Some notations: $C_n = E_{n,1} + \sum_{i=1}^{n-1} E_{i,i+1}$ is the adjacency matrix of the elementary cycle $(12\dots n)$. Then $W_n = C_n + E_{n-1,1}$ is the Wielandt matrix.

$\mathcal{W} = \{A \in M_n(\mathbf{B}) \mid \exists\ P \in \mathcal{P}_n \colon P^T A P = W_n\}$ – Wielandt like
$\mathcal{C} = \{A \in M_n(\mathbf{B}) \mid \exists\ P \in \mathcal{P}_n \colon P^T A P = C_n\}$ – cycles
$\mathcal{E} = \{E_{ij} \in M_n(\mathbf{B}) \mid 1 \leqslant i, j \leqslant n\}$ – cells
$\mathcal{D} = \{E_{ii} \in M_n(\mathbf{B}) \mid 1 \leqslant i \leqslant n\}$ – diagonal cells
$\mathcal{N} = \mathcal{E} \setminus \mathcal{D} = \{E_{ij} \in \mathcal{E} \mid i \neq j\}$ – off-diagonal cells

4. By 2. $A \in \mathcal{W} \Rightarrow T(A) \in \mathcal{W}$.

Steps of the proof

5. T is bijective on $\mathcal{W}$.
6. Let $n \geqslant 4$, $E_{ij} \in \mathcal{N}$. Then there exist two distinct matrices $W_1, W_2 \in \mathcal{W}$ such that $W_1 \circ W_2 = E_{ij}$, i.e. W_1 and W_2 have a unique non-zero entry in the position (i, j).
7. For any pair $E_{ij}, E_{kl} \in \mathcal{N}$, $E_{ij} \neq E_{kl}$, there exists a matrix $W \in \mathcal{W}$ such that $W \geqslant E_{ij}$, $W \ngeqslant E_{kl}$. 8. Let $A \in M_n$. Then $T(A) = O$ iff $A = 0$. 9. $T(\mathcal{N}) \subseteq \mathcal{N}$, and moreover, $T(\mathcal{N}) = \mathcal{N}$.
10. For any digraph G the edge number $|E(G)| = |E(G(T(A(G))))|$.
11. G does not have loops iff $G(T(A(G)))$ does not have loops.
12. $T(\mathcal{C}) = \mathcal{C}$
13. $T(\mathcal{D}) \subseteq \mathcal{D}$, and moreover, $T(\mathcal{D}) = \mathcal{D}$.

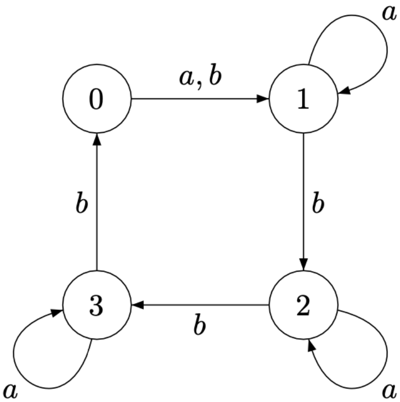
Hence T is bijective!

Application to minimal synchronizing automaton

Application to minimal synchronizing automaton

Definition

A word w is called a synchronizing (reset) word of a deterministic finite automaton DFA if w brings all states of the automaton to some specific state.



abbbabbba

Conjecture (Černý, 1964)

The shortest synchronizing word for any n-state complete DFA has length $\leq (n-1)^2$.

Theorem (Černý, 1964)

There are DFAs with minimal synchronizing words of length exactly

$$(n-1)^2.$$

Theorem

All known bounds are of order n^3.

Graphs of large exponent and/or scrambling index lead to examples of slowly synchronizing automata.

Thank you!