Curvature controls beading in soft elastic cylinders

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Matsuo and Tanaka (1) revealed how a thin, cylindrical hydrogel can accommodate a variety of patterns in response to external stimuli. One of these patterns, corresponding to a beaded or undulating axisymmetric morphology, has caught the attention of researchers due to its similarity to the Rayleigh-Plateau instability in thin jets of fluid. Current modelling of this instability has focused on the competition between bulk elasticity and surface tension. However, this competition only admits the emergence of an infinite-wavelength instability, which is inconsistent with experimental observations.

Similar beaded patterns are seen in biological contexts (2) and in artificial materials for electronics (3). This latter example reveals how coated cylindrical systems can exhibit hierarchical beading patterns, thus justifying the need for a better understanding of the underlying phenomena so they can be exploited in functional materials.

In this work we consider a soft elastic cylinder coated by a thin membrane. The membrane responds to stretching, bending, and a mismatch against an imposed natural curvature. We minimise the potential energy to obtain the bulk equilibrium equations and the boundary conditions generated by the superficial contributions. A key parameter that appears in the model is the bendoelasto numbr, which controls the ratio of the bending energy of the coating to the elastic energy of the cylindrical core. A linear stability analysis reveals that for sufficiently large bendoelastic numbers, the beaded pattern can emerge with a finite wavelength. For small but non-zero bendoelastic numbers, this system admits infinite-wavelength instabilities that are analogous to those seen when only elastocapillarity is considered. We refer to these latter instabilities as curvature-dependent capillary-like instabilities. Our study reveals that it is possible to bridge the finite-wavemode instability seen in the shape equation, which is obtained by minimising the Helfrich functional, and the zero-wavemode elastocapillary instability. We also carry out a weakly nonlinear analysis to investigate how the localized envelope of the beaded pattern depends on the properties of the coating.

To obtain insights in the behaviour of the fully non-linear regime, we study the asymptotic limit of a slender cylinder. The simpified model reveals how two key mechanisms, the finite-wave mode instability and phase separation, interact to produce patterns with two length scales.

(1) Matsuo, E., Tanaka, T. Patterns in shrinking gels. Nature 358, 482–485 (1992).
(2) Ochs, S. e t al. The origin and nature of beading: A reversible transformation of the shape of nerve fibers. Progress in Neurobiology 95, 391–426 (1997).
(3) Liu, Z.F. et al. Hierarchically buckled sheath-core fibers for superelastic electronics, sensors, and muscles. Science 349, 400-404 (2015).