Positive Maps and Entanglement in Real Hilbert Spaces

Vern Paulsen
joint with G. Chiribella, K. Davidson, and M. Rahaman

ICMS

May 22, 2023

Motivations

What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function.
Letter from Schrodinger to Lorentz. June 6th, 1926.

Even though quantum theory is based on complex Hilbert spaces and they play a key "tidying" role in the theory, physicists have wondered if the quantum world is inherently complex.
In *Quantum theory based on real numbers can be experimentally falsified*(Renou, Navascues et al), Dec. 27, 2021, a Bell-like experiment based on a network scenario is proposed that numerically detects if underlying state spaces are real or complex. Trials of this experiment have now shown they are complex.

This motivated our work which looks for other similarities/differences between the real and complex case for various concepts like, entanglement and separability, positive maps, p-positive maps, the various characterizations of entanglement breaking maps and the PPT^2 conjecture.
In the real case many authors study these concepts for maps defined on the space of real symmetric matrices, but we prefer to look at maps on the full space of real matrices. Why?
Still have the Choi-Jamiolkowski isomorphism:

$$\Phi \in \mathcal{L}(M_d(\mathbb{R}), M_r(\mathbb{R})) \leftrightarrow C_\Phi = \sum_{i,j=1}^{d} E_{i,j} \otimes \Phi(E_{i,j}) \in M_d(\mathbb{R}) \otimes M_r(\mathbb{R}).$$

Also the canonical way to extend maps from the symmetric matrices to all matrices is to project onto the symmetric,

$$X \to 1/2(X + X^t),$$

which is not CP.

Separable and Entangled

Throughout $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. We set

$$SEP(\mathbb{K}^d, \mathbb{K}^r) := \{\sum_l P_l \otimes Q_l : P_l \in PSD_d(\mathbb{K}), Q_l \in PSD_r(\mathbb{K})\}$$
$$\subseteq M_d(\mathbb{K}) \otimes M_r(\mathbb{K}).$$

Any matrix in $\big(M_d(\mathbb{K}) \otimes M_r(\mathbb{K})\big)^+ \setminus SEP(\mathbb{K}^d, \mathbb{K}^r)$ is called $\mathbb{K}$-*entangled*.

If a real matrix is separable as a complex matrix, is it necessarily also separable as a real matrix? That is, is

$$SEP(\mathbb{R}^d, \mathbb{R}^r) \stackrel{?}{=} SEP(\mathbb{C}^d, \mathbb{C}^r) \cap \big(M_d(\mathbb{R}) \otimes M_r(\mathbb{R})\big) := CSEP(\mathbb{R}^d, \mathbb{R}^r).$$

NO!

$SEP(\mathbb{R}^k, \mathbb{R}^n) \subsetneq CSEP(\mathbb{R}^k, \mathbb{R}^n)$

Let $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ then

$$X = I_2 \otimes I_2 + A \otimes A = \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right)$$

$$= 1/2(I_2 + A) \otimes (I_2 + A) + 1/2(I_2 - A) \otimes (I_2 - A),$$

is $\mathbb{C}$-separable.
But note that if $Y = \sum_l P_l \otimes Q_l$ is $\mathbb{R}$-separable, then $id \otimes T(Y) = \sum_l P_l \otimes T(Q_l) = Y$. Since $id \otimes T(X) \neq X$, X is not $\mathbb{R}$-separable.

More generally we let

$$SEP_p(\mathbb{K}^d, \mathbb{K}^r) = \{\sum_j |u_j >< u_j| \ : u_j \in \mathbb{K}^d \otimes \mathbb{K}^r, \text{ rank at most p } \},$$

and

$$CSEP_p(\mathbb{R}^d, \mathbb{R}^r) = SEP_p(\mathbb{C}^d, \mathbb{C}^r) \cap \big(M_d(\mathbb{R}) \otimes M_r(\mathbb{R})\big).$$

Stormer's Theorem and Its Generalizations

Theorem: Let $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ be linear with $\Phi(X^*) = \Phi(X)^*$. Then Φ is p-positive if and only if

$$Tr(C_\Phi X) \geq 0, \forall X \in SEP_p(\mathbb{K}^d, \mathbb{K}^r).$$

In particular, if X is $\mathbb{K}$-entangled, then there is a positive map such that $Tr(C_\Phi X) < 0$.

Complexifications of Real Linear Maps

Note that every $X \in M_n(\mathbb{C})$ can be written uniquely as $X = A + iB$ with $A, B \in M_n(\mathbb{R})$. Given a real linear map $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ its *complexification* is the complex linear map $\tilde{\Phi} : M_d(\mathbb{C}) \to M_r(\mathbb{C})$ given by

$$\tilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B)$$

and $C_\Phi = C_{\tilde{\Phi}}$

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be linear. Then $\tilde{\Phi}$ is p-positive if and only if

$$Tr(C_\Phi X) \geq 0, \forall X \in CSEP_p(\mathbb{R}^d, \mathbb{R}^r).$$

In particular, if $X \in CSEP_p(\mathbb{R}^d, \mathbb{R}^r) \backslash SEP_p(\mathbb{R}^d, \mathbb{R}^r)$, then there is a p-positive map Φ such that $\tilde{\Phi}$ is not p-positive and $Tr(C_\Phi X) < 0$.

Entanglement Breaking Maps

Definition: A CP map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is called $\mathbb{K}$-**entanglement p-breaking** if for every n and every $P = (P_{i,j}) \in M_n(M_d(\mathbb{K}))^+ := \big(M_n(\mathbb{K}) \otimes M_d(\mathbb{K})\big)^+$ we have that

$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in SEP_p(\mathbb{K}^n, \mathbb{K}^r).$$

Horodecki-Schor-Ruskai and Its Generalizations

Theorem: Let $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ be a linear map. Then the following are equivalent.

(i) Φ is $\mathbb{K}$-entanglement p-breaking,

(ii) $C_\Phi \in SEP_p(\mathbb{K}^d, \mathbb{K}^r)$,

(iii) there exist matrices $(A_i)_{i=1}^k \subset M_{r,d}(\mathbb{K})$ such that $rank(A_i) \leq p$ for every i and $\Phi(X) = \sum_{i=1}^k A_i X A_i^*$,

(iv) for every m and every p-positive map $\Psi : M_r(\mathbb{K}) \to M_m(\mathbb{K})$ the map $\Psi \circ \Phi$ is completely positive,

(v) for every n and every p-positive map $\Psi : M_n(\mathbb{K}) \to M_d(\mathbb{K})$ the map $\Phi \circ \Psi$ is completely positive,

(vi) $\Phi = \Delta \circ \Gamma$ where $\Gamma : M_d(\mathbb{K}) \to l_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K})$ and $\Delta : l_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K}) \to M_r(\mathbb{K})$ are completely positive maps for some $k \geq 1$.

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be a CP map. Then the following are equivalent:

1. for every n and every $P = (P_{i,j}) \in \big(M_n(\mathbb{R}) \otimes M_d(\mathbb{R})\big)^+$ we have that

$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in CSEP_p(\mathbb{R}^n, \mathbb{R}^r).$$

2. $\tilde{\Phi}$ is $\mathbb{C}$-entanglement p-breaking,
3. $C_\Phi \in CSEP_p(\mathbb{R}^d, \mathbb{R}^r)$,
4. there exist matrices of rank at most p, $A_i \in M_{r,d}(\mathbb{C})$ such that $\Phi(X) = \sum_i A_i X A_i^*$,
5. there exists a finite dimensional real C*-algebra $\mathcal{C} \subseteq \ell^\infty_k(\mathbb{C}) \otimes M_p(\mathbb{C})$ and CP maps $\Delta : M_d(\mathbb{R}) \to \mathcal{C}$, $\Psi : \mathcal{C} \to M_r(\mathbb{R})$ such that $\Phi = \Psi \circ \Delta$.

Here by "real C*-algebra", we mean a real vector subspace, closed under product and adjoint.

The PPT^2 Conjecture

A map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is called **PPT**(positive partial transpose) if Φ and $T \circ \Phi$ are both CP, where T denotes the transpose map.

PPT^2 **Conjecture:** If $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is PPT, then $\Phi \circ \Phi$ is EB.

Here are a few things that we know:

- It is true for $d = 2, 3$ and for many families of maps.
- It is equivalent to the conjecture that if $\Phi_i : M_{d_i}(\mathbb{C}) \to M_{d_{i+1}}(\mathbb{C})$ are PPT, then $\Phi_2 \circ \Phi_1$ is EB.
- (Kennedy-Manor-P) If Φ is PPT, unital and idempotent, then its range is an abelian C*-algebra in the Choi-Effros product, $\Phi(X) \star \Phi(Y) = \Phi(\Phi(X)\Phi(Y))$, and hence $\Phi \circ \Phi = \Phi$ is EB.
- (Kennedy-Manor-P) If Φ is PPT and either unital or trace-preserving, then $\lim_n d(\Phi^n, EB) = 0$.
- (Jaques-P-Rahaman) If Φ is PPT and both unital and trace-preserving, then there exists k such that Φ^k is EB.

The direct real analogue of this conjecture is false.
Define $\Phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ with

$$C_\Phi = \frac{1}{2}\begin{bmatrix} I_2 & \gamma \\ -\gamma & I_2 \end{bmatrix} \quad \text{where} \quad \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then Φ is PPT, unital, trace preserving and idempotent, $\Phi \circ \Phi = \Phi$, but $\Phi \circ \Phi$ is not $\mathbb{R}$-entanglement breaking.
However, $\tilde{\Phi} \circ \tilde{\Phi}$ is $\mathbb{C}$-entanglement breaking.

The ITP^2 Conjecture

Seeking a reasonable real version.
A map $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ is **ITP** if it is CP and $T \circ \Phi = \Phi$.

ITP^2 **Conjecture:** If $\Phi : M_d(\mathbb{R}) \to M_d(\mathbb{R})$ is ITP, then $\Phi \circ \Phi$ is $\mathbb{R}$-EB.
Things we know:

- If complex PPT^2 true, then IPT^2 true.
- Do not know if they are equivalent.
- If Φ ITP, unital and idempotent, then the range of Φ in the Choi-Effros product is isomorphic to $\ell^\infty_k(\mathbb{R})$ for some k and $\Phi \circ \Phi$ is EB.
- If Φ ITP and either unital or trace-preserving, then $\lim_n d(\Phi^n, \mathbb{R} - EB) = 0$.
- If Φ ITP, both unital and trace-preserving, then there exists k such that Φ^k is $\mathbb{R}$-EB.

Questions

- Can a state in $CSEP(\mathbb{R}^d, \mathbb{R}^r) \setminus SEP(\mathbb{R}^d, \mathbb{R}^r)$ yield a quantum advantage in real quantum theory?
- One proof of Hayden-Van Dam shows that embezzlement works in the real case as well. What about exact embezzlement, i.e., catalytic production of entanglement in real quantum theory?
- Given a real quantum channel, is there a way to guarantee that noise is real? Is there a potentially useful theory of real quantum repeaters?

Thanks!