Robust Quantum Control

What is robust control and why does it matter?

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May 25, 2023

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Overview of Talk

- What is robust control? Why does it matter?
- Review of classical robust control techniques
- Why we need different tools for quantum systems?
- Quantifying robustness for quantum systems
  1. Structured singular value or μ-analysis
  2. Log-sensitivity
  3. Robustness infidelity measure
- Conclusions and future work

What is robust control?

Robust control — designing control systems able to function properly in the presence of uncertainties and variations in the system or environment.

Uncertainties can arise from a variety of sources — measurement noise, parameter variations, and disturbances.

Prerequisite for technology — taking experiments from lab to real world applications

Classical robust control

Goal — ensure control system remains stable and provides satisfactory performance in the presence of uncertainties

Design controllers less sensitive to changes in the system or environment, that can adapt to unexpected disturbances

Popular approaches

- H_∞ **control:** minimize the worst-case performance of the system over a range of unstructured uncertainties.
- μ**-synthesis:** optimize controller to achieve a desired level of performance while simultaneously guaranteeing robust stability in the face of structured uncertainties.

Classical Control for LTI Systems

LTI-system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

Laplace Trans.

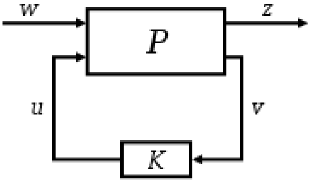
$$s\hat{\mathbf{x}} = A\hat{\mathbf{x}} + B\hat{\mathbf{u}}$$
$$\hat{\mathbf{y}} = C\hat{\mathbf{x}} + D\hat{\mathbf{u}}$$

Transfer function

$$T_{\hat{\mathbf{y}},\hat{\mathbf{u}}} = C(sI - A)^{-1}B + D$$

Laplace trans. $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{u}}$

$\mathbf{x}(t)$ state vector, $\mathbf{u}(t)$ control, $\mathbf{y}(t)$ observables



Noisy, closed loop system

$$\begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}} \\ \hat{\mathbf{u}} \end{bmatrix}$$

State feedback: $\hat{\mathbf{u}} = K(s)\hat{\mathbf{v}}$

Transfer function: noisy, closed-loop

$$T_{\hat{\mathbf{z}},\hat{\mathbf{w}}}(P, K, s) = P_{11}(s) + P_{12}(s)K(s)[I - P_{22}(s)K(s)]^{-1}P_{21}(s)$$

H_∞ Control

Aims: Minimize worst-case sensitivity of the system to unstructured uncertainties & disturbances, ensuring stability

Optimization problem: find controller K that minimizes $||T_{\hat{\mathbf{z}},\hat{\mathbf{w}}}(P,K,s)||_\infty = \sup_\omega \bar{\sigma}(T_{\hat{\mathbf{z}},\hat{\mathbf{w}}}(P,K,i\omega))$ subject to stability constraints

Pros: can handle a wide range (too wide?) of unstructured uncertainties, unmodeled dynamics, external disturbances

Cons:

- Not always easy to tune the cost functions to achieve simultaneous performance and stability requirements
- Computationally very sensitive near optimal solution
- Framework to formalize stability of linear systems

μ Synthesis

Noisy, K-closed-loop system

$$\begin{bmatrix} v \\ z \end{bmatrix} = \begin{bmatrix} G_{11}(K) & G_{12}(K) \\ G_{21}(K) & G_{22}(K) \end{bmatrix} \begin{bmatrix} \eta \\ w \end{bmatrix}$$

w disturbance, z error,
$\eta = \Delta \boldsymbol{v}$, $\Delta \in \boldsymbol{\Delta}$
structured perturbation

Structured singular value (μ)

$$[\mu_{\boldsymbol{\Delta}}(G)]^{-1} = \min_{\Delta \in \boldsymbol{\Delta}, \Delta_f} \{\|\Delta\| : \det(I - \mathrm{diag}(\Delta, \Delta_f)G) = 0\}$$

used to bound performance degradation under bounded, structured uncertainties

$$\|T_{z \leftarrow w}\| \leq \mu_{\boldsymbol{\Delta}}(G) \quad \textit{for} \quad \|\Delta\| < 1/\mu_{\boldsymbol{\Delta}}(G)$$

μ **value** measures system's worst-case stability margin over a range of structured uncertainties and frequencies

μ-Synthesis

Design: find stabilizing controller K minimizing $\mu_{\mathbf{\Delta}}(G(K))$

Pros:

- can handle linear and probably non-LTI systems
- structured uncertainties such as parametric uncertainties, unmodeled dynamics, time delays
- provides more detailed information about what controllers can achieve than H-infinity control

Cons: Design computationally intensive, may require specialized tools and expertise to design and analyze the resulting controllers.

Measuring robust performance

H_∞-norm of transfer function

Structured singular value μ

Gain and phase margins: determine how much a system's gain or phase can be increased before it becomes unstable

Sensitivity and log-sensitivity to structured perturbations

Statistical techniques often based on MC simulations.

Why Stability is Classically Desirable

Stability is a fundamental property of control systems, desirable as it ensures safe and predictable operation.

Instability can lead to unpredictable behavior, oscillations, and even system failure.

Unstable aircraft may experience uncontrollable roll & yaw (Dutch roll) and pitch (phugoid, short periodic) oscillations, loss of control and potentially catastrophic consequences.

Unstable chemical process: runaway reactions or explosions, serious harm to people and the environment.

Criteria for Stability of LTI systems

Pole-zero criterion. LTI system stable if and only if all the poles of its transfer function lie in the open left half-plane of the complex plane, i.e., real parts of all poles negative.

Hurwitz criterion. LTI system stable iff all coefficients of characteristic polynomial and 1st column of Routh array are positive.

Nyquist stability. LTI system closed-loop stable iff Nyquist plot does not encircle $(-1, 0)$ in complex plane.

Bode stability. LTI system is stable iff its phase margin is positive — amount by which phase of transfer function falls short of -180° at the frequency where magnitude is unity.

Why can't we apply this to quantum systems?

Non-LTI systems: Quantum control systems generally have non-linear (bilinear) input-output relationship — controls time-dependent

Time-domain problems: Worst case performance over range of frequencies not useful under decoherence.

Marginal stability: Closed quantum systems subject to coherent control marginally stable, stability is undesirable, kills quantum advantage!

Nonlinear performance measures e.g., entanglement

Goal: Theory of robust performance without stabilization for marginally stable, non-LTI systems

Quantum Control Systems

Quantum dynamics modeled in different ways. Schrödinger or Liouville equation formulation common but real rep. easy

$$\boxed{\frac{d}{dt}\boldsymbol{x}(t) = (A_S + A_E + A_C)\boldsymbol{x}(t)}$$

- $\boldsymbol{x}(t) \in \mathbb{R}^N$ — quantum state
  $x_n = \mathrm{Tr}(\rho\sigma_n)$, $\{\sigma_n\}$ basis for $\mathfrak{u}(N)$
- A_S — system dynamics (independent of control)
- A_E — effect of environment on dynamics
- A_C — controller $\boxed{A_C(t) = \sum_m f_m(t) A_m}$

Most quantum systems are not LTI

Quantum LTI Systems

Bilinear systems can be transformed to linear systems with full state feedback though not generally time-invariant

Time-invariance if f_m, $A_C = \sum_m f_m A_m$ TI.

LTI-system	State feedback	Transfer function
$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ $\mathbf{y} = C\mathbf{x}$	Control $\mathbf{u}(t) = A_C\mathbf{x}(t)$	$T_{\hat{\mathbf{y}},\hat{\mathbf{u}}} = C(sI - A)^{-1}B$ Laplace trans. $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{u}}$

Poles of transfer function — eigenvalues of $A = A_S + A_C + A_E$

- **Hamiltonian system:** A_S antisymmetric
- **Coherent control:** A_C anti-symmetric
- **Environment dynamics** A_E not antisymmetric

Curse of Stability for Quantum Systems

Closed system $A_E = 0$

Eigenvalues of A

imaginary axis

Marginal stability

Open system $A_E \neq 0$

Eigenvalues of A have

negative real parts

Stability

Curse of Stability

- Environment acts as stabilizing controller for coherent quantum dynamics
- Stabilized states and dynamics mostly classical
- Stability margins related to rates of entanglement loss

Loss of Quantum Advantage

Robust Performance without Stabilization?

Is robust performance without stability possible?

What qualifies as robust performance for QS?
Must consider marginal stability, non-LTI systems

How can we measure robust performance?
Measures that require stability not useful.

Quantification of robust performance involves assessing the system's ability to cope with a wide range of uncertainties and disturbances — challenging

Performance metrics may need to include non-linear performance measures (e.g., entanglement measures, concurrence)

μ Analysis for Reservoir Engineering

Reservoir engineering: Class of QC problems that can be formulated as LTI systems and meet stability criteria

Two qubits in lossy cavity designed to generate entanglement [PRA 94 (3), 032313]

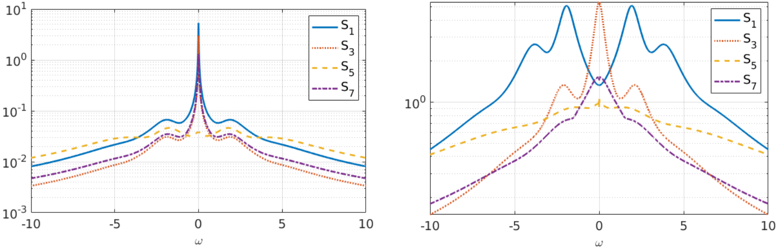
$$\frac{d}{dt}\rho(t) = -\imath[H_{\alpha,\Delta}, \rho(t)] + \sum_k \gamma_k^2 \mathfrak{L}\left(\sigma_-^{(k)}\right)\rho(t)$$
$$H_{\alpha,\Delta} = \sum_{n=1}^2 \left(\alpha_n^* \sigma_+^{(n)} + \alpha_n \sigma_-^{(n)} + \Delta_n \sigma_+^{(n)} \sigma_-^{(n)}\right)$$

Robustness wrt Structured Perturbations

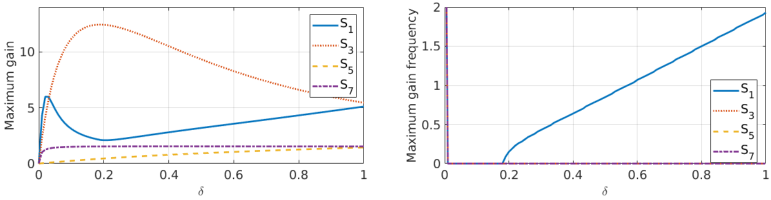
$\mathcal{S}_1, \mathcal{S}_2$ — Effective qubit-cavity coupling
$\mathcal{S}_3, \mathcal{S}_4$ — Detuning of individual qubits from cavity
$\mathcal{S}_5$ — Collective decay, $\mathcal{S}_6, \mathcal{S}_7$ — Single qubit decay

μ Analysis for Reservoir Engineering

IEEE TAC 67 (11), 6012-6024

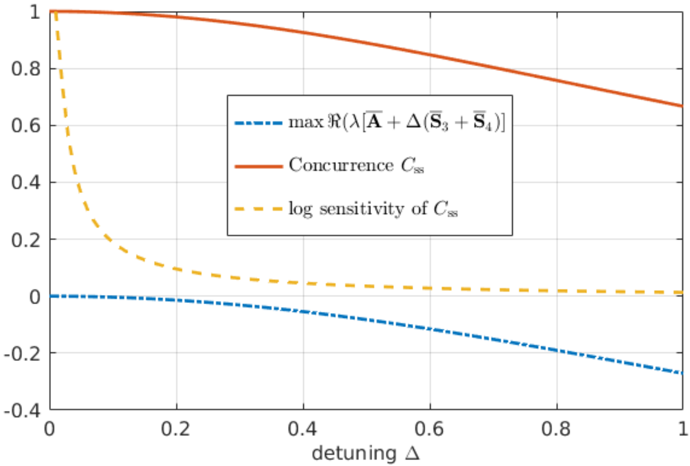


Error gain $||T_{z,w}(i\omega, \delta S_k)||$ for $\delta = 0.1$ (left) and 1 (right)



Error gain vs δ (left) and maximum gain frequency vs δ (right)

Robustness of Reservoir Engineering



Transfer functions for Error Dynamics arxiv:2305.03918v1

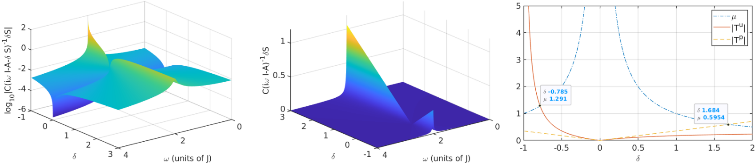
Model $\boldsymbol{r}_u$ vs Perturbed $\boldsymbol{r}_p$

$$\frac{d}{dt}\boldsymbol{r}_u = \mathcal{A}\boldsymbol{r}_u + \boldsymbol{Bd}$$
$$y_u = \mathcal{C}_u\boldsymbol{r}_u$$
$$\frac{d}{dt}\boldsymbol{r}_p = (\mathcal{A} + \delta\mathcal{S})\boldsymbol{r}_p + \boldsymbol{Bd}$$
$$y_p = \mathcal{C}_p\boldsymbol{r}_p$$

Error Dynamics $\boldsymbol{z} = \boldsymbol{r}_u - \boldsymbol{r}_p$

$$\frac{d}{dt}\boldsymbol{z} = (\mathcal{A} + \delta\mathcal{S})\boldsymbol{z} + \delta\mathcal{S}\boldsymbol{w}_u$$
$$\boldsymbol{e} = \mathcal{C}_p\boldsymbol{z} + (\mathcal{C}_p - \mathcal{C}_u)\boldsymbol{w}_u$$
$$\frac{d}{dt}\boldsymbol{z} = \mathcal{A}\boldsymbol{z} + \delta\mathcal{S}\boldsymbol{w}_p$$
$$\boldsymbol{e} = \mathcal{C}_u\boldsymbol{z} + (\mathcal{C}_p - \mathcal{C}_u)\boldsymbol{w}_p$$

Calculate norm of transfer function for error dynamics



(a) Unperturbed transfer function (b) Perturbed transfer function (c) μ vs δ

Fig. 7: Transfer function relative to unperturbed and perturbed state and μ for uncertainty in $\gamma = \gamma_0 + \delta_{(\gamma)}$ with $\gamma_0 = 0.01$, $J = 1$, $\Delta = 0$. The intersection point of $\delta_{(\gamma)} = \mu(\delta_{(\gamma)})$ determines $\delta_{(\gamma),\max}$ and $\mu_{(\gamma),\infty}$.

Log-Sensitivity: Basics

Definition: dimensionless differential sensitivity measure that quantifies the effect of parameter uncertainty or disturbances (perturbations) on a performance metric.

LTI system: with uncertainty structured as S

$$\frac{d}{dt}\tilde{\mathbf{x}}(t) = (A_S + A_E + A_C + (\delta - \delta_0)\, S)\tilde{\mathbf{x}}(t)$$

with magnitude $\delta \in \mathbb{R}$ and nominal value δ_0

Performance metric: function of state $\tilde{\mathbf{y}}(\tilde{\mathbf{x}}(t), \delta)$ — fidelity error for gate operation or state transfer

$$s(\delta_0, t) = \left.\frac{\partial \tilde{\mathbf{y}}}{\partial \delta} \frac{\delta}{\tilde{\mathbf{y}}}\right|_{\delta=\delta_0} = \left.\frac{\partial \ln \tilde{\mathbf{y}}}{\partial \ln \delta}\right|_{\delta=\delta_0}$$

Log-Sensitivity: The Good and the Bad

Advantages

- Analytically tractable expression when the performance measure is linear in the state
- Applicable to time-domain control problems
- Potential for extension to non-linear performance measures and time-varying systems (non-trivial)

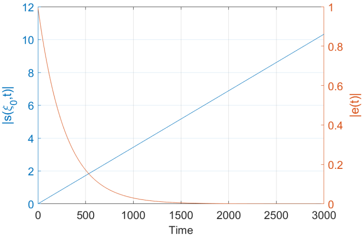
Limitations/Challenges

- Only provides local measure of sensitivity/robustness
- No bounds on maximum allowable perturbations for given performance bounds unlike μSSV
- Utility currently limited to analysis — any potential for use in synthesis as of yet unrealized

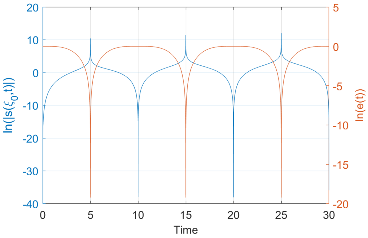
Log-Sensitivity: Recent Results [arXiv:2210.15783]

Trade-off between robustness and performance for a fixed controller; as performance measure approaches target value, log-sensitivity increases:

- For systems with an asymptotic steady-state results indicate polynomial divergence in $s(\delta_0, t)$.
- In cases where the target is achieved in finite time, results indicate divergence of $s(\delta_0, t)$ in finite time.



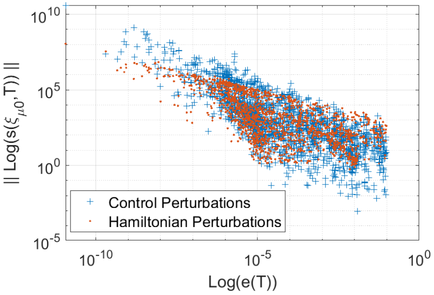
Two-qubits in a cavity
Performance: steady state overlap



Excitation transfer in spin chains
Performance: transfer fidelity

Log-Sensitivity: Recent Results

- Trade-off is not a fundamental limitation on the robustness and fidelity of controllers across the optimization landscape
- Controllers that provide the same performance may vary widely in terms of robustness



Left: Log-sensitivity versus fidelity error for excitation transfer in a 5-ring

arXiv.2303.0951
arXiv:2303.05649
arXiv:2303.00142

IJRNC 28, 2383-2403

RIM — Robustness measure based on Wasserstein distance [PRA 107 (3), 032606]

Stochastic robustness measure based on *p*th Wasserstein distance between observed and ideal probability distribution

$$\mathrm{RIM}_p = \mathbb{E}_{f\sim\mathbf{P}(\mathcal{F})}\left[(1-f)^p\right]^{\frac{1}{p}}$$

if fidelity distribution $\mathbf{P}(\mathcal{F})$ approximated by samples $\{F_n\}$

Average Infidelity: $\mathrm{RIM}_1 = 1 - \mathbb{E}_{f\sim\mathbf{P}(\mathcal{F})}[f]$

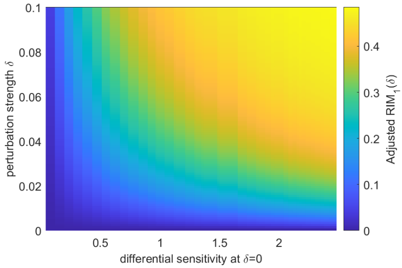
Differential sensitivity of error expection agrees with the derivative of the RIM_1 at $\delta = \delta_0$:

$$\mathbb{E}_{\mathbf{P(S)}}\left[\frac{\partial\tilde{\mathbf{e}}(T;S_\mu,\delta)}{\partial\delta}\Big|_{\delta=\delta_0}\right] = \frac{\partial\mathrm{RIM}_1(\delta)}{\partial\delta}\Big|_{\delta=\delta_0}$$

RIM - Results [arXiv.2303.0951]

RIM_1 has the ability to confirm more global robustness properties indicated by the differential sensitivity at $\delta = 0$.

Chart shows $(\mathrm{RIM}_1(\delta) - \mathbf{e}(T))$ as a function of δ versus $\frac{\partial \tilde{\mathbf{e}}(T;S_\mu,\delta)}{\partial \delta}$ for excitation transfer in a 5-ring $1 \to 2$ transfer.



Suggests agreement of local robustness captured by differential sensitivity at $\delta = 0$ and RIM_1 for larger δ

Summary and Outlook

- Robust system & control design prerequisite for technology
- Classical robust control offers tools but insufficient
- H_∞ control and μ-analysis may be applicable for reservoir engineering and open system control
- Differential (log) sensitivity useful
- Statistical robustness measures such as RIM_p based on distributions useful
- Framework for robust system & control design for quantum systems needed, taking into account
  - lack of time-invariance
  - marginal stability
  - non-linear performance measures

Research Directions

Quantum Technologies

It all starts with a question...Quantum Technologies is a new journal that brings together scientists, engineers, and leaders in industry to answer important questions to advance the field.

Questions like:

- How do we quantify the utility of quantum algorithms?
- What are the full capabilities of relativistic quantum cryptography?
- How will challenges in micro- and nanofabrication impact the development of quantum technologies?
- What is robust control in quantum technology?
- Can the microfabrication of atomic and optical components open new capabilities in quantum technologies?
- How can quantum technologies be used for testing fundamental physics?

Instead of submitting one fully formed research paper, researchers submit incremental pieces of results and analysis that contribute towards answering the question. Finally, an impact paper will summarise what has been published in response to the question.

No publication fees will be charged on submissions received before 1 October 2023, so if you have research that could help us answer important questions in this area, we welcome you to submit your research.

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