Quantum Zeno effect and strong damping for infinite dimensional open systems

Robert Salzmann

Joint work with Simon Becker and Nilanjana Datta

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Dynamics in closed quantum systems

For a **closed system** with associated Hilbert space $\mathcal{H}$ the time evolution is governed by Schrödinger's equation

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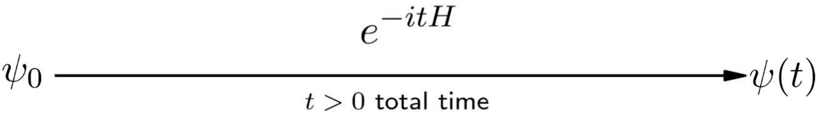
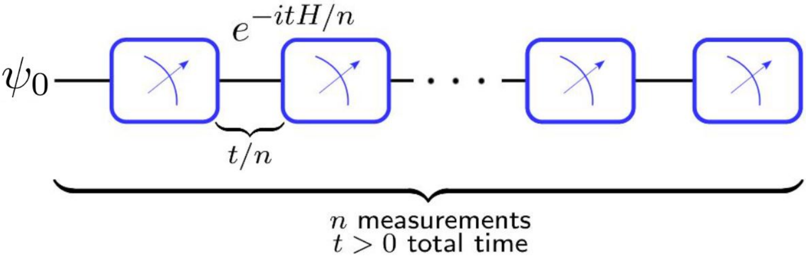
Solution given by unitary group $\left(e^{-itH}\right)_{t\in\mathbb{R}}$,
i.e. $\psi(t) = e^{-itH}\psi_0$.



Quantum Zeno effect in closed systems

Simplest Setup: Frequently perform projective measurement $\{|\psi_0\rangle\langle\psi_0|, \mathbb{1} - |\psi_0\rangle\langle\psi_0|\}$ in time intervals t/n:



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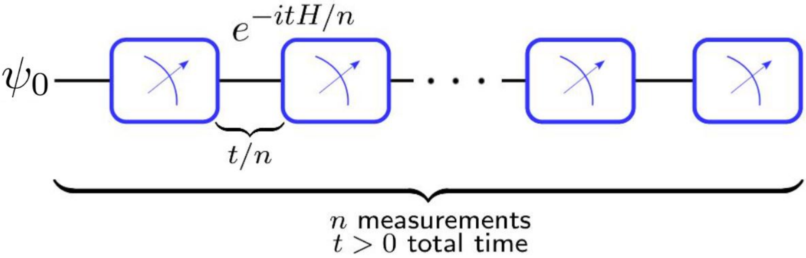

Quantum Zeno effect:

$$\mathrm{Prob}(\text{Measure } \psi_0 \ n \text{ times})$$
$$= \left\| \left(|\psi_0\rangle\langle\psi_0|e^{-itH/n}\right)^n \psi_0 \right\|^2 = \left|\langle\psi_0, e^{-itH/n}\psi_0\rangle\right|^{2n} \xrightarrow[n\to\infty]{} 1.$$

System is frozen in its initial state.

More generally:

- Mixed intial state $\rho_0 \in \mathcal{T}(\mathcal{H})$, i.e. $\rho_0 \geq 0$ and $\mathrm{Tr}(\rho_0) = 1$.
- General binary projective measurement $\{P, \mathbb{1} - P\}$, $P \in \mathcal{B}(\mathcal{H})$ projection.

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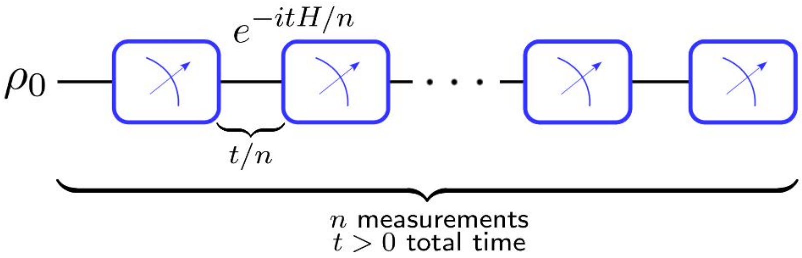
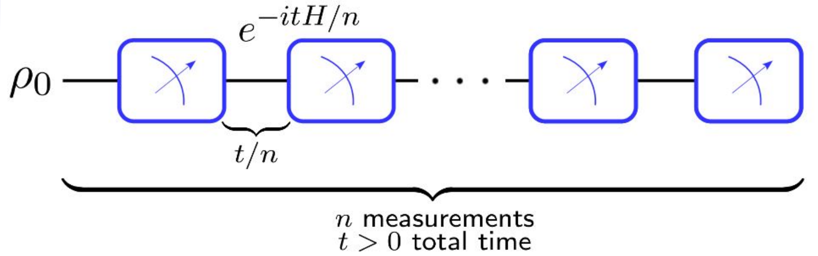
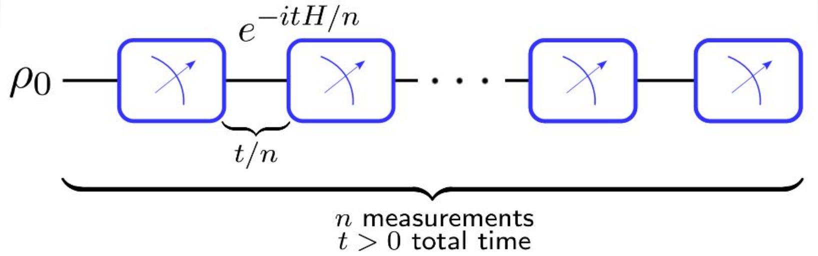




What is the **effective dynamics** of this process?



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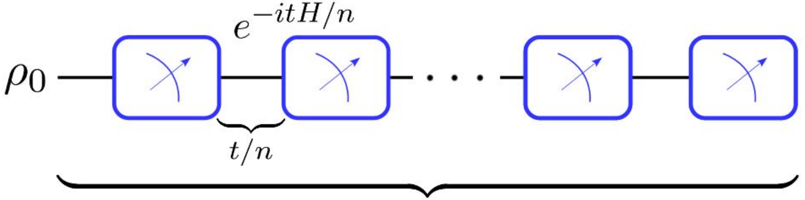
$$\left(Pe^{-itH/n}\right)^n \xrightarrow[n\to\infty]{} e^{-itPHP}P. \tag{1}$$



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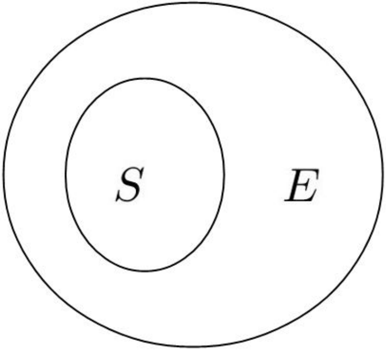
Note (1) implies freezing of measurement probabilities (QZE):

$$\mathrm{Prob}(\text{Measure } P \ n \text{ times}) = \mathrm{Tr}\Big((Pe^{-itH/n})^n \rho_0 (e^{itH/n}P)^n\Big)$$
$$\xrightarrow[n\to\infty]{} \mathrm{Tr}\Big(e^{-itPHP}P\rho_0 P e^{itPHP}\Big) = \mathrm{Tr}(P\rho_0)$$
$$= \mathrm{Prob}(\text{Measure } P \text{ at } t = 0).$$

Quantum Zeno setup in open quantum systems

What about open quantum systems, generalised measurements/applications of general quantum operations?



Composite Hilbert space $\mathcal{H} \otimes \mathcal{H}_E$.

Consider **open quantum system** with time evolution governed by Lindblad equation

$$\begin{cases} \partial_t \rho(t) = \mathcal{L}(\rho(t)) \\ \rho(0) = \rho_0, \end{cases}$$

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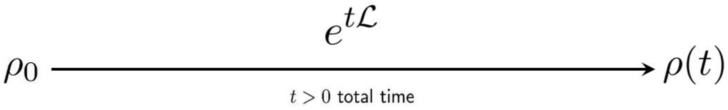

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$$\rho_0 \xrightarrow[t>0 \text{ total time}]{e^{t\mathcal{L}}} \rho(t)$$

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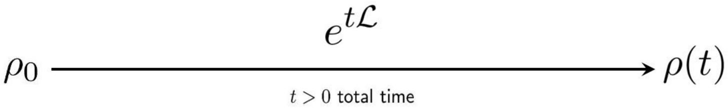

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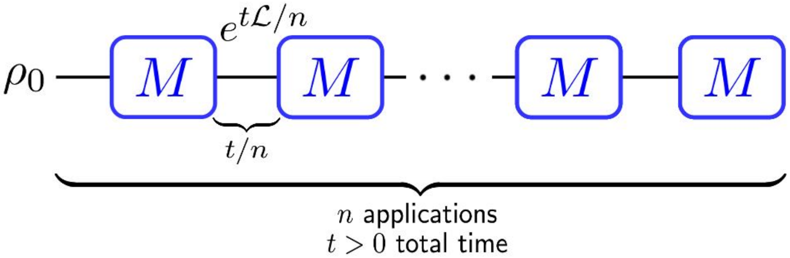
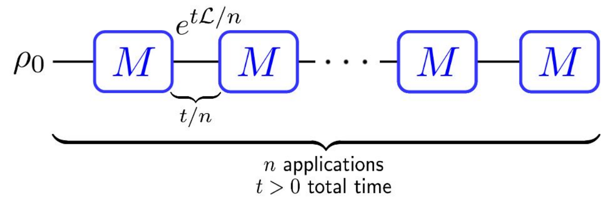
Quantum Zeno setup in open systems: Frequently interleave dynamics by applying quantum operation M





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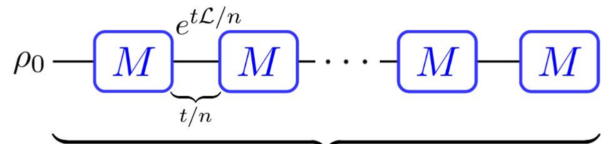

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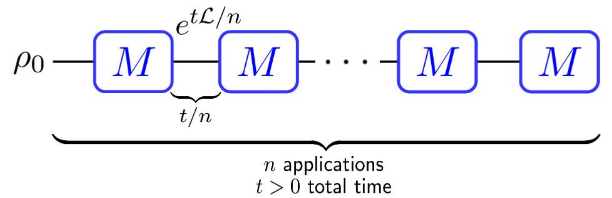

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- Möbus and Wolf extended in 2019 to quantum operations M satisfying a certain spectral condition.

$$\lim_{n\to\infty} (Me^{t\mathcal{L}/n})^n = e^{tP\mathcal{L}P}P,$$

with P being projector on invariant subspace of M, i.e. $\ker(\mathbb{1} - M)$.

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- Using similar techniques we also derive strong damping.

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If $M^n \xrightarrow[n\to\infty]{} P$ then P projector on invariant subspace $\ker(\mathbb{1} - M)$ on X.

Uniform convergence: Spectral condition on M

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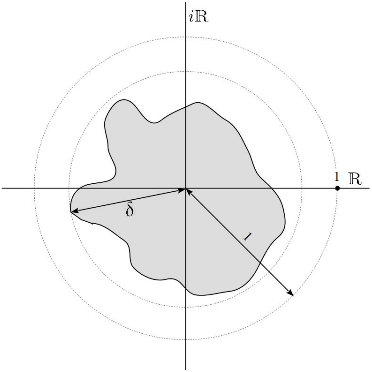

Γ closed, counterclockwise oriented curve in complex plane surrounding 1 but no other point in $\mathrm{Spec}(M)$.
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$$P = \frac{1}{2\pi i} \oint_\Gamma (z - M)^{-1} \, dz$$

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Here

$$\|(M(\mathbb{1} - P))^n\| = \left\| \frac{1}{2\pi i} \oint_\gamma z^n (z - M)^{-1}\, dz \right\| \leq C\tilde{\delta}^n.$$

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Hence, assuming additionally $N = 0$ gives

$$\lim_{n\to\infty} M^n = P$$

in operator norm.

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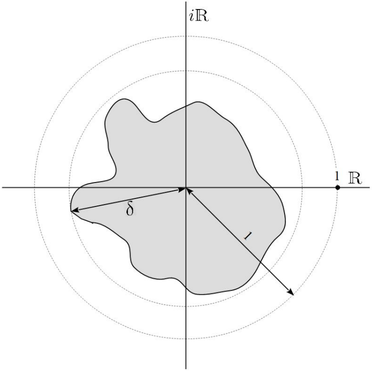

Example

For $\sigma \in \mathcal{T}(\mathcal{H})$ state and $p \in [0,1]$ consider M to be the **generalised depolarising channel**, i.e.

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One can prove that M satisfies spectral condition with $\mathrm{Spec}(M) = \{1-p, 1\}$.
Projector on invariant subspace given by

$$P(\rho) = \mathrm{Tr}(\rho)\sigma.$$

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In infinite dimensions this is not the case! There exists contraction having a spectral gap but non-vanishing quasi-nilpotent operator corresponding to spectral point on unit circle.
Open question: Can one also find quantum operation or channel with spectral gap but non-trivial quasi-nilpotent operator?

Spectral condition is necessary for uniform convergence

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Let M be a contraction.
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Theorem

Let $\mathcal{L}$ bounded and M be a contraction satisfying the spectral condition. Then

$$\left\| \left(M e^{t\mathcal{L}/n} \right)^n - e^{tP\mathcal{L}P} P \right\| \leq C \left(\frac{\|\mathcal{L}\|}{\sqrt{n}} + \tilde{\delta}^{n+1} \right),$$

for some $0 \leq \delta < \tilde{\delta} < 1$ and $C > 0$ independent of $\mathcal{L}$ and n. Here P projector on invariant subspace of M.

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Möbus and Rouzé (2021) improved to tight convergence rate $\mathcal{O}(1/n)$.

Quantum Zeno limit for unbounded generators

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M *contraction satisfying the* **spectral condition** *and furthermore* $M\mathcal{L}$ *and* $\mathcal{L}M$ *are densely defined and bounded.*

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Use quantitiative result for bounded generators $\mathcal{L}_k$ and coupled limit $k, n \to \infty$.

Beyond the spectral condition:

Example

Consider for $\lambda \in [0,1)$ **(bosonic quantum-limited) attenuator channel** Φ_λ^{att} which is defined on coherent states $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{n!} |n\rangle$ as

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$\sup_{|\alpha\rangle\langle\alpha|} \||\lambda^n\alpha\rangle\langle\lambda^n\alpha| - |0\rangle\langle 0|\|_1 = 2 \quad$ for all $n \in \mathbb{N}$. Hence, $(M^n)_{n\in\mathbb{N}}$ does not converge in operator norm and does therefore not satisfy the spectral condition.

Quantum Zeno dynamics without spectral condition

Theorem

Let $\mathcal{L}$ be bounded and M be a contraction which satisfies for all $x \in X$

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Idea of the proof

Let $\mathcal{L}_n := \left(e^{t\mathcal{L}/n} - \mathbb{1}\right) n$, which satisfies

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$$\Delta^k_{\mathsf{disc}}(n) = \left\{(i_1,\ldots,i_k)\in\mathbb{N}^k \;\middle|\; \sum_{l=1}^k i_l \le n\right\}.$$

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Consider $(\mathcal{L}_n)_{n\in\mathbb{N}}$ *and* M *contraction such that* $\lim_{n\to\infty}\mathcal{L}_n = t\mathcal{L}$ *and* $s\text{-}\lim_{n\to\infty} M^n = M$. *Then for all* $x \in X$ *and* $k \in \mathbb{N}$ *we have*

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$$\xrightarrow[n\to\infty]{} Px + \sum_{k=1}^{\infty}\frac{(tP\mathcal{L}P)^k}{k!}x = e^{tP\mathcal{L}P}Px.$$

Strong damping

Consider dynamics governed by generator

$$\mathcal{L}_{\text{total}} = \gamma\mathcal{K} + \mathcal{L}$$

where $\mathcal{K}$ being, possibly unbounded, generator of strongly continuous contraction semigroup, $\mathcal{L} \in \mathcal{B}(X)$ and $\gamma \geq 0$.

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$\mathcal{L}_{\text{total}}$ generates a strongly continuous semigroup $\left(e^{t(\gamma\mathcal{K}+\mathcal{L})}\right)_{t\geq 0}$ which satisfies

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We consider strong interaction limit, i.e. $\gamma \to \infty$.

Theorem

Let $\mathcal{K}$ with domain $\mathcal{D}(\mathcal{K}) \subset X$ be the generator of a strongly continuous contraction semigroup which satisfies

$$\lim_{\gamma\to\infty} e^{\gamma\mathcal{K}}x = Px \tag{2}$$

for all $x \in X$ and some $P \in \mathcal{B}(X)$. Furthermore, let $\mathcal{L} \in \mathcal{B}(X)$. Then for all $t > 0$

$$\lim_{\gamma\to\infty} e^{t(\gamma\mathcal{K}+\mathcal{L})}x = e^{tP\mathcal{L}P}Px. \tag{3}$$

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Note

$$\begin{aligned} e^{t(\mathcal{K}+\mathcal{L}/n)} - e^{t\mathcal{K}} &= \int_0^1 \frac{d}{ds}\left(e^{st(\mathcal{K}+\mathcal{L}/n)} e^{(1-s)t\mathcal{K}}\right) ds \\ &= \frac{t}{n}\int_0^1 e^{st(\mathcal{K}+\mathcal{L}/n)} \mathcal{L} e^{(1-s)t\mathcal{K}} ds. \end{aligned}$$

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Noting by the above that $\lim_{n\to\infty} e^{t(\mathcal{K}+\mathcal{L}/n)} = e^{t\mathcal{K}}$ we see

$$\lim_{n\to\infty} \mathcal{L}_n = t \int_0^1 e^{st\mathcal{K}} \mathcal{L} e^{(1-s)t\mathcal{K}} ds.$$

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Thanks for your attention!