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Inductive limits of quantum systems, equilibrium states and dynamics

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ICMS Workshop
Mathematical Physics &Quantum Technology:
From finite to infinite dimensions

Bayes Centre, Edinburgh
May 25, 2023

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limits of quantum systems,

ICMS Workshop
Mathematical Physics &Quantum Technology:
From finite to infinite

Examples

- Mean field limit ($N\to\infty$, permutation symmetry)
- Classical limit ($\hbar\to 0$)
- Infinite volume limit (lattice system)
- Continuum limits (lattice spacing $\to 0$, renormalization)
- Continuum limits (repeated $\to$ continuous measurement)

Features

- Not just a parameter, structure may change in the limit
- Limit in norm for states and/or observables
- General results do some of the work for
  - Limit space
  - Dynamics
  - Equilibrium states

Soft inductive limits

Each term in a sequence/net
lives in a different universe

Then how can they be called similar?

Introduce comparison maps
These are the conceptual basis of
the limit scheme

Soft inductive limits

Indices for the limit: a directed set $(N,>)$

Nets $(x_n)_{n\in N}$, $x_n \in E_n$ = Banach space

Comparison maps $j_{nm}:E_m \to E_n$ for $n>m$, such that

- each j_{nm} is linear, $\|j_{nm}\|\leq 1$
- system is asymptotically transitive:

$$\lim_k \limsup_n \|(j_{nm} - j_{nk}\, j_{km})x_m\| = 0 \quad \forall\, x_m \in E_m$$

More (or less) structure as needed: j_{nm} then completely positive, unital, homomorphism

Soft inductive limits

A nets $(x_n \in E_n)_{n \in N}$ is called j-convergent,
if $\lim_k \limsup_n ||x_n - j_{nm} x_m|| = 0$.

j-convergent sets x_n, y_n are said to have the same limit if $\lim_n ||x_n - y_n|| = 0$.

$j\text{-}\lim_n x_n = x_\infty \in E_\infty$ = quotient space
="abstract limit space"

- E_∞ is complete with the norm $||x_\infty|| = \limsup_n ||x_n||$
- For fixed $x_m \in E_m$ the basic net $y_n = j_{nm} x_m$ converges
- Define $j_{\infty m}: E_m \to E_\infty$ by $j_{\infty m} x_m := j\text{-}\lim_n y_n$
- Basic nets are dense

Soft inductive limits: general remarks

The notion of convergent net is more fundamental than the comparison maps:
Can change j_{nm} for any finite n,m. Only asymptotic $n >> m \to \infty$ enter.

Two families j_{nm} and j'_{nm} are called equivalent, if they have same convergent nets

Construction jointly generalizes strict inductive limits ($j_{nm} = j_{nk} j_{km}$) and the completion construction (all j_{nm}=id, any seminorm on E_1).

Sometimes may compare directly E_n and E_∞
A split inductive system is of the form $j_{nm} = p_n i_m$ with
$i_n : E_n \to E_\infty$, $p_n : E_\infty \to E_n$, $p_n i_n = id_n$, $i_n p_n \to id_\infty$ strongly.

Example 1: Mean Field

Fix one-site algebra $\mathcal{A}$ (C*, often finite matrix algebra)

$E_n=\mathcal{A}^{\otimes n}$ $j_{nm}(A)=\mathrm{sym}_n(A\otimes 1^{\otimes(n-m)})$
=average over permutations

Strict inductive system, unknown if equivalent to split

A_n, B_n convergent, $C_n= A_n B_n \Rightarrow C_n$ convergent, $[A_n,B_n]\to 0$
Hence E_∞ is an abelian C*-algebra.

$E_\infty \cong C(\Sigma)$, Σ= state space of $\mathcal{A}$: $A_\infty(\rho)=\lim_n \rho^{\otimes n}(A_n)$

Example 2: Classical Limit

Fix a finite dimensional phase space (Ξ,σ)

$\mathcal{H}_\hbar$ carries Weyl operators $W_\hbar(\xi)$,

$$W_\hbar(\xi)\, W_\hbar(\eta)=\exp(-i\sigma(\xi,\eta)/(2\hbar))\, W_\hbar(\xi+\eta)$$

$$\alpha^\hbar{}_\xi(A)= W(\xi)_\hbar{}^* A W(\xi)_\hbar$$

$E_\hbar=T(\mathcal{H}_\hbar)$ or $E_\hbar=\{A\in B(\mathcal{H}_\hbar) \mid \xi\to \alpha^\hbar{}_\xi(A) \text{ norm contin.}\}$

j= α-covariant / not too noisy / $w_\hbar(\xi):=W_\hbar(\hbar\xi)$ convergent
can be chosen split or strict (not both)

$E_0=L^1(\Xi,d\xi)$ or $E_0=\{f\in L^\infty(\Xi,d\xi) \mid \xi\to \alpha^\hbar{}_\xi(f) \text{ norm contin.}\}$
Get $j_{0\hbar}$ by expectation with coherent vector
Aymptotically homomorphic.

j-convergence $\Rightarrow$ j*-convergence

Example 3: Quasilocal algebra

X= $\mathbf{Z}^d$, lattice with observables $\mathcal{A}_x$ at site $x \in X$
N= family of (some) finite regions, $\subset$

$\mathcal{A}_n = \bigotimes_{x \in n} \mathcal{A}_x$; j_{nm}= inclusion by tensoring with **1**

$\mathcal{A}_\infty$ = quasilocal algebra [Bratteli-Robinson]

Examples …:

- renormalization/lattice refinement
- Finite Weyl systems $\rightarrow$ $[x,p]=i$
- Continuous measurement
- Tensor products of inductive systems

Dynamics

Hamiltonians usually not convergent:

Mean field: $H_n = n h_n$ must be extensive

Classical: dynamics generated by $U_t = \exp(-iH_\hbar t/\hbar)$ must not converge to implement anything

Lattice: Hamiltonian is not quasilocal

Mean field/Classical:

get Poisson bracket as next order commutator:

$\text{j-lim}_n \, n i [A_n, B_n] = \{A_\infty, B_\infty\}$

Aim: jj-convergence $T_{n,t}$ = semigroup on E_n such that x_n convergent $\Rightarrow$ $T_{n,t} x_n$ convergent.

$T_{\infty,t} x_\infty := \text{j-lim}_n \, T_{n,t} x_n$

Dynamics

Our theory neatly separates:

(A) Limit for the generator
equations of motion

(B) Limit for the semigroup
solution

Nettified version of
semigroup theory

(A) Densely defined
generator G

(B) (s-G) has
dense range

(A) Uses net structure j_{nm}
(B) Can be done in E_{∞} (often simpler!)

Dynamics: Theorem

Given soft inductive system (E,j) and semigroups $T_{n,t}=\exp(t\, G_n)$.
Then equivalent

(1) $T_{n,t}$ preserves j-convergence and $T_{\infty,t}$ is a strongly continuous semigroup with generator G_∞

(2) The resolvents $R_n(\lambda)=(\lambda-G_n)^{-1}$ ($\lambda>0$) preserve j-convergence and $R_\infty(\lambda)$ has dense range.

(3) There is a dense subset $\mathcal{D}$ of convergent nets such that $(\lambda-G_n)\mathcal{D}$ is also a dense subspace of convergent nets

(4) G_∞ is well-defined and generates a semigroup

Mean field dynamics, Hamiltonian case

$G_n(A)=i[nh_n,A]$ $\Rightarrow$energy density h_∞ =j-lim$_n h_n$

$G_\infty(f) = \{h_\infty, f\}$, with suitable definition of Poisson bracket
$(T_{\infty,t} f)(\sigma)=f(\mathcal{F}_t(\sigma))$, $\mathcal{F}_t(\sigma)$= Hamiltonian flow

For $f\in C(\Sigma)$: gradient $df(\sigma)\in\mathcal{A}$:

$$tr(\rho\, df(\sigma)) := d/(dt)\, f(t\rho+(1-t)\sigma)\,|_{t=0}$$

$$\{f,g\}(\sigma) = tr(\sigma\, i[df(\sigma),dg(\sigma)])$$

Mean field tagged dynamics, Hamiltonian case

Modified inductive system: leave M sites out of sym-operation
$E_\infty=C(\Sigma)\otimes\mathcal{A}^{\otimes M} \cong \mathcal{A}^{\otimes M}$-valued functions on Σ

For dynamics get $(T_{\infty,t} f)(\sigma)=U^*(t,\sigma)^{\otimes M} f(\mathcal{F}_t(\sigma))\, U(t,\sigma)^{\otimes M}$

$$d/(dt)U(t,\sigma)=iH(t)U(t,\sigma) \quad \text{with} \quad H(t)= i\, dh_\infty(\mathcal{F}_t(\sigma))$$

Mean field dynamics, Lindblad case

$G_n(A)$=n Sym_n(Lindblad terms) (operator permutations)

Special simple case:

$G_n(A) = i[nh_n, A] + n\sum_j (a^*_{j,n}[A, a_{j,n}] + [a^*_{j,n}, A]\, a_{j,n})$ (*)

Tagged dynamics:

$(T_{\infty,t} f)(\sigma) = \Lambda_{t,\sigma}^{\otimes M}(f(\mathcal{F}_t(\sigma)))$

$\mathcal{F}_t(\sigma)$= non-Hamiltonian flow (arbitrary)

$\Lambda_{t,\sigma}$ = cp cocycle

$S(\Lambda^*_{t,\sigma}\rho, \mathcal{F}_t(\sigma))$ decreases

Assuming (*)

Spectrum of $\mathcal{F}_t(\sigma)$ conserved (otherwise arbitrary)

Λ generated by Hamiltonian

$dh_\infty + \sum_j \mathrm{Im}(a^*_{j,\infty}\, da_{j,\infty})$

Classical limit dynamics, Hamiltonian case

H_n= Schrödinger operator with $\leq$quadratic potential
or $\xi \rightarrow \alpha^{\hbar}_{\xi}(H_{\hbar})$ has j-convergent (bounded!) 2^{nd} derivatives

This ensures Hamiltonian vector field of H_0
uniformly Lipschitz, hence with global existence

Project: extend this to incomplete dynamics
(escape$\rightarrow\infty$ and non-esa Hamiltonians)

This approach avoids WKB artefacts:

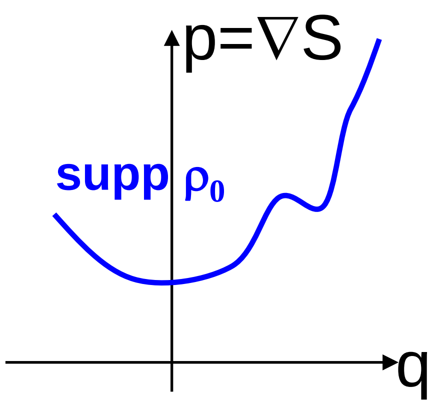
$\psi(x)= \varphi(x) \exp(iS(x)/\hbar)$; $\rho_{\hbar}=|\psi\rangle\langle\psi|$

ρ_0=j*-lim $_{\hbar}$ $\rho_{\hbar}$, i.e., for all j-convergent A:

i.e., $\lim_{\hbar} \mathrm{tr}(\rho_{\hbar} A_{\hbar})= \int\rho_0(d\xi)\, A_0(\xi)$

$\rho_0(dp\, dq)= |\varphi(x)|^2\, \delta(p-dS(x))\, dp\, dq$

Form not stable under flow



Classical limit dynamics, monitored

The hallmark of classical theory is the possibility to monitor the system continuously, essentially without disturbance.

How does this arise in the limit?

Interlace the dynamics with (“mild$_\hbar$”) measurements:

- at times fixed independently of the state, or
- as jump events in a Lindblad evolution (e.g. GRW).

Then, with suitable choice of parameters,

- limiting dynamics with same Hamiltonian
- monitoring outcomes correct.

Classical limit dynamics, quasifree (divisible)

Quasifree evolution: $T_t(W(\xi))=f_t(\xi)\, W(S_t\xi)$ on (Ξ,σ)

Probable **Theorem** (Alberto Barchielli, RFW, yesterday)

This defines a dynamical semigroup iff

- $S_t = \exp(tZ)$ is a matrix semigroup
- $f_t(\xi) = \exp \int_0^t d\tau\, \Psi(S_\tau \xi)$
- $\Psi(\xi) = i a^T\xi - \frac{1}{2}\, \xi^T A \xi + \int \mu(d\eta)\, (e^{i\eta^T\xi} - 1 - i\eta^T\xi\, h(\eta))$
- μ =Lèvy measure: $\min\{|\xi|^2,1\}$ integrable, h=cutoff
- $A + \frac{i\hbar}{2}(\sigma Z + Z^T\sigma) \geq 0$

Limit $\hbar\to 0$ of semigroups with data (a,A,μ) gives same for $\hbar=0$: Generalized Ornstein-Uhlenbeck process, which also describes the evolution of Wigner functions for all $\hbar$

Lattice dynamics: some features

Nothing really new relative to Bratteli&Robinson,
but some features simplified/more natural:

N=(regions,$\subset$), directed index set

- Translations act in no finite E_n, but do map j-convergent sequences to j-convergent sequences. Hence they act on the infinite lattice.

- Independence of boundary conditions is automatic:

  > (3) There is a dense subset $\mathcal{D}$ of convergent nets such that $(\lambda\text{-}G_n)\mathcal{D}$ is also a dense subspace of convergent nets.

  Choose $\mathcal{D}$ so observables see only interior points.

Equilibrium states: Gibbs variational principle

Start from j-limit of observables,
H_n=Hamiltonian density

Take $s_n : E_n^* \to \mathbf{R} \cup \{-\infty\}$ entropy density functional
upper semicont. and concave

Local equilibrium state ω_n minimizes "free energy"

$g(\omega) = \omega(H_n) - \beta^{-1} s_n(\omega)$ ($\beta^{-1} = kT$)

Find: $\lim_n g(\omega_n)$ and j*-limits or cluster points of the net ω_n

<u>Idea</u>: understand limit $s_n \to s_\infty$
Then free energies converge, satisfy Gibbs_∞, and minimizers converge to minimizers