Peripheral Poisson Boundary

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Mathematical Physics in Quantum Technology: From finite to infinite dimensions, ICMS, Edinburgh, UK

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- Peripherally automorphic completely positive maps, https://arxiv.org/abs/2212.07351.

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$$= \{\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{11} \end{bmatrix} : a_{ij} \in \mathbb{C}, \quad \forall i, j\}.$$

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- Here is another example.

Unilateral shifts

- Consider the sequence space $l^2 = l^2(\mathbb{Z}_+)$ with the standard orthonormal basis $\{e_0, e_1, e_2, \ldots,\}$.
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- Let V be the unilateral right shift defined by $Ve_n = e_{n+1}, \quad \forall n$ and extended linearly and continuously.
- Then V is an isometry and has the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

- The adjoint V^* is the unilateral left shift. It has the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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- The set of fixed points of τ are precisely the Toeplitz operators.

Toeplitz Operators

- These are bounded operators on l^2, whose matrices with respect to the standard basis have the form:

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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- But it is not an algebra. As product of two Toeplitz operators need not be Toeplitz.
- For instance, V, V^* are Toeplitz but VV^* is not Toeplitz.

A different product

- We may naturally identify the Toeplitz operator

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

with a function f on the unit circle whose nth Fourier coefficient is a_n. Here f is in the L^∞ of the unit circle and it is known as the symbol of the Toeplitz operator A.

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- Going back, using the identification made above, one can define a new product on Toeplitz operators, which makes it a commutative algebra!

Fixed points and noncommutative Poisson boundary

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- The original formula for the product was complicated. We will come back to this.
- Izumi called the von Neumann algebra $(F(\tau), \circ)$ (or its explicit realization) as the non-commutative Poisson boundary of τ.

The Dynamics

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- How are these Poisson boundaries related?
- If $\tau(X_0) = -X_0$ and $X_0 \neq 0$. Then $X_0 \in F(\tau^2)$ but $X_0 \notin F(\tau^3)$.

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- How to compute the product '$\circ$'?

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- (iii) $\theta : \mathcal{B} \to \mathcal{B}$ is a normal, unital $*$-endomorphism;
- (iv) (dilation property):

$$\tau^n(X) = P\theta^n(X)P, \quad \forall X \in \mathcal{A}, n \in \mathbb{Z}_+;$$

Dilation theory

- Dilation Theorem: Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra.
- Let $\tau : \mathcal{A} \to \mathcal{A}$ be a normal UCP map.
- Then there exists a triple $(\mathcal{K}, \mathcal{B}, \theta)$, where
- (i) $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ as a closed subspace;
- (ii) $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ is a von Neumann algebra, satisfying $\mathcal{A} = P\mathcal{B}P$ where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$;
- (iii) $\theta : \mathcal{B} \to \mathcal{B}$ is a normal, unital $*$-endomorphism;
- (iv) (dilation property):

$$\tau^n(X) = P\theta^n(X)P, \quad \forall X \in \mathcal{A}, n \in \mathbb{Z}_+;$$

- That is,

$$\theta^n(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \tau^n(X) & * \\ * & * \end{bmatrix}.$$

- The dilation is unique up to unitary equivalence under a natural minimality condition.

What is dilation theory?

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- The minimal one, called 'Weak Markov Flow' is constructed using Stinespring's theorem. The Time shift gives a semigroup of endomorphisms.
- Further dilation to automorphisms, may or may not exist (depending upon the set-up) and when it exists it is typically not unique.

Lifting of peripheral eigenvectors

- Let $\theta : \mathcal{B} \to \mathcal{B}$ be minimal dilation of an UCP map $\tau : \mathcal{A} \to \mathcal{A}$. The $*$-endomorphism property of θ implies that $\mathcal{P}(\theta)$ is a C^*-algebra under multiplication.

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- We set $X \circ Y = P\hat{X}\hat{Y}P$ as the modified product. This defines the peripheral Poisson boundary $(\mathcal{P}(\tau), \circ)$. As a C^*-algebra it is isomorphic to $(\mathcal{P}(\theta), \cdot)$.

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- The dilation is given by $\theta(Z) = U^*ZU$, where U is the bilateral shift.
- The von Neumann algebra generated by $\mathcal{P}(\theta)$ is the algebra of all bounded operators.
- Remark: In general, it is not possible to lift non-peripheral eigenvectors.

A formula for the new product

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- We do not know of any proof the existence of this limit without using dilation theory.

Consequences

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- Corollary 3: The map $X \mapsto \tau(X)$ is an automorphism on the peripheral boundary $(\mathcal{P}(\tau), \circ)$.

The dynamics revisited

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for every $n \geq 1$.
- Proof: From elementary linear algebra the linear span of peripheral eigenvectors of τ and τ^n are same for every $n \geq 1$. Now the result is not hard to prove from the formula for the Choi-Effros product proved before.

Peripherally automorphic maps in finite dimensions

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- Theorem: Let $\tau : M_d \to M_d$ be a UCP map with a Choi-Kraus decomposition $\tau(X) = \sum_{i=1}^{r} L_i^* X L_i, \ \ \forall X \in M_d$. Then the following are equivalent.

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- (i) τ is peripherally automorphic.
- (ii) For $\lambda \in \mathbb{T}$, $\tau(Y) = \lambda Y$ if and only if $YL_i = \lambda L_i Y$ for every $1 \leq i \leq r$.

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- and $\mathcal{N}(\tau) = \{X \in M_d : \lim_{n\to\infty} \tau^n(X) = 0\}$.
- Furthermore, $\mathcal{P}(\tau^m) = \mathcal{P}(\tau)$ and $\mathcal{N}(\tau^m) = \mathcal{N}(\tau)$ for every $m \geq 1$.

THANKS