Peripheral Poisson Boundary

B V Rajarama Bhat, Indian Statistical Institute, Bangalore

Mathematical Physics in Quantum Technology: From finite to infinite dimensions, ICMS, Edinburgh, UK

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Thanks to the organisers for the invitation and all the arrangements.

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- Peripheral Poisson boundary, https://arxiv.org/abs/2209.07731
- Peripherally automorphic completely positive maps, https://arxiv.org/abs/2212.07351.



► Two examples.



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Fixed points and noncommutative Poisson Boundary.

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- Finite dimensional case.

• Let M_3 be the C*-algebra of 3×3 complex matrices.

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• Let $\tau: M_3 \to M_3$ be the UCP map defined by

$$\tau\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{11} \end{bmatrix}$$

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• The fixed point space of au is given by

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▶ We note that $\mathcal{F}(\tau)$ is a subspace of M_3 but not a subalgebra, as in general for $A, B \in \mathcal{F}$, AB may not be in \mathcal{F} .

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New product

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Choi and Effros observed this phenomenon. They showed that under a very general context, it is possible to modify the product to get an algebra. This product is now known as Choi-Effros product.

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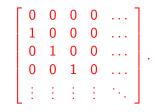
Unilateral shifts

- Consider the sequence space l² = l²(ℤ₊) with the standard orthonormal basis {e₀, e₁, e₂,...,}.
- ▶ Let V be the unilateral right shift defined by $Ve_n = e_{n+1}$, $\forall n$ and extended linearly and continuously.

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- ▶ Let V be the unilateral right shift defined by $Ve_n = e_{n+1}$, $\forall n$ and extended linearly and continuously.
- Then V is an isometry and has the matrix:



▶ The adjoint V^{*} is the unilateral left shift. It has the matrix:



A natural UCP map

• Consider $\tau : \mathfrak{B}(l^2) \to \mathfrak{B}(l^2)$ defined by

 $\tau(X) = V^* X V, \quad \forall X \in \mathcal{B}(I^2).$

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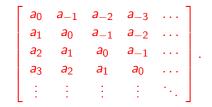
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<i>x</i> ₁₀	<i>x</i> ₁₁	<i>x</i> ₁₂	<i>x</i> ₁₃			<i>x</i> 21	<i>x</i> ₂₂	<i>x</i> 23	<i>x</i> ₂₄		
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The set of fixed points of \(\tau\) are precisely the Toeplitz operators.

These are bounded operators on l², whose matrices with respect to the standard basis have the form:

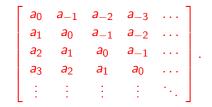


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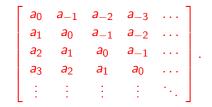
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These are bounded operators on l², whose matrices with respect to the standard basis have the form:



- Clearly the collection of Toeplitz operators forms a vector space.
- But it is not an algebra. As product of two Toeplitz operators need not be Toeplitz.
- For instance, V, V^* are Toeplitz but VV^* is not Toeplitz.

A different product

We may naturally identify the Toeplitz operator

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with a function f on the unit circle whose *n*th Fourier coefficient is a_n . Here f is in the L^{∞} of the unit circle and it is known as the symbol of the Toeplitz operator A.

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- If f, g are two such functions, we have the usual pointwise product (f.g)(z) = f(z)g(z), defined almost everywhere.
- Going back, using the identification made above, one can define a new product on Toeplitz operators, which makes it a commutative algebra!

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- The original formula for the product was complicated. We will come back to this.
- Izumi called the von Neumann algebra (F(τ), •) (or its explicit realization) as the non-commutative Poisson boundary of τ.

• Question: Suppose $\tau : \mathcal{A} \to \mathcal{A}$ is a normal UCP map. Consider the discrete dynamics:

$$\{\tau, \tau^2, \tau^3, \ldots\}.$$

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- How are these Poisson boundaries related?
- ▶ If $\tau(X_0) = -X_0$ and $X_0 \neq 0$. Then $X_0 \in F(\tau^2)$ but $X_0 \notin F(\tau^3)$.

• Let $\tau : \mathcal{A} \to \mathcal{A}$ be a normal UCP map.

 $E(\tau) = \text{ span } \{X : \tau(X) = \lambda X, \text{ for some } \lambda \in \mathbb{T}\}.$

 $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$

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- ► A vector X as above is called a peripheral eigenvector.
- E(τ) is an operator system which may not be closed under multiplication.
- Theorem: Take

$$\mathcal{P}(\tau) := \overline{E(\tau)}^{\|\cdot\|}.$$

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Then $\mathcal{P}(\tau)$ has a new product \circ , which makes it a C^* -algebra.

 $E(\tau) = \text{ span } \{X : \tau(X) = \lambda X, \text{ for some } \lambda \in \mathbb{T} \}.$

$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

- ► A vector X as above is called a peripheral eigenvector.
- E(τ) is an operator system which may not be closed under multiplication.
- Theorem: Take

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- How to compute the product 'o'?

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That is,

$$\theta^{n}\left(\left[\begin{array}{cc} X & 0 \\ 0 & 0 \end{array}\right]\right) = \left[\begin{array}{cc} \tau^{n}(X) & * \\ * & * \end{array}\right]$$

The dilation is unique up to unitary equivalence under a natural minimality condition.

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- Long history. L. Accardi, Hudson and Parthasarathy,...
- The minimal one, called 'Weak Markov Flow' is constructed using Stinespring's theorem. The Time shift gives a semigroup of endomorphisms.
- Further dilation to automorphisms, may or may not exist (depending upon the set-up) and when it exists it is typically not unique.

 Let θ : B → B be minimal dilation of an UCP map τ : A → A. The *-endomorphism property of θ implies that P(θ) is a C*-algebra under multiplication.

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- Theorem: Every peripheral eigenvector X of τ lifts uniquely to a peripheral eigenvector of θ : That is, if $\tau(X) = \lambda X$ with $|\lambda| = 1$, then there exists unique \hat{X} such that

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We set X ∘ Y = PXŶP as the modified product. This defines the peripheral Poisson boundary (P(τ), ∘). As a C*-algebra it is isomorphic to (P(θ), ·). In general the peripheral Poisson boundary does not have von Neumann algebra structure.

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Remark: In general, it is not possible to lift non-peripheral eigenvectors.

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We do not know of any proof the existence of this limit without using dilation theory.



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- Corollary 2: If $\tau(X) = \lambda X$ and $\tau(Y) = \mu Y$ with $|\lambda| = |\mu| = 1$. Then $\tau(X \circ Y) = \lambda . \mu(X \circ Y)$. [Note that, if $\lambda . \mu$ is not in the point spectrum of τ then $X \circ Y = 0$.]

- Corollary 1: If the original von Neumann algebra A is abelian then (P(τ), •) is also abelian.
- Corollary 2: If $\tau(X) = \lambda X$ and $\tau(Y) = \mu Y$ with $|\lambda| = |\mu| = 1$. Then $\tau(X \circ Y) = \lambda . \mu(X \circ Y)$. [Note that, if $\lambda . \mu$ is not in the point spectrum of τ then $X \circ Y = 0$.]
- Corollary 3: The map X → τ(X) is an automorphism on the peripheral boundary (P(τ), ∘).

• Theorem: Let \mathcal{A} be a von Neumann algebra and let $\tau : \mathcal{A} \to \mathcal{A}$ be a normal UCP map.

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Proof: From elementary linear algebra the linear span of peripheral eigenvectors of *τ* and *τⁿ* are same for every *n* ≥ 1. Now the result is not hard to prove from the formula for the Choi-Effros product proved before.

• Definition: Let $\tau : M_d \to M_d$ be a UCP map. Then τ is said to be peripherally automorphic if $X \circ Y = XY$ for every X, Y in $\mathcal{P}(\tau)$.

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Remark: If τ has a faithful invariant state then τ is peripherally automorphic.

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- Remark: If τ has a faithful invariant state then τ is peripherally automorphic.
- ► Theorem: Let $\tau : M_d \to M_d$ be a UCP map with a Choi-Kraus decomposition $\tau(X) = \sum_{i=1}^r L_i^* X L_i$, $\forall X \in M_d$. Then the following are equivalent.

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- ► Theorem: Let $\tau : M_d \to M_d$ be a UCP map with a Choi-Kraus decomposition $\tau(X) = \sum_{i=1}^r L_i^* X L_i$, $\forall X \in M_d$. Then the following are equivalent.
- (i) τ is peripherally automorphic.
- (ii) For $\lambda \in \mathbb{T}$, $\tau(Y) = \lambda Y$ if and only if $YL_i = \lambda L_i Y$ for every $1 \le i \le r$.

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- where $\mathcal{P}(\tau)$ is the peripheral space of τ ,
- and $\mathcal{N}(\tau) = \{X \in M_d : \lim_{n \to \infty} \tau^n(X) = 0\}.$
- Furthermore, $\mathcal{P}(\tau^m) = \mathcal{P}(\tau)$ and $\mathcal{N}(\tau^m) = \mathcal{N}(\tau)$ for every $m \ge 1$.

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THANKS