Optimal estimation of quantum Markov chains

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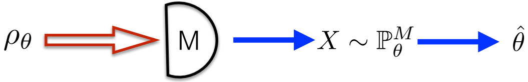
Quantum parameter estimation



- Estimation problem: estimate θ by performing a measurement M on system in state ρ_θ

- What is quantum about this ?
  - fixed measurement: "classical stats" problem with special probabilistic structure
  - "optimal" measurement: need to understand structure of quantum statistical model

- Classical and quantum Cramér-Rao bounds[1]: if $\hat{\theta}$ is unbiased

$$\mathbb{E}\left[(\hat{\theta}-\theta)^T \cdot (\hat{\theta}-\theta)\right] \geq \underbrace{I^M(\theta)^{-1}}_{\text{Classical Fisher info}} \geq \underbrace{F(\theta)^{-1}}_{\text{Quantum Fisher info}}$$

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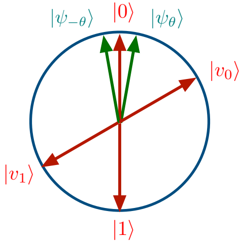
[1] A. Holevo. *Probabilistic and Statistical Aspects of Quantum Theory* (1982); S. L. Braunstein, C. M. Caves, P.R.L. (1994)

Measurements and Fisher information

One parameter pure qubit model: $|\psi_\theta\rangle = e^{-i\theta\sigma_y} = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$

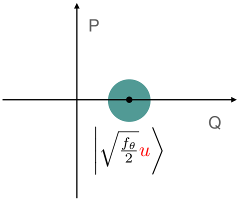
- QFI: $F(\theta) = 4\|\dot{\psi}\|^2 = 4$
- At $\theta = 0$:
    - FI=QFI for any basis $\{|v_i\rangle = e^{-i\tau\sigma_y}|i\rangle\}$ with $\tau \neq 0$
    - classical FI =0 for the standard basis

      for any neighbourhood of $\theta = 0$
      model is not identifiable : $p_\theta(i) = p_{-\theta}(i)$



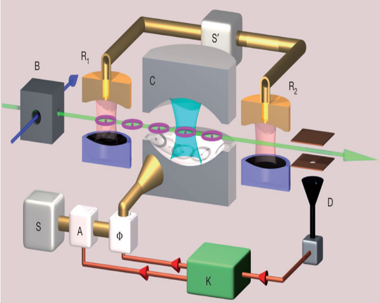
Gaussian shift model: coherent state $\left|\sqrt{\frac{f}{2}}u\right\rangle$

- QFI: $F(u) = f$
- At $u = 0$
    - FI=QFI for quadrature measurement of Q
    - FI =0 for number operator

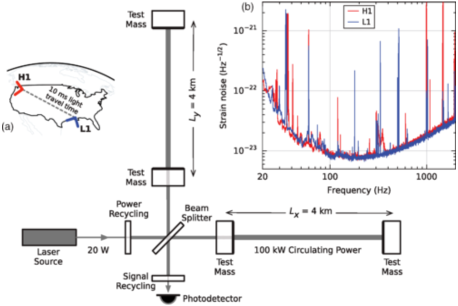


Quantum input-output systems[2]

- Input-output formalism describes controlled open system dynamics
- Quantum filtering, feedback control, quantum networks
- Control and estimation (system identification): two sides of the coin



Feedback control of cavity state in the atom maser
C. Sayrin *et al*, *Nature* (2011)



Advanced LIGO
B. P. Abbott *et al.* *Phys. Rev. Lett.* (2016)

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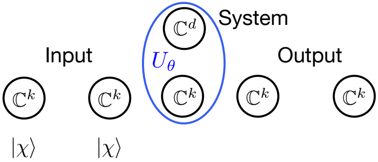
[2] C. W. Gardiner and P. Zoller, *Quantum Noise* (2004)
H. M. Wiseman and G. J. Milburn, *Quantum measurements and control* (2010)

Quantum open system in the input-output formalism (discrete time)



- Unitary interaction U on $\mathbb{C}^d \otimes \mathbb{C}^k$ depends on unknown parameter θ

- System-output state at time n

$$\begin{aligned} |\Psi_\theta(n)\rangle &= U_\theta^{(n)} \cdot \dots \cdot U_\theta^{(1)} |\phi \otimes \chi^{\otimes n}\rangle \\ &= \sum_{i_1,\dots,i_n=1}^{k} K_{i_n}^\theta \dots K_{i_1}^\theta |\phi\rangle \otimes |i_1\rangle \otimes \cdots \otimes |i_n\rangle, \qquad K_i^\theta = \langle i | U_\theta | \chi \rangle \end{aligned}$$

- Assume transition operator $\mathcal{T}_\theta : M_d \to M_d$ is aperiodic and has a unique stationary state $\rho_{\rm ss}$

$$\mathcal{T}_\theta : \rho \mapsto \sum_i K_i^\theta \rho K_i^{\theta *}, \qquad \mathcal{T}_\theta^n(\rho) \longrightarrow \rho_{\rm ss}$$

- Output state becomes stationary for large n

$$\rho_\theta^{(out)}(n) = \mathrm{Tr}_s(|\Psi_\theta(n)\rangle\langle\Psi_\theta(n)|)$$

Output quantum Fisher information

Theorem [M.G., J. Kiukas, J. Math. Phys. (2017), Commun. Math. Phys. (2015)]

The quantum Fisher information $F_\theta(n)$ of the output state scales linearly with n at rate

$$\lim_{n\to\infty}\frac{1}{n}F_\theta(n) = f_\theta = 4\sum_{i=1}^{k}\left[\mathrm{Tr}\left[\rho_{ss}\dot{K}_i^*\dot{K}_i\right] + 2\mathrm{Tr}\left[\mathrm{Im}(K_i\rho_{ss}\dot{K}_i^*)\cdot\mathcal{R}(\mathrm{Im}\sum_j \dot{K}_j^* K_j)\right]\right]$$

where $\mathcal{R}$ is the Moore-Penrose inverse of $\mathcal{I}-\mathcal{T}_\theta$.

- System-ouptut QFI scales quadratically for times shorter than coherence time of $\mathcal{T}_\theta$[3]
- Heisenberg (quadratic) scaling is possible for systems with multiple stationary states[4]
- Full parametrisation: QFI provides information geometry metric on the manifold of identifiable parameters[5]
- Standard sequential measurements typically do not achieve the QFI (even asymptotically)

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[3] K. Macieszczak, MG, I. Lesanovsky, J. P. Garrahan, Phys. Rev. Lett. (2016)

[4] F. Girotti, MG, in preparation

[5] M.G., J. Kiukas, J. Math. Phys. (2017), Commun. Math. Phys. (2015)

Local Asymptotic Normality[6]

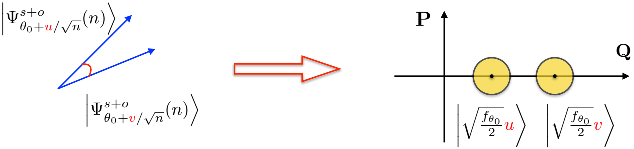
- Suppose θ is known to lie in a $n^{-1/2+\epsilon}$ neighbourhood of θ_0, so $\theta = \theta_0 + u/\sqrt{n}$

Theorem (Local asymptotic normality)

The quantum model $|\Psi^{s+o}_{\theta_0+u/\sqrt{n}}(n)\rangle$ *converges locally to coherent states model (Gaussian shift) .*

$$\lim_{n\to\infty} \left\langle \Psi^{s+o}_{\theta_0+u/\sqrt{n}}(n) \,\middle|\, \Psi^{s+o}_{\theta_0+v/\sqrt{n}}(n) \right\rangle = \left\langle \sqrt{\frac{f_{\theta_0}}{2}}u \,\middle|\, \sqrt{\frac{f_{\theta_0}}{2}}v \right\rangle$$

where $|u\rangle$ *is a one mode coherent state of mean* $(u, 0)$.



- Multiparameter LAN: convergence to multimode coherent state model
- Canonical coordinates of limit CV system are not explicit and may be hard to access sequentially

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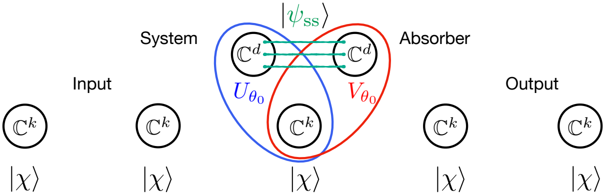
[6] M.G., J. Kiukas, J. Math. Phys. (2017), Commun. Math. Phys. (2015)

Output post-processing with quantum coherent absorber



- Each output unit interacts with a quantum coherent absorber [7] [8] [9] $\mathbb{C}^d$ via V_{θ_0} such that system+absorber have a pure stationary state $|\psi_{\rm ss}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ at $\theta = \theta_0$, i.e.

$$W_{\theta_0} := V_{\theta_0} U_{\theta_0} : |\psi_{\rm ss}\rangle \otimes |\chi\rangle \mapsto |\psi_{\rm ss}\rangle \otimes |\chi\rangle$$

- At $\theta = \theta_0$ system-output state is uncorrelated

$$|\Psi_{\theta_0}\rangle = W^{(n)}_{\theta_0} \dots W^{(1)}_{\theta_0} |\psi_{\rm ss}\rangle \otimes |\chi^{\otimes n}\rangle = |\psi_{\rm ss}\rangle \otimes |\chi^{\otimes n}\rangle, \qquad W_\theta = V_{\theta_0} U_\theta$$

- At $\theta \approx \theta_0$ output carries info about θ as ‘vacuum excitations’

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[7] K. Stannigel, P. Rabl, and P. Zoller, New J. Phys. (2012)

[8] A. Godley, M.G., Quantum (2023)

[9] D. Yang, S. F. Huelga, M. B. Plenio, arXiv:2209.08777

Output translationally invariant modes

- Translationally invariant modes given by 'excitation patterns'
  - Excitation pattern $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k$ with $\alpha_1 = \alpha_k = 1$

    $A^*_\alpha(n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n-k+1} (\sigma_i^+)^{\alpha_1} \cdot \ldots \cdot (\sigma_{i+k-1}^+)^{\alpha_k}$
  - Level 1:

    $A^*_1(n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_i^+$
  - Level 2

    $A^*_{11}(n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \sigma_i^+ \sigma_{i+1}^+, \qquad A^*_{101}(n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n-2} \sigma_i^+ \sigma_{i+2}^+, \quad \ldots$
- Fock space of pattern number states $\mathcal{P} = (k_1, \alpha^{(1)}; \ldots; k_m, \alpha^{(m)})$

$$|\mathcal{P}; n\rangle := \frac{1}{\sqrt{k_1! \ldots k_m!}} \left(A^*_{\alpha^{(1)}}(n)\right)^{k_1} \ldots \left(A^*_{\alpha^{(m)}}(n)\right)^{k_m} |0\rangle^{\otimes n}$$

- Asymptotic orthogonality of number states basis

$|2, 1; n\rangle = \frac{1}{\sqrt{2}} \left(A_1{}^*(n)\right)^2 |0\rangle^{\otimes n} = \frac{1}{\sqrt{2n}} \sum_{i \neq j} |0 \ldots 010 \ldots 010 \ldots 0\rangle$

$|1, 11; n\rangle = A^*_{11}(n)|0\rangle^{\otimes n} = \frac{1}{\sqrt{n}} \sum_i |0 \ldots 0110 \ldots 0\rangle$

$\langle 1, 11; n | 2, 1; n\rangle = O(\frac{1}{\sqrt{n}})$

Central Limit, Poisson Limit & LAN

- Known abs. parameter θ_0. Unknown sys. parameter $\theta = \theta_0 + u/\sqrt{n}$ for some local par. u
- Assume that system+absorber is primitive

Theorem

The output state converges to a (joint) coherent state of all translationally invariant modes

1. The quadratures satisfy CLT:

$$Q_\alpha(n) := \frac{1}{\sqrt{2}}(A_\alpha(n) + A_\alpha^*(n)) \longrightarrow N\left(\mathrm{Re}\sqrt{2}\mu_\alpha u, \frac{1}{2}\right)$$

$$P_\alpha(n) := \frac{1}{\sqrt{2}i}(A_\alpha(n) - A_\alpha^*(n)) \longrightarrow N\left(\mathrm{Im}\sqrt{2}\mu_\alpha u, \frac{1}{2}\right)$$

2. The number operators $N_\alpha(n) = A_\alpha^*(n)A_\alpha(n)$ have asymptotic Poisson distributions

$$N_\alpha(n) \longrightarrow \mathrm{Poisson}(|\mu_\alpha|^2 u^2)$$

where $\mu_\alpha = \mathrm{Tr}\left(\left[\mathcal{J}_\alpha(\mathrm{Id} - \mathcal{T})^{-1} \circ \dot{\mathcal{T}} + \widetilde{\mathcal{J}}_\alpha\right](\rho)\right)$ *with* $\mathcal{J}_\alpha = \mathcal{J}_{\alpha_1} \circ \cdots \circ \mathcal{J}_{\alpha_k}$, $\widetilde{\mathcal{J}}_\alpha = \mathcal{J}_{\alpha_1} \circ \cdots \circ \dot{\mathcal{J}}_{\alpha_k}$ *and* $\mathcal{J}_0 = \mathcal{T}$ *and* $\mathcal{J}_1(\cdot) = K_0 \cdot K_1^*$

3. Output QFI is carried by the Gaussian TIM: $f_{\theta_0} = 4\sum_\alpha |\mu_\alpha|^2$

Law of the counting process

- Counting process $X_1, X_2, \dots, X_n \in \{0, 1\}$ of sequential standard basis measurements
- Total counts operator: $N(n) = X_1 + X_2 + \dots + X_n$

Theorem

The total counts operator converges in distribution to the sum

$$N(n) \xrightarrow{\mathcal{L}} \sum_{\alpha} |\alpha| \cdot Y_{\alpha}$$

where Y_{α} are independent Poisson variables with intensity $|\mu_{\alpha}|^2$.

- This suggests that the counting process performs a joint measurement of all $N_{\alpha}(n)$!
- For large n the trajectory consists of $O(1)$ patterns separated by long sequences of 0s.

  Extract (low order) pattern excitations numbers $C : (x_1, x_2, \dots, x_n) \longrightarrow \mathcal{P} = \{k_{\alpha}\}_{\alpha}$

$$\begin{array}{l}
0,0,0,\boxed{1,0,1},0,0,0,0,0,0,\boxed{1},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\boxed{1,1},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
\boxed{1},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\boxed{1},\\
0,0,0,\boxed{1},0,0,0,0,0,0,0,0,0,0,0,0,\boxed{1},0,0,0,0,0,0,0,0,0,0,\boxed{1,0,1},0,0,0,0,0,0,0,0,0,0,0,\\
0,0,0,0,\boxed{1},0,0,0,0,0,0,\boxed{1},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\boxed{1},
\end{array}$$

Compatibility between modes counting operators and the counting process

$$\begin{array}{l}
0, 0, 0, \boxed{1, 0, 1}, 0, 0, 0, 0, 0, 0, \boxed{1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \boxed{1, 1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
\boxed{1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \boxed{1}, \\
0, 0, 0, \boxed{1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \boxed{1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \boxed{1, 0, 1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
0, 0, 0, 0, \boxed{1}, 0, 0, 0, 0, 0, 0, \boxed{1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \boxed{1},
\end{array}$$

Claim: asymptotically in n...

Counting process effectively measures the TIM number operators and therefore the distribution of $\mathcal{P}$ is given by multidimensional Poisson $\prod_\alpha \mathrm{Poisson}(|\mu_\alpha|^2)$.

- Coarse grained counting measurement with POVM $\{P_\mathcal{P}\}_\mathcal{P}$ where $P_\mathcal{P}$ is the projection onto

$$\mathcal{H}_\mathcal{P} := \mathrm{Lin}\{|x_1, \ldots, x_n\rangle : C(x_1, \ldots, x_n) = \mathcal{P}\}, \qquad \mathcal{P} = \{k_\alpha\}_\alpha$$

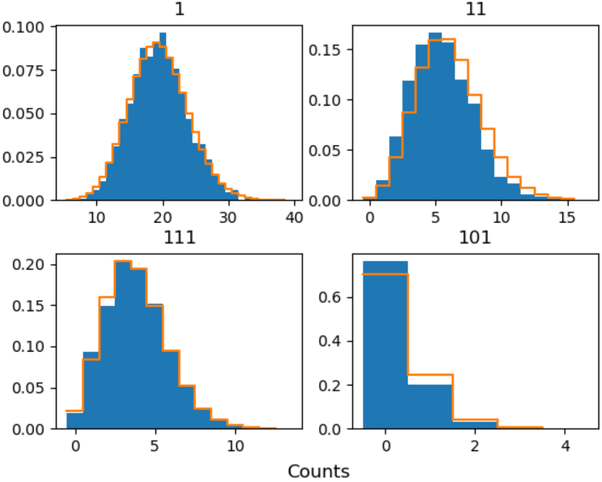
- Output state is a superposition of translationally invariant multiple excitation states

$$|\Psi^{(o)}(n)\rangle = \sum_\mathcal{P} |\mathcal{P}; n\rangle + o(1), \qquad |\mathcal{P}; n\rangle = \prod_\alpha \frac{1}{\sqrt{k_\alpha !}} \left(A^*_\alpha(n)\right)^{k_\alpha} |0\rangle^{\otimes n}$$

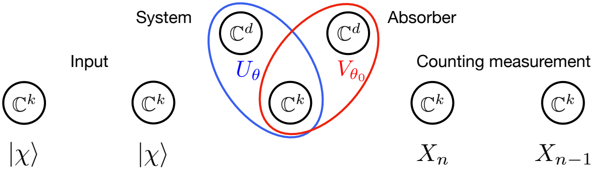
Compatibility between modes counting operators and the counting process

Claim: asymptotically in n...

Counting process effectively measures the TIM number operators and therefore the distribution of $\mathcal{P}$ is given by multidimensional Poisson $\prod_\alpha \mathrm{Poisson}(|\mu_\alpha|^2)$.



Two stage adaptive optimal measurement strategy



- Optimal measurement depends on the unknown parameter θ
- Design two stage adaptive measurement procedure:
- Stage 1: use standard measurements on a small portion of output $\tilde{n} = n^{1-\epsilon}$ to compute a preliminary estimator $\tilde{\theta}_n$
- Stage 2: Set absorber parameter to $\theta_0 := \tilde{\theta}_n - \delta_n$ where $\delta_n = n^{-1/2+3\epsilon}$ and perform counting measurement on output of sys+abs
  - collect pattern statistics $X_1, \ldots, X_n \longrightarrow \mathcal{P} = \{K_\alpha\}_\alpha$.
  - Optimal linear estimator based on the homodyne limit of counting

$$\hat{\theta}_n = \tilde{\theta}_n + \frac{2}{f_\theta \delta_n} \frac{1}{n} \left(\sum_\alpha K_\alpha \right) - \frac{\delta_n}{2}$$

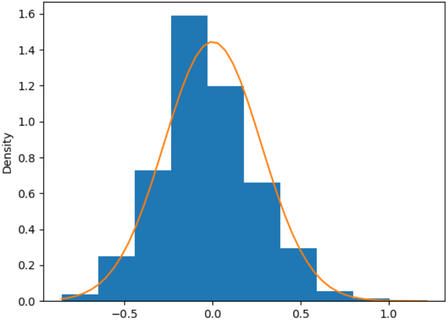
Asymptotic normality and optimality of linear estimator

- Optimal linear estimator for local parameter u with $\theta = \tilde{\theta}_n + u/\sqrt{n}$

$$\hat{u}_n = \sqrt{n}\left[\frac{2}{f_\theta \delta_n}\frac{1}{n}\left(\sum_\alpha K_\alpha\right) - \frac{\delta_n}{2}\right]$$

- Asymptotic normality and achievability of QCRB

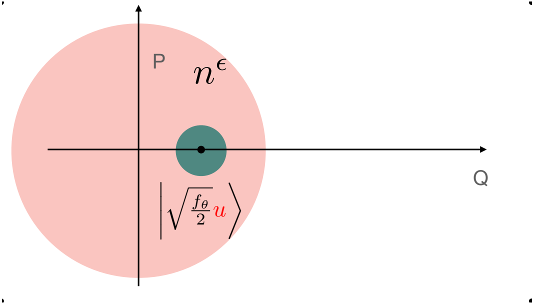
$$\hat{u}_n \longrightarrow N\left(0, \frac{1}{f_\theta}\right)$$



Null measurement is problematic

- Localisation:
  use $n^{1-\epsilon} \ll n$ samples to compute preliminary estimator $\tilde{\theta}_n$
- Concentration $\longrightarrow \theta = \tilde{\theta}_n + u/\sqrt{n}$ with $|u| \leq n^{\epsilon}$
- Asymptotic normality:

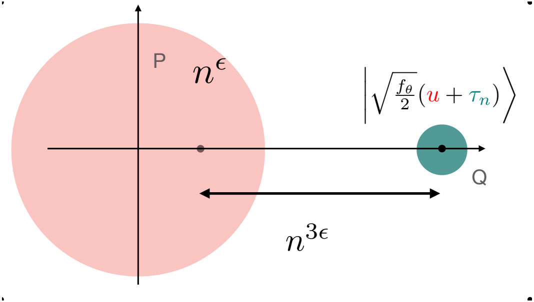
  with absorber set at $\theta_0 = \tilde{\theta}_n$, output state model is equivalent to $\left| \sqrt{\frac{f_\theta}{2}} u \right\rangle$
- Quadrature Q measurement is optimal : $\hat{u} \sim N(u, \frac{1}{2})$
- Number (null) measurement cannot distinguish u and $-u$ (non-identifiability)



Displaced-null measurement is asymptotically optimal

- Set absorber to $\theta_0 = \tilde{\theta}_n - \frac{\tau_n}{\sqrt{n}}$ with $\tau_n = n^{3\epsilon} \longrightarrow$ local parameter $u + \tau_n$
- Asymptotic normality: output state model is equivalent to $\left| \sqrt{\frac{f_\theta}{2}} \left(u + \tau_n \right) \right\rangle$
- Counting measurement becomes linear (homodyne):

$$Y \sim \text{Poisson}\left(\frac{f_\theta}{4} (u + \tau_n)^2 \right) \xrightarrow{\qquad} \hat{u} := \frac{2}{\tau_n f_\theta} Y - \frac{\tau_n}{2} \sim N\left(u, \frac{1}{f_\theta} \right)$$



Conclusions and Outlook

- Primitive quantum Markov chains satisfy LAN with closed form formula for QFI
- Coherent absorber rotates output to vacuum state at matching parameters
- Translationally invariant modes have asymptotically coherent states
- Counting performs simultaneous measurement of all modes number operators
- Two step adaptive procedure with displaced-null strategy achieves QFI
- Simple final estimator based on pattern counts
- Extension to multidimensional parameters, continuous time