Eternal Adiabaticity and Long-Term Stability of Perturbed Quantum Symmetries

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with

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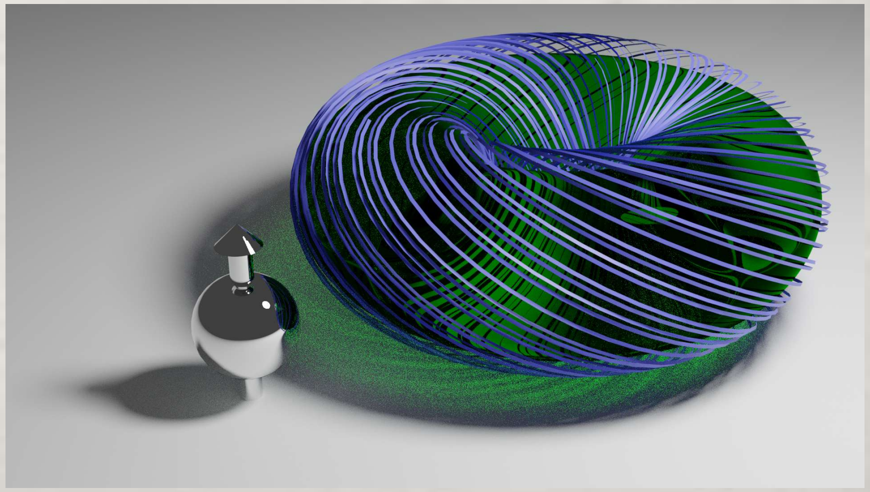
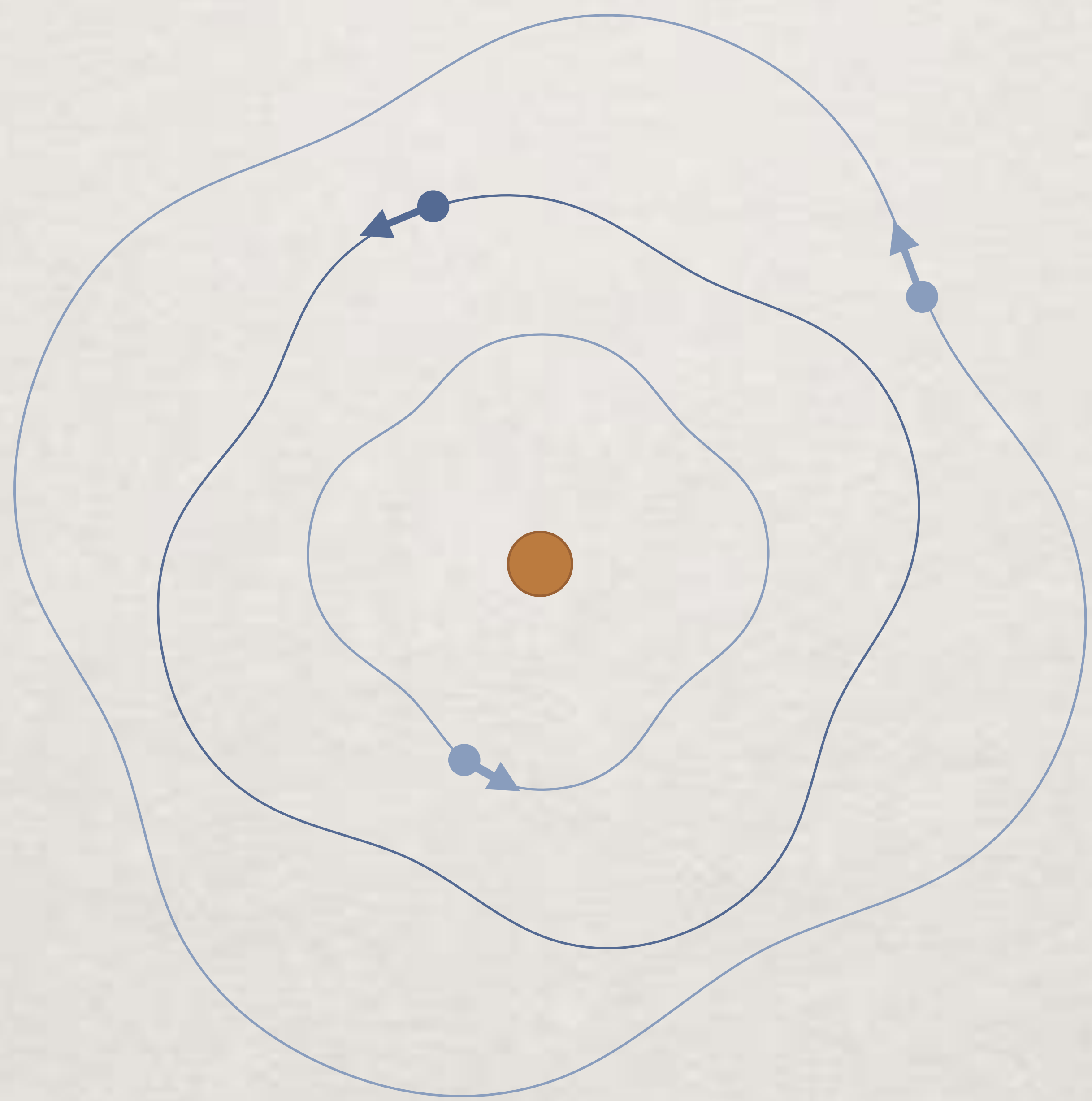
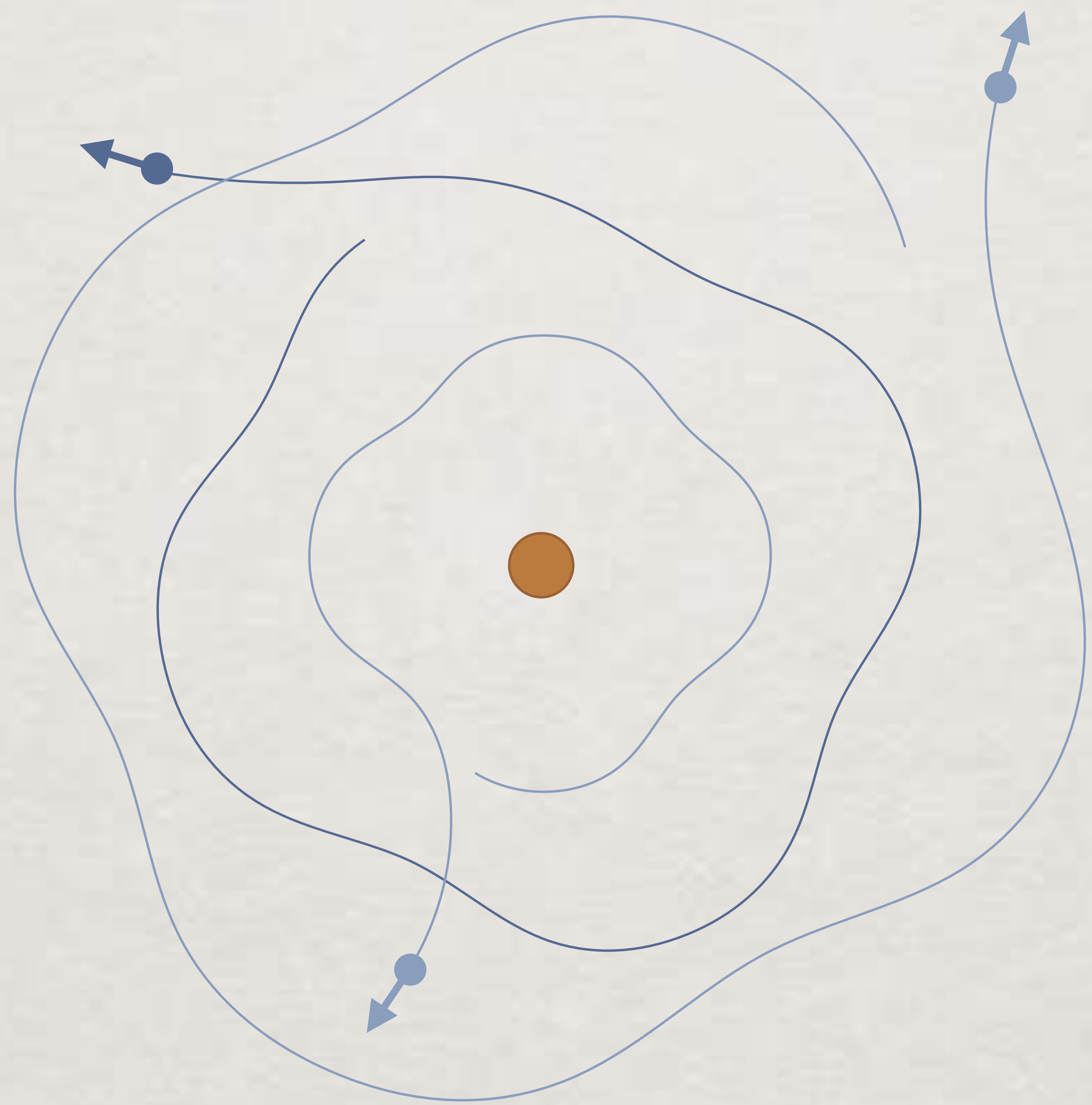


Long-Term Stability against Small Perturbations



stability of the solar system

Long-Term Stability against Small Perturbations



stability of the solar system

KAM Theorem in Classical Mechanics

Kolmogorov, Arnold, and Moser, 1950s-1960s

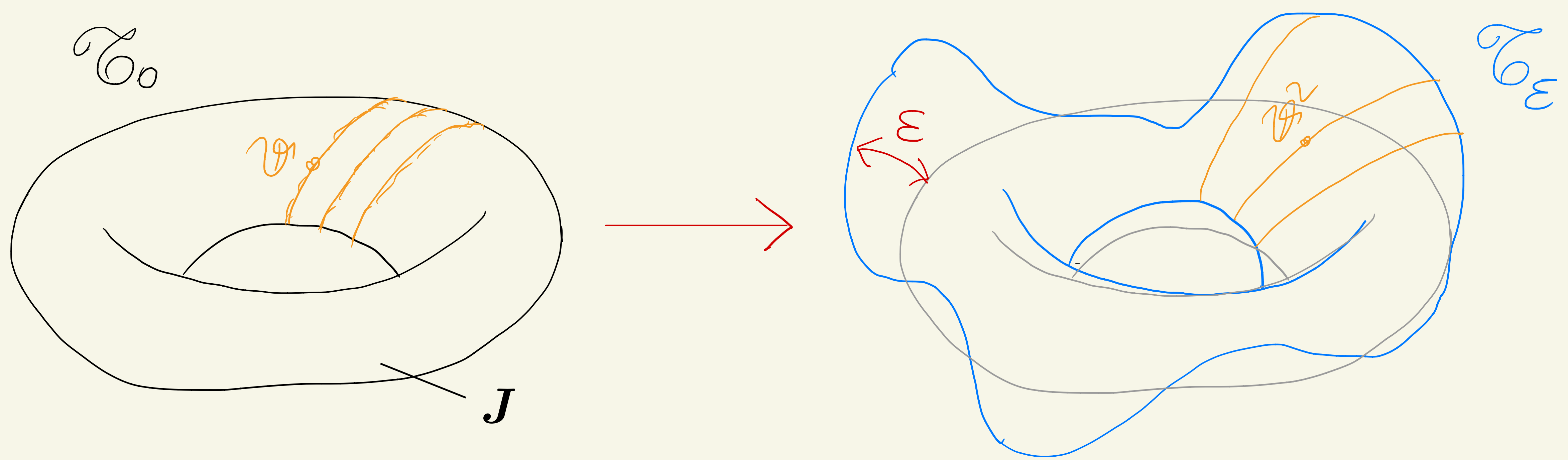
long-term stability against small perturbations

integrable system $H_0(\boldsymbol{J})$

$$\begin{cases} \dot{\boldsymbol{J}} = -\nabla_{\boldsymbol{\theta}} H_0(\boldsymbol{J}) = 0 \\ \dot{\boldsymbol{\theta}} = \nabla_{\boldsymbol{J}} H_0(\boldsymbol{J}) = \boldsymbol{\omega}(\boldsymbol{J}) \end{cases}$$

perturbed system $H_0(\boldsymbol{J}) + \varepsilon V(\boldsymbol{\theta}, \boldsymbol{J})$

$$\begin{cases} \dot{\boldsymbol{J}} = -\nabla_{\boldsymbol{\theta}} H(\boldsymbol{\theta}, \boldsymbol{J}) = -\varepsilon \nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}, \boldsymbol{J}) \neq 0 \\ \dot{\boldsymbol{\theta}} = \nabla_{\boldsymbol{J}} H(\boldsymbol{\theta}, \boldsymbol{J}) = \boldsymbol{\omega}(\boldsymbol{J}) + \varepsilon \nabla_{\boldsymbol{J}} V(\boldsymbol{\theta}, \boldsymbol{J}) \end{cases}$$



drawing by Paolo Facchi

Outline

- **KAM-like stability** for finite-dimensional quantum systems

- **robust symmetries** vs **fragile symmetries**

  All symmetries are conserved,
  but some symmetries are more conserved than others.

- **eternal adiabaticity** of the evolutions of perturbed finite-dimensional quantum systems

"KAM Stability for Conserved Quantities in Finite-Dimensional Quantum Systems," PRL **126**, 150401 (2021);
"Eternal Adiabaticity in Quantum Evolution," PRA **103**, 032214 (2021).

Adiabatic Theorem

Adiabatic Theorem

A. Messiah, *Quantum Mechanics* (Dover, New York, 2017).

- slow driving

$$\frac{\mathrm{d}}{\mathrm{d}t}U_\varepsilon(t) = -\mathrm{i}H(\varepsilon t)U_\varepsilon(t)$$

- instantaneous eigenstates

$$H(t) = \sum_k E_k(t)P_k(t)$$

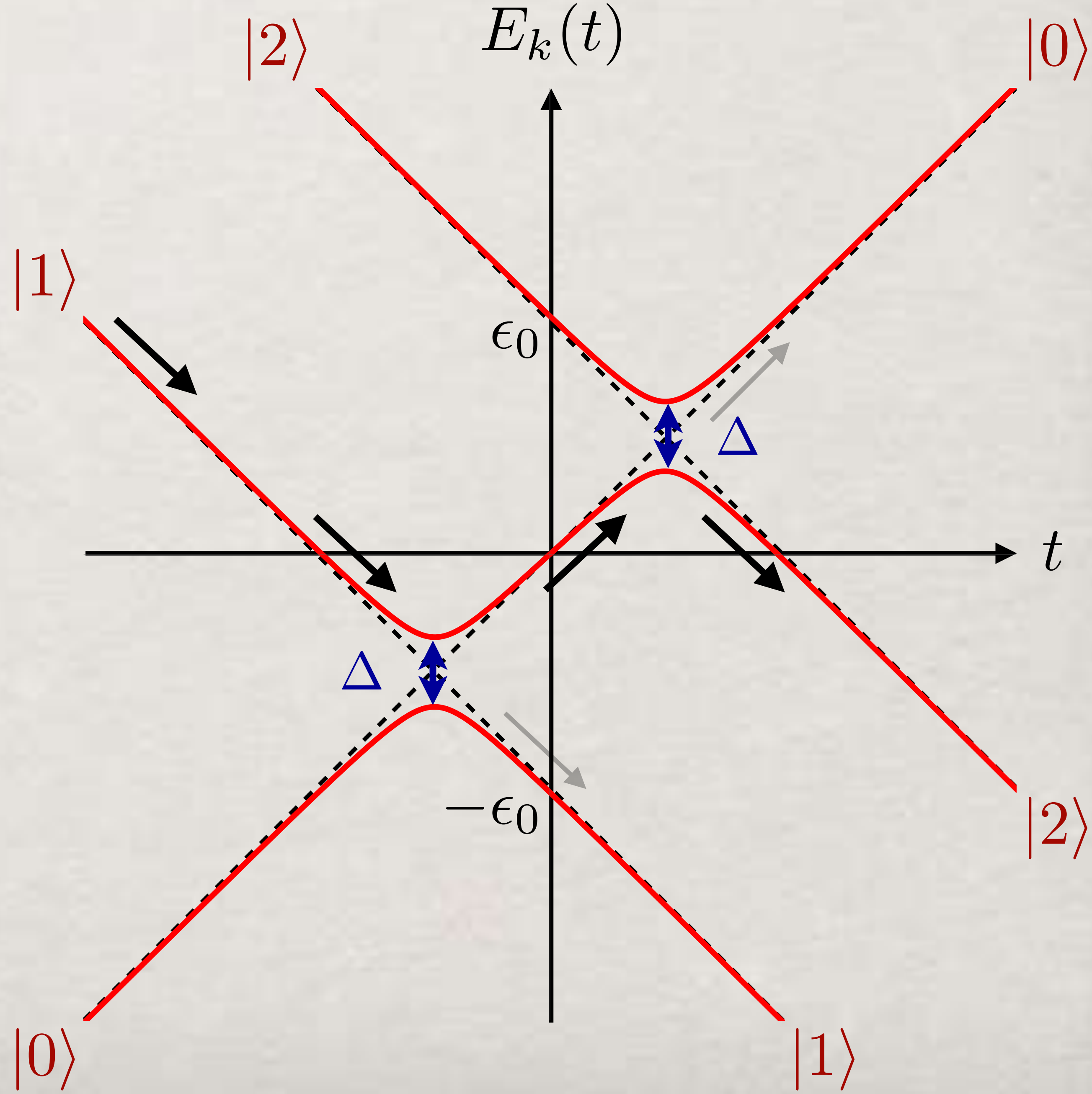
- adiabatic evolution

$$U_\varepsilon(t)P_\ell(0) = P_\ell(\varepsilon t)U_\varepsilon(t) + O(\varepsilon)$$

$$(0 \le t \le T/\varepsilon)$$

- example

$$H(t) = \begin{pmatrix} \alpha t & \Delta & \Delta \\ \Delta & -\alpha t - \epsilon_0 & 0 \\ \Delta & 0 & -\alpha t + \epsilon_0 \end{pmatrix}$$



Perturbed Evolution and Adiabaticity

perturbed evolution

$$\frac{\mathrm{d}}{\mathrm{d}t} U_\varepsilon(t) = -\mathrm{i}(H + \varepsilon V) U_\varepsilon(t)$$

$$\tilde{U}_\varepsilon(t) = \mathrm{e}^{\mathrm{i}t\varepsilon V} U_\varepsilon(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{U}_\varepsilon(t) = -\mathrm{i} H(\varepsilon t) \tilde{U}_\varepsilon(t), \quad H(t) = \mathrm{e}^{\mathrm{i}tV} H \mathrm{e}^{-\mathrm{i}tV}$$

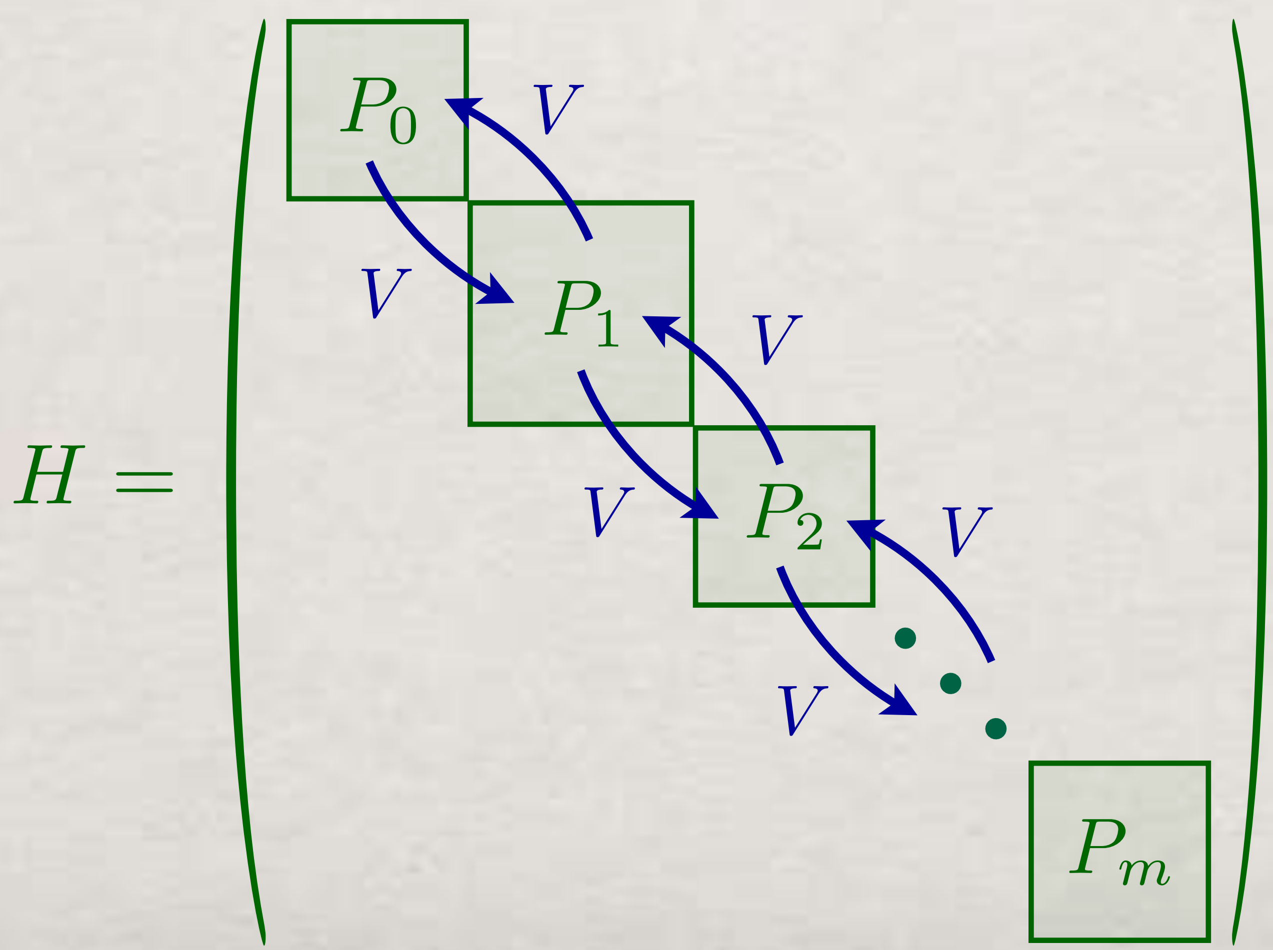
isospectral driving

Perturbed Evolution and Adiabaticity

$$\frac{\mathrm{d}}{\mathrm{d}t}U_\varepsilon(t) = -\mathrm{i}(H + \varepsilon V)U_\varepsilon(t)$$

- spectral representation

$$H = \sum_\ell e_\ell P_\ell, \quad P_k P_\ell = \delta_{k\ell} P_k, \quad \sum_\ell P_\ell = 1, \quad e_k \neq e_\ell \text{ for } k \neq \ell$$



m distinct eigenvalues

Perturbed Evolution and Adiabaticity

- zeroth order

$$\delta(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}tH}\| \le \varepsilon t \|V\|$$

$$t \lesssim 1$$

- first order (Zeno)

$$\delta_Z(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_Z)}\| \le \frac{2\sqrt{m}}{\eta}\varepsilon\|V\|(1+\varepsilon\|V\|t)$$

$$V_Z = \sum_{\ell=0}^{m-1} P_\ell V P_\ell, \quad [H, V_Z] = 0, \quad \eta = \min_{k\neq\ell} |e_k - e_\ell|$$

$$t \lesssim 1/\varepsilon$$

T. Kato, "On the Adiabatic Theorem of Quantum Mechanics," J. Phys. Soc. Jpn. **5**, 435 (1950);
Quantum **3**, 152 (2019);
PRL **126**, 150401 (2021).

Perturbed Evolution and Adiabaticity

- zeroth order

$$\delta(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}tH}\| \le \varepsilon t \|V\|$$

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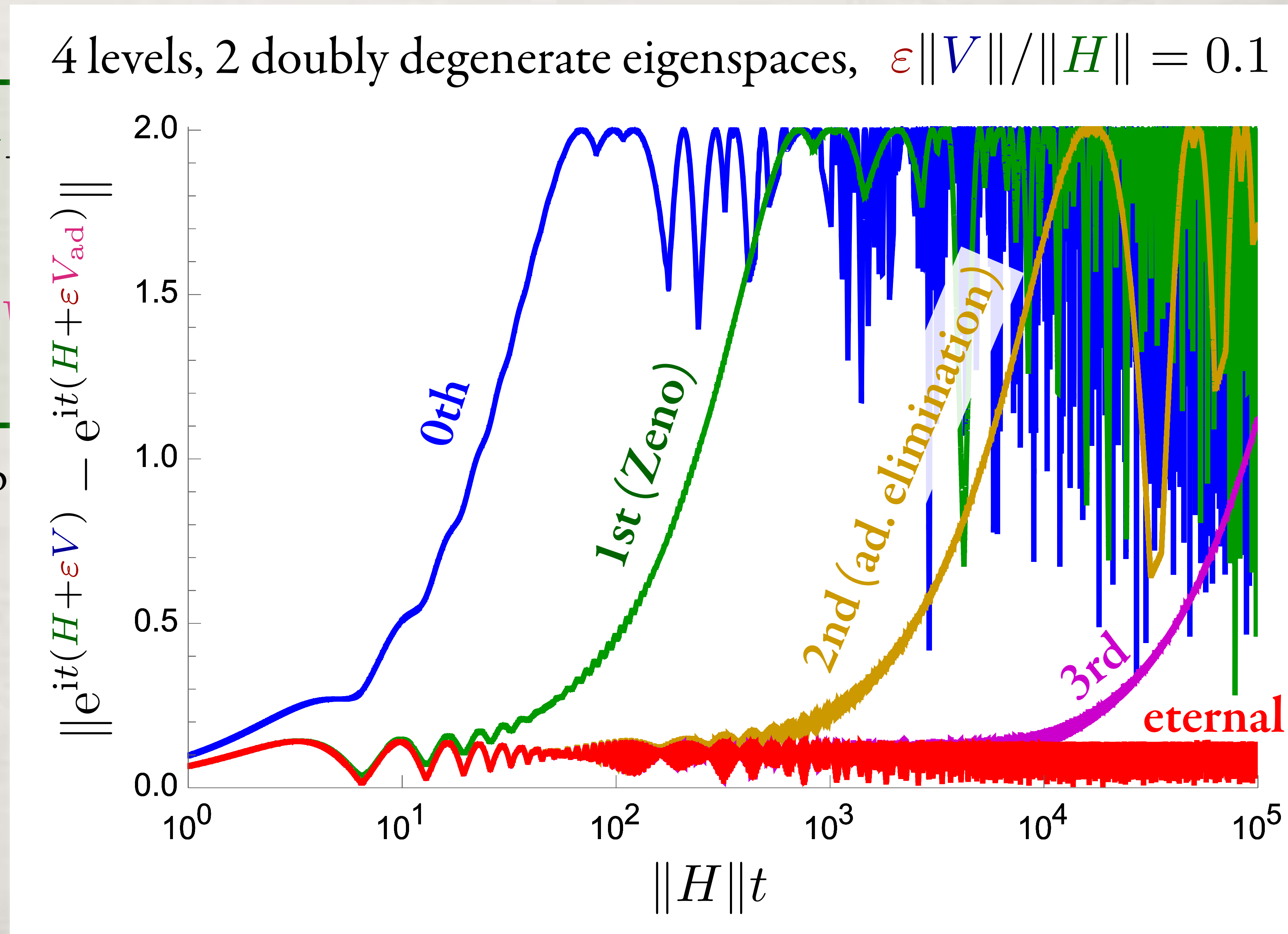
$$V_Z = \sum_{\ell=0}^{m-1} P_\ell$$

T. Kato, “On the Adiab



Perturbed Evolution and Adiabaticity

- zeroth order

$$\delta(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}tH}\| \le \varepsilon t \|V\| \qquad t \lesssim 1$$

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$$V_Z = \sum_{\ell=0}^{m-1} P_\ell V P_\ell, \quad [H, V_Z] = 0, \quad \eta = \min_{k\ne\ell} |e_k - e_\ell| \qquad t \lesssim 1/\varepsilon$$

- eternal adiabaticity

$$\delta_\infty(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V_\infty(\varepsilon)]}\| < \frac{7\sqrt{m}}{\eta}\varepsilon\|V\| \qquad \forall t$$

$$H + \varepsilon V_\infty(\varepsilon) = U_\varepsilon^\dagger (H + \varepsilon V) U_\varepsilon, \quad [H, V_\infty(\varepsilon)] = 0, \quad U_\varepsilon = 1 + O(\varepsilon)$$

"KAM Stability for Conserved Quantities in Finite-Dimensional Quantum Systems," PRL **126**, 150401 (2021); "Eternal Adiabaticity in Quantum Evolution," PRA **103**, 032214 (2021).

Eternal Adiabaticity

$$\delta_\infty(t) = \|e^{it(H+\varepsilon V)} - e^{it[H+\varepsilon V_\infty(\varepsilon)]}\| < \frac{7\sqrt{m}}{\eta}\varepsilon\|V\| \quad \forall t$$

$$H + \varepsilon V_\infty(\varepsilon) = U_\varepsilon^\dagger(H+\varepsilon V)U_\varepsilon, \quad [H, V_\infty(\varepsilon)] = 0, \quad U_\varepsilon = 1 + O(\varepsilon)$$

$$\begin{aligned}
&\|e^{it(H+\varepsilon V)} - e^{it[H+\varepsilon V_\infty(\varepsilon)]}\| \\
&= \|e^{it(H+\varepsilon V)} - U_\varepsilon^\dagger e^{it(H+\varepsilon V)}U_\varepsilon\| \\
&= \|U_\varepsilon e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon V)}U_\varepsilon\| \\
&= \|[U_\varepsilon, e^{it(H+\varepsilon V)}]\| \\
&= \|[U_\varepsilon - 1, e^{it(H+\varepsilon V)}]\| \\
&\le 2\|U_\varepsilon - 1\| = O(\varepsilon)
\end{aligned}$$

Schrieffer-Wolff transformation

$$U_\varepsilon = \sum_\ell W_\ell|W_\ell|^{-1}, \quad W_\ell = W_\ell P_\ell$$

Bloch equation

$$W_\ell + \varepsilon S_\ell(VW_\ell - W_\ell V W_\ell) - P_\ell = 0$$

reduced resolvent

$$S_\ell = \sum_{k\neq\ell}\frac{1}{e_k - e_\ell}P_k$$

PRL **126**, 150401 (2021);
PRA **103**, 032214 (2021).

C. Bloch, Nucl. Phys. **6**, 329 (1958);
J. des Cloizeaux, Nucl. Phys. **20**, 321 (1960).

Isospectral Perturbations

$$\tilde{H} = H + O(\varepsilon), \quad \tilde{H} = \sum_{\ell=0}^{\tilde{m}-1} \tilde{e}_\ell \tilde{P}_\ell, \quad H = \sum_{\ell=0}^{\tilde{m}-1} e_\ell P_\ell$$

$$\delta_\infty(t) = \|\mathrm{e}^{\mathrm{i}t\tilde{H}} - \mathrm{e}^{\mathrm{i}tH}\| = \left\| \sum_{\ell=0}^{\tilde{m}-1} (\mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} \tilde{P}_\ell - \mathrm{e}^{\mathrm{i}te_\ell} P_\ell) \right\| \leq \underline{\left\| \sum_{\ell=0}^{\tilde{m}-1} \mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} (\tilde{P}_\ell - P_\ell) \right\|} + \underline{\left\| \sum_{\ell=0}^{\tilde{m}-1} (\mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} - \mathrm{e}^{\mathrm{i}te_\ell}) P_\ell \right\|}$$

First term:

$$\left\| \sum_{\ell=0}^{\tilde{m}-1} \mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} (\tilde{P}_\ell - P_\ell) \right\| \leq \sum_{\ell=0}^{\tilde{m}-1} \|\tilde{P}_\ell - P_\ell\| = O(\varepsilon)$$

Second term:

$$\left\| \sum_{\ell=0}^{\tilde{m}-1} (\mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} - \mathrm{e}^{\mathrm{i}te_\ell}) P_\ell \right\| = \max_\ell |\mathrm{e}^{\mathrm{i}t\tilde{e}_\ell} - \mathrm{e}^{\mathrm{i}te_\ell}| = 2 \max_\ell \left| \sin\left(t \frac{\tilde{e}_\ell - e_\ell}{2} \right) \right| = 2$$

at $t = \dfrac{\pi}{\tilde{e}_\ell - e_\ell}$

$$\delta_\infty(t) = O(\varepsilon), \ \forall t$$

$$\Updownarrow$$

$$\mathrm{Spec}(H) = \mathrm{Spec}(\tilde{H})$$

A Previous Work

W. Scherer, "Superconvergent Perturbation Method in Quantum Mechanics," PRL **74**, 1495 (1995);
W. Scherer, "Quantum Averaging II: Kolmogorov's Algorithm," JPA **30**, 2825 (1997).

- A complete analogue of Kolmogorov's perturbation algorithm in classical mechanics is constructed for self-adjoint operators.

$$H + \varepsilon V^{(n)}(\varepsilon) = U_\varepsilon^{(n)\dagger}(H + \varepsilon V) U_\varepsilon^{(n)} + O(\varepsilon^{2^n})$$

$$[H, V^{(n)}(\varepsilon)] = 0$$

$$\Rightarrow \quad \delta_n(t) = \| \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V^{(n)}(\varepsilon)]} \| = O(\varepsilon)$$

$$\text{for} \quad t \lesssim O(\varepsilon^{2^n - 1})$$

Robustness of Symmetries

Conserved Quantities and Symmetries

$$[S, H] = 0$$

conservation law ⟷ symmetry

$$S_t = \mathrm{e}^{\mathrm{i}tH} S \mathrm{e}^{-\mathrm{i}tH} = S \qquad \mathrm{e}^{\mathrm{i}\alpha S} H \mathrm{e}^{-\mathrm{i}\alpha S} = H$$

$$H = \sum_\ell e_\ell P_\ell \qquad S = \sum_\ell P_\ell S P_\ell$$

$$= \begin{pmatrix} \begin{matrix} e_0 & \\ & e_0 \end{matrix} & & & \\ & \begin{matrix} e_1 & & \\ & e_1 & \\ & & e_1 \end{matrix} & & \\ & & \begin{matrix} e_2 & \\ & e_2 \end{matrix} & \\ & & & \ddots \end{pmatrix} \qquad = \begin{pmatrix} S_0 & & & \\ & S_1 & & \\ & & S_2 & \\ & & & \ddots \end{pmatrix}$$

Robustness of Symmetries

$$S_t^\varepsilon = \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} = S + O(\varepsilon), \quad \forall \|V\| = 1$$

at least for $t \lesssim O(1)$

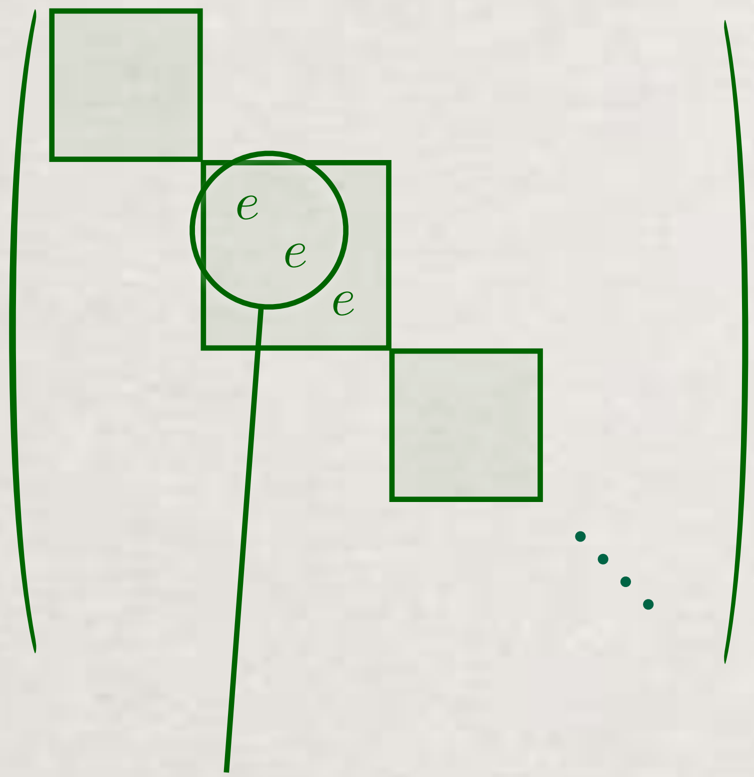
$$\begin{aligned}
\|S_t^\varepsilon - S\| &= \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}tH} S \mathrm{e}^{-\mathrm{i}tH}\| \\
&= \|\mathrm{e}^{-\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S - S \mathrm{e}^{-\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}\| \\
&= \|[\mathrm{e}^{-\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}, S]\| \\
&= \|[\mathrm{e}^{-\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - 1, S]\| \\
&\leq 2\|S\| \|\mathrm{e}^{-\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - 1\| \\
&= 2\|S\| \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}tH}\| \\
&\leq 2\varepsilon t \|V\| \|S\| \\
&= O(\varepsilon), \quad \text{for} \quad t \lesssim O(1)
\end{aligned}$$

$$\|UAW\| = \|A\|$$

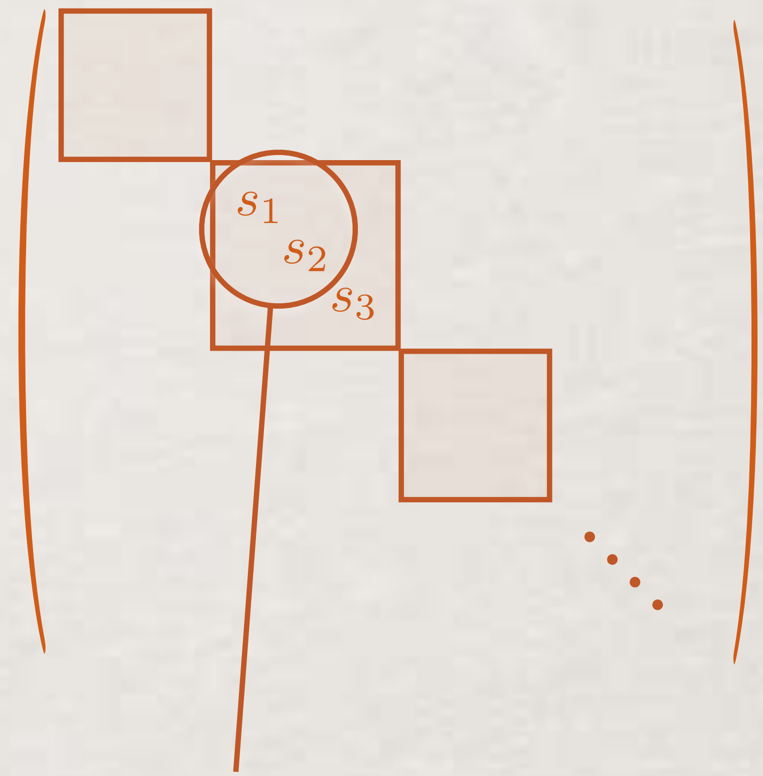
for longer times??

Symmetries Breaking the Degeneracy

$H =$



$S =$



$$H = e\Big(|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|\Big)$$

$$S = s_1|u_1\rangle\langle u_1| + s_2|u_2\rangle\langle u_2|$$

$\Delta = s_1 - s_2 \neq 0$ breaking the degeneracy

perturbation

$$V = |u_1\rangle\langle u_2| + |u_2\rangle\langle u_1|$$

initial state

$$|\psi_0\rangle = |u_1\rangle$$

$$\langle S_t^\varepsilon\rangle - \langle S\rangle = -\Delta\sin^2\varepsilon t = O(1)$$

at $t = \pi/(2\varepsilon)$

This type of symmetries are fragile.

Symmetries Respecting the Degeneracy

$$H = \begin{pmatrix} e_0 & & & & & & & \\ & e_0 & & & & & & \\ & & e_1 & & & & & \\ & & & e_1 & & & & \\ & & & & e_1 & & & \\ & & & & & e_2 & & \\ & & & & & & e_2 & \\ & & & & & & & \ddots \end{pmatrix} \qquad S = \begin{pmatrix} s_0 & & & & & & & \\ & s_0 & & & & & & \\ & & s_1 & & & & & \\ & & & s_1 & & & & \\ & & & & s_1 & & & \\ & & & & & s_2 & & \\ & & & & & & s_2 & \\ & & & & & & & \ddots \end{pmatrix} = f(H)$$

$$\begin{aligned} \|S_t^\varepsilon - S\| &= \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - S\| \\ &= \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S - S\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}\| \\ &= \|[\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}, S]\| \\ &= \|[\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon \hat{V})}, S]\| \quad \longleftarrow \quad [H, \hat{V}] = 0 \\ &\le 2\|S\|\|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon \hat{V})}\| \end{aligned}$$

Symmetries Respecting the Degeneracy

$$H = \begin{pmatrix} e_0 & & & & & & \\ & e_0 & & & & & \\ & & e_1 & & & & \\ & & & e_1 & & & \\ & & & & e_1 & & \\ & & & & & e_2 & \\ & & & & & & e_2 \\ & & & & & & & \ddots \end{pmatrix} \qquad S = \begin{pmatrix} s_0 & & & & & & \\ & s_0 & & & & & \\ & & s_1 & & & & \\ & & & s_1 & & & \\ & & & & s_1 & & \\ & & & & & s_2 & \\ & & & & & & s_2 \\ & & & & & & & \ddots \end{pmatrix} = f(H)$$

$$\begin{aligned} \|S_t^\varepsilon - S\| &= \|e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S\| \\ &= \|e^{it(H+\varepsilon V)} S - S e^{it(H+\varepsilon V)}\| \\ &= \|[e^{it(H+\varepsilon V)}, S]\| \\ &= \|[e^{it(H+\varepsilon V)} - e^{it[H+\varepsilon V_\infty(\varepsilon)]}, S]\| \\ &\le 2\|S\|\|e^{it(H+\varepsilon V)} - e^{it[H+V_\infty(\varepsilon)]}\| \\ &\le \frac{14\sqrt{m}}{\eta}\varepsilon\|V\|\|S\|, \quad \forall t \end{aligned}$$

$\longleftarrow \quad [H, V_\infty(\varepsilon)] = 0$

robust symmetries

Robust / Fragile Symmetries

PRL **126**, 150401 (2021)

- S is a symmetry

$$\Leftrightarrow \quad S \in \{H\}' = \{A : [A, H] = 0\}$$

- symmetry S is *fragile* if

$$\exists V \quad \text{s.t.} \quad \sup_{t\in\mathbb{R}} \|e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S\| = O(1), \quad \forall \varepsilon > 0.$$

- symmetry S is *robust*

$$\Leftrightarrow \quad S = f(H) \in \{H\}'' = \{A : [A, B] = 0, \ \forall B \in \{H\}'\}$$

$$\Leftrightarrow \quad \|e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S\| \le \frac{14\sqrt{m}}{\eta} \varepsilon \|V\| \|S\|,$$

$$\forall \varepsilon > 0, \ \forall V, \ \forall t \in \mathbb{R}.$$

Example: Four-Level System

$$H = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & e & \\ & & & e \end{pmatrix}$$

perturbation V
observable M
initial state $|\psi_0\rangle$

} randomly chosen

$$M = M_{\text{off}} + S_{\text{fragile}} + S_{\text{robust}}$$

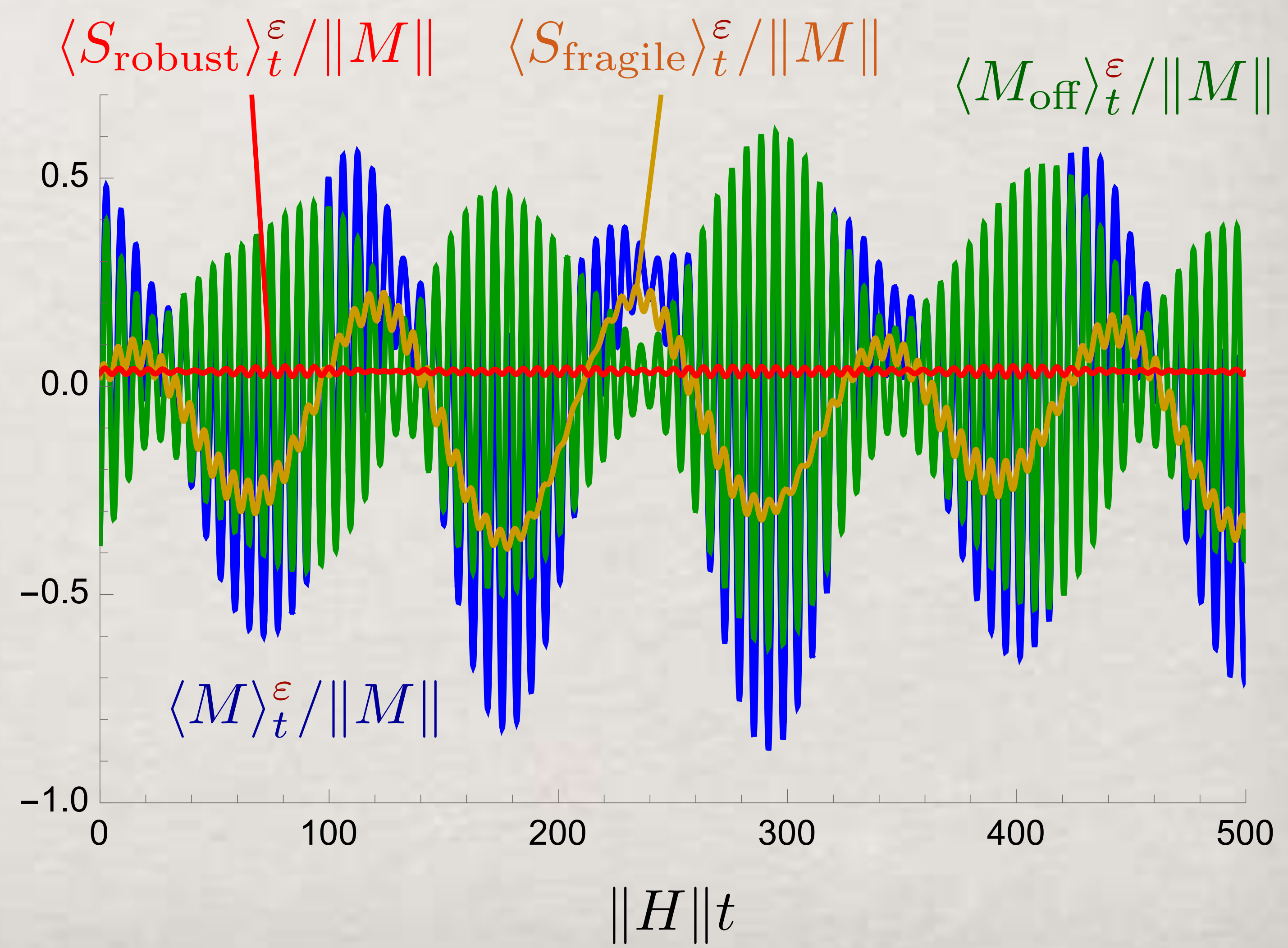
$$S = \sum_{\ell=0}^{m-1} P_\ell M P_\ell$$

$$S_{\text{robust}} = \sum_{\ell} \frac{1}{d_k} \mathrm{Tr}(P_\ell M P_\ell) P_\ell$$

$$S_{\text{fragile}} = S - S_{\text{robust}}$$

$$M_{\text{off}} = M - S$$

$\varepsilon \|V\| / \|H\| = 0.1$



Example: Spin Chain

$$H = -J \sum_{n=1}^{N} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}, \quad \boldsymbol{\sigma}_{N+1} = \boldsymbol{\sigma}_1$$

$N = 4$, V and $|\psi_0\rangle$ randomly chosen

$$\varepsilon \|V\| = 0.02 \|H\|$$

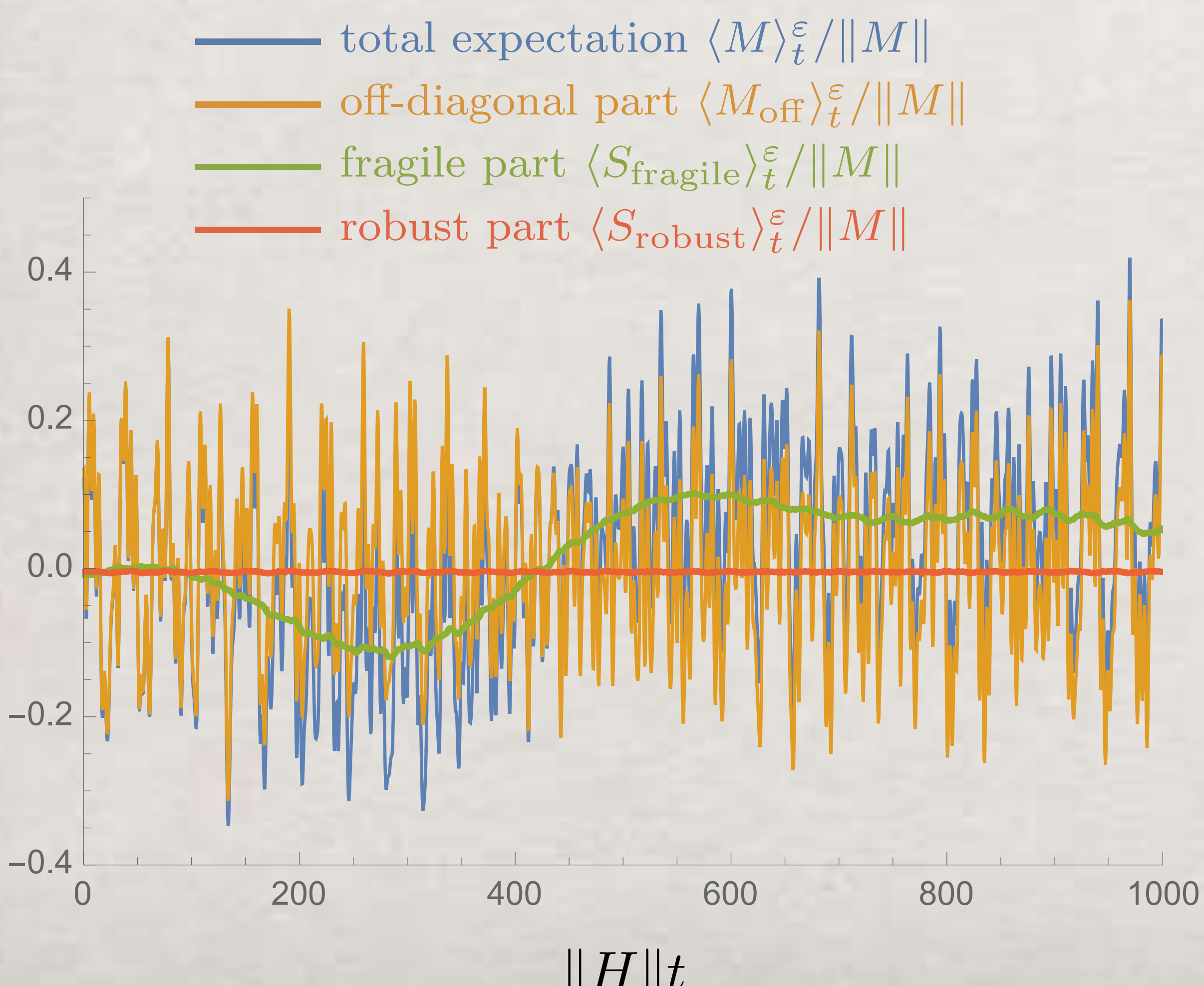
Integral Charges $\{Q_1, \dots, Q_N\}$

$$Q_1 = \sum_{n=1}^{N} \sigma_{n,z}, \quad Q_2 = H$$

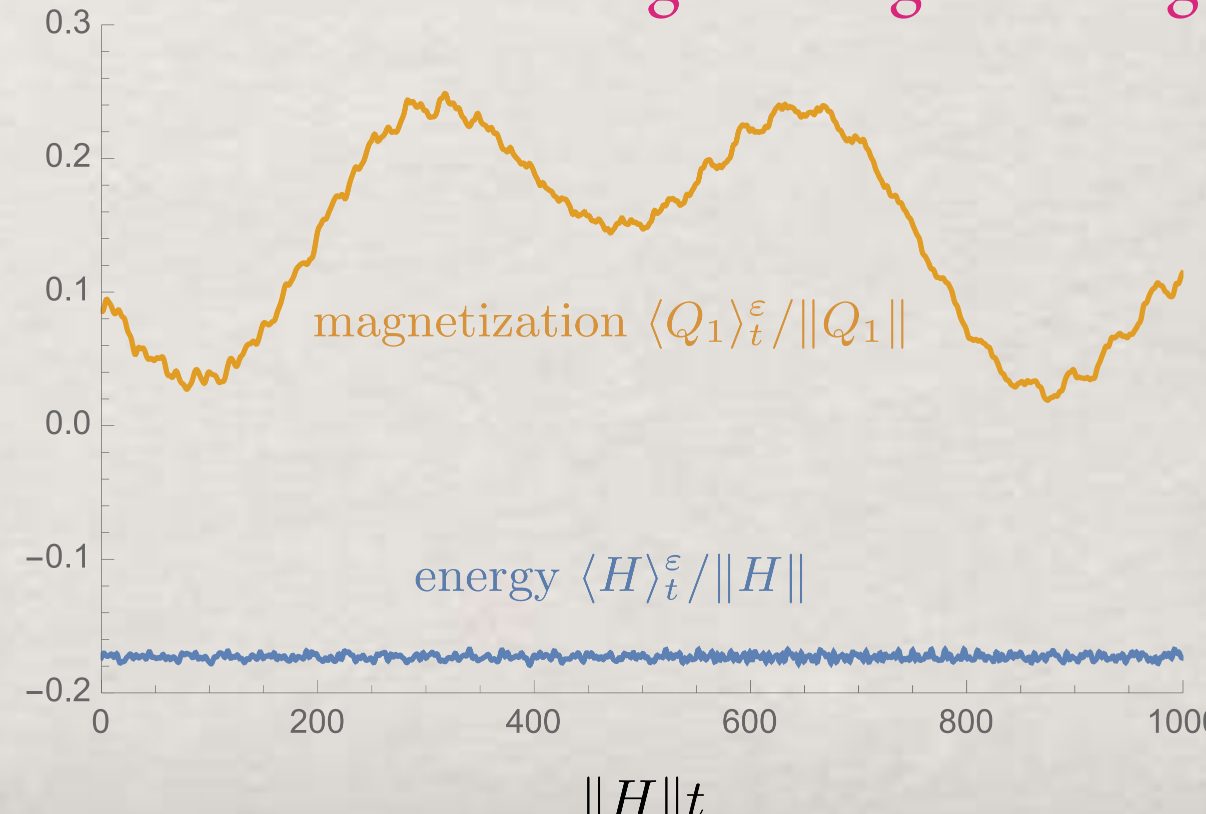
$$Q_{j+1} = -\mathrm{i}[B, Q_j] \quad (j = 2, \dots, N-1)$$

$$B = \frac{1}{2} \sum_{n=1}^{N} n \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}$$

randomly chosen M



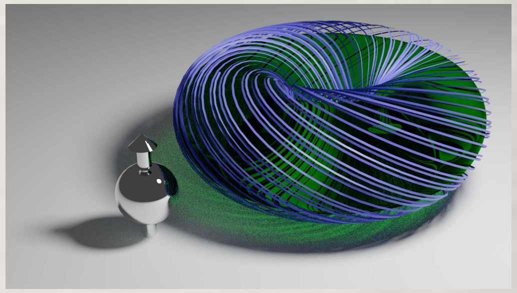
Integral charges are fragile.



Summary

“KAM-Stability for Conserved Quantities in Finite-Dimensional Quantum Systems,” PRL **126**, 150401 (2021);
“Eternal Adiabaticity in Quantum Evolution,” PRA **103**, 032214 (2021).

- **KAM stability** for finite-dimensional quantum systems



All symmetries are conserved,
but some symmetries are more conserved than others.

fragile symmetry $S \neq f(H)$

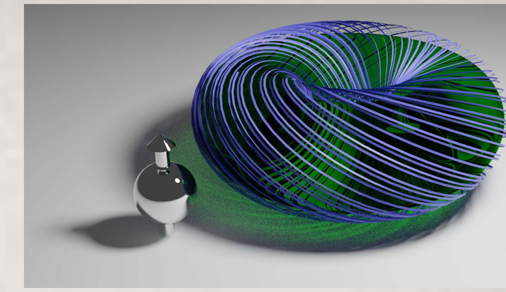
$$\Rightarrow \quad \exists V \quad \text{s.t.} \quad \sup_{t\in\mathbb{R}} \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - S\| = O(1), \quad \forall \varepsilon > 0.$$

robust symmetry $S = f(H)$

$$\Rightarrow \quad \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} S \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - S\| \le \frac{14\sqrt{m}}{\eta} \varepsilon \|V\| \|S\|,$$

$$\forall \varepsilon > 0, \ \forall V, \ \forall t \in \mathbb{R}.$$

Problems



- **quantum KAM theorem** and/or **eternal adiabaticity** hold in infinite dimensions?
- under what conditions?

- eternal adiabaticity for **open quantum systems**
- GKLS structure of an effective generator