Are infinite-dimensional closed quantum systems generically controllable?

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Mathematical Physics in QT: From Finite to Infinite Dimensions
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Table of Contents

What's known?

Let $\mathbb{H} \cong \mathbb{C}^n$ be "the" n-dimensional Hilbert space and U_n its unitary group.

Given a **controlled Schrödinger Equation** (lifted form)

$$\dot{U}(t) = -\mathrm{i}H(t)U(t)\,, \quad U(0) = I_n \in \mathrm{U}_n\,, \tag{S_n}$$

where the control Hamiltonian $H(t)$ is given by

$$H(t) = H_0 + \sum_{j=1}^{m} v_j(t)H_j$$

with real-valued (piecewise constant) controls $v_1(t), \ldots, v_m(t) \in \mathbb{R}$.

Then ...

Goal: generic controllability – from finite to infinite dimensions

Theorem (Jurdjevic, Kupka, ...)

Then (S_n) *is generically controllable, i.e. for all* $m \geq 1$ *the set of* $(m+1)$*-tuples* $(\mathrm{i}H_0, \mathrm{i}H_1, \ldots, \mathrm{i}H_m)$ *such that* (S_n) *is controllable on* U_n *is a generic subset of* $\mathfrak{u}_n \times \cdots \times \mathfrak{u}_n$.

Corollary ($m = 1$)

For a generic pair (H_0, H_1) *of Hamiltonians the Schrödinger Equation* (S_n) *is controllable on* U_n.

Remark

- The same result holds for SU_n and $\mathfrak{su}_n$.

Goal: generic controllability – from finite to infinite dimensions

Recall:

- Controllability of (S_n) means that the reachable set of (S_n), i.e.

$$\mathcal{R}(I_n) := \{U(t) \ : \ t \geq 0\,,\ U(\cdot) \text{ solves } (\mathrm{S}_n) \text{ for some controls } v_j(\cdot)\}$$

  coincides with U_n.

What is the precise meaning of generic? – There are different definitions:

- (MT): generic $=$ complement has measure zero;
- (TOP): generic $=$ complement is meager (i.e. the complement is contained in a countable union of nowhere dense subsets);
- A useful simplification:

$$G \text{ open and dense} \quad \Longrightarrow \quad G \text{ (top)-generic}$$

In the following, we will work with the topological notion of genericity.

Goal:

Let $\mathbb{H}$ denote "the" ∞-dim. separable Hilbert space and $\mathrm{U}(\mathbb{H})$ its unitary group.

Given a **controlled Schrödinger Equation**

$$\dot{U}(t) = -\mathrm{i}H(t)U(t)\,, \quad U(0) = \mathrm{I}_{\mathbb{H}} \in \mathrm{U}(\mathbb{H})\,, \qquad (\mathrm{S}_\infty)$$

where the control Hamiltonian $H(t)$ is given by

$$H(t) = H_0 + \sum_{j=1}^{m} v_j(t)H_j$$

with real-valued piecewise constant controls $v_1(t), \ldots, v_m(t) \in \mathbb{R}$, bounded control Hamiltonians $H_1, \ldots, H_m$ and possibly unbounded drift H_0.

Then ...

Goal: generic controllability – from finite to infinite dimensions

Goal

Under some suitable assumption on H_0 Eq. (S_∞) is generically controllable in the following sense, for all $m \geq 1$ the set of m tuples $(\mathrm{i}H_1, \dots, \mathrm{i}H_m)$ such that (S_∞) is controllable on $\mathrm{U}(\mathbb{H})$ is a generic subset of $\mathfrak{u}(\mathbb{H}) \times \cdots \times \mathfrak{u}(\mathbb{H})$.

Remark

- Due to technical reasons, we have fixed H_0 in contrast to the finite dimensional case.

The finite dimensional case – a brief overview

Consider again the **controlled Schrödinger Equation**

$$\dot{U}(t) = -\mathrm{i}\Big(H_0 + \sum_{j=1}^{m} v_j(t) H_j\Big) U(t)\,, \quad U(0) = I_n \in \mathrm{U}_n \qquad (\mathrm{S}_n)$$

with real-valued (piecewise) controls $v_1(t), \dots, v_m(t)$.

Fundamental controllability result [Jurdjevic, Sussmann, Brockett, ...]

(S_n) is controllable on U_n if and only if the *system Lie algebra* given by $\mathfrak{g} := \langle \mathrm{i}H_0, \mathrm{i}H_1 \dots, \mathrm{i}H_m \rangle_{\mathrm{LA}}$ coincides with $\mathfrak{u}_n$. The same holds for SU_n and $\mathfrak{su}_n$.

Recall:

- $\langle \mathrm{i}H_0, \mathrm{i}H_1 \dots, \mathrm{i}H_m \rangle_{\mathrm{LA}}$ denotes the Lie algebra generated by $\mathrm{i}H_0, \mathrm{i}H_1, \dots$ $\dots, \mathrm{i}H_m$, i.e. the linear span of $\mathrm{i}H_0, \mathrm{i}H_1 \dots, \mathrm{i}H_m$ and all their commutators.

The finite dimensional case – a brief overview

Sketch of proof:

- The closure to the *systems group* $\mathcal{G}$, i.e. the group generated by all unitaries of the form

$$\mathrm{e}^{\mathrm{i}t(H_0+\sum_{j=1}^m v_j H_j)}, \quad t \in \mathbb{R}, \quad (v_1, \dots, v_m) \in \mathbb{R}^m \qquad (*)$$

  contains all one-parameter groups $\mathrm{e}^{\mathrm{i}tH_j}$, $t \in \mathbb{R}$, $j = 0, 1, \dots, m$.

- If the Lie-algebra-rank-condition, i.e.

$$\mathfrak{g} = \mathfrak{u}_n \quad \text{or} \quad \mathfrak{g} = \mathfrak{su}_n \qquad \text{(LARC)}$$

  is satisfied then the closure of $\mathcal{G}$ coincides with U_n (or SU_n).

- Due to compactness of U_n (and SU_n) we conclude (S_n) is (exactly) controllable, i.e. $\mathcal{R}(\mathrm{I}_n) = \mathcal{G} = \mathrm{U}_n$ (or $\mathcal{R}(\mathrm{I}_n) = \mathcal{G} = \mathrm{U}_n$).

The finite dimensional case – a brief overview

How to prove generic controllability in finite dimensions:

- First note that one can restrict to the case $m = 1$.
- Then the following observation is essential:

  Given a pair (H_0, H_1) such that the $\mathrm{ad}_{\mathrm{i}H_0}$-invariant subspace

  $$\mathrm{span}\left\{ \mathrm{ad}_{\mathrm{i}H_0}^k(\mathrm{i}H_1) \;:\; k \in \mathbb{N}\right\} \tag{G}$$

  has dimension $n^2 - n$. Then

  $$\langle \mathrm{i}H_0, \mathrm{i}H_1 \rangle_{\mathrm{LA}} = \begin{cases} \mathfrak{su}_n & \text{if } \mathrm{tr}\, H_0 = \mathrm{tr}\, H_1 = 0, \\ \mathfrak{u}_n & \text{else.} \end{cases}$$

- Hence it suffices to show that the set of pairs $(H_0, H_1) \in \mathfrak{su}_n \times \mathfrak{su}_n$ which satisfy condition (G) are generic in $\mathfrak{su}_n \times \mathfrak{su}_n$.

The finite dimensional case – a brief overview

How to prove generic controllability in finite dimensions:

- To this end consider the maps

$$(H_0, H_1) \mapsto P_I(H_0, H_1) := \det \Big(\mathrm{ad}_{\mathrm{i}H_0}(\mathrm{i}H_1) \quad \cdots \quad \mathrm{ad}_{\mathrm{i}H_0}^{n^2-n}(\mathrm{i}H_1) \Big)_I \,,$$

  where $I = (k_1, \ldots, k_{n^2-n})$ is a fixed multi-index of the from

$$1 \leq k_1 < k_2 \cdots < k_{n^2-n} \leq n^2 - 1 \,.$$

- Finally, set

$$P(H_0, H_1) := \big(P_I(H_0.H_1) \big)_{I \in \mathcal{I}} \,,$$

  where $\mathcal{I}$ denotes the set of multi-indices of the above from.
- Then one has the equivalence

$$(H_0, H_1) \text{ satisfies (G)} \quad \Longleftrightarrow \quad P(H_0, H_1) \neq 0$$

- Since P is polynomial (and not identical zero) we conclude that the set $P \neq 0$ is open and dense and thus we conclude generic controllability.

First results in infinite dimensions: setup & basic notions

Key issues:

- What is the “right” setup for an operator-theoretic approach?

In particular,

- What is the “right” topology on $\mathrm{U}(\mathbb{H})$?
- How to generalized/adapt LARC.

Recall: There are at least three commonly used topologies on $\mathrm{U}(\mathbb{H})$: the uniform (= norm) topology, the strong and the weak operator topology.

Here we favor the strong operator topology for several reasons:

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Here we favor the strong operator topology for several reasons:

- If some of the Hamiltonians e.g. H_0 is unbounded, then the corresponding one-parameter group $\mathrm{e}^{\mathrm{i}tH_0}$ is only strongly (but not uniformly) continuous.
- For $\dim \mathbb{H} = \infty$ the set $\mathrm{U}(\mathbb{H})$ is non-separable with respect to the uniform topology; thus (S_∞) is **never** (uniformly) **approximately controllable**.
- $\mathrm{U}(\mathbb{H})$ is still a topological/metric group with respect to the strong operator topology.

First results in infinite dimensions: setup & basic notions

Consider over again the **controlled Schrödinger Equation:**

$$\dot{U}(t) = -\mathrm{i}\underbrace{\Big(H_0 + \sum_{j=1}^{m} v_j(t) H_j\Big)}_{=:H(t)} U(t)\,, \quad U(0) = \mathrm{I}_{\mathbb{H}} \in \mathrm{U}(\mathbb{H})\,. \qquad (\mathrm{S}_\infty)$$

Assumption A:

1. Drift Hamiltonian H_0: (possibly unbounded) self-adjoint operator;
2. Control Hamiltonians $H_1, \dots, H_m$: bounded self-adjoint operators;
3. Admissible controls $v_1(t), \dots, v_m(t)$: real-valued and piecewise constant;

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Solution concept: The above assumptions guarantee the for all admissible controls $v_1(t), \ldots, v_n(t)$ there exists of a unitary evolution family $U(t,s)$ which satisfies (S_∞) in the strong sense, i.e.

- $(t,s) \to U(t,s) \in \mathrm{U}(\mathbb{H})$ is strongly continuous.
- $U(t,s)U(s,r) = U(t,r)$ for all $t, s \in \mathbb{R}$
- There exists a dense subset $\mathbb{D} \subset \mathbb{H}$ s.t. $\partial_t^+ U(t,s)\psi\big|_{t=s} = \mathrm{i}H(t)\psi$ for all ψ.

First results in infinite dimensions: setup & basic notions

Let $H_0, \dots, H_m$ be as above.

1. The *strongly reachable set* $\mathcal{R}(\mathrm{I}_{\mathbb{H}})$ of (S_∞) is the smallest strongly closed subsemigroup of $\mathrm{U}(\mathbb{H})$ which contains all exponentials of the form

$$\mathrm{e}^{\mathrm{i}t(H_0+\sum_{j=1}^m v_j H_j)}\,, \quad t \geq 0\,, \quad (v_1, \dots, v_m) \in \mathbb{R}^m\,.$$

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2. The *system group* $\mathcal{G}$ of (S_∞) is the smallest strongly closed subgroup of $\mathrm{U}(\mathbb{H})$ which contains all exponentials of the form

$$\mathrm{e}^{\mathrm{i}t(H_0+\sum_{j=1}^N v_j H_j)}\,, \quad t \in \mathbb{R}\,, \quad (v_1, \dots, v_m) \in \mathbb{R}^m\,.$$

First results in infinite dimensions: setup & basic notions

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3. (S_∞) is called *strongly approximately controllable* if $\mathcal{R}(\mathrm{I}_{\mathbb{H}}) = \mathrm{U}(\mathbb{H})$.

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3. (S_∞) is called *strongly approximately controllable* if $\mathcal{R}(\mathrm{I}_{\mathbb{H}}) = \mathrm{U}(\mathbb{H})$.
4. The *system Lie algebra* $\mathfrak{g}$ of (S_∞) is defined as

$$\mathfrak{g} := \{\Omega \in \mathfrak{u}(\mathbb{H}) \;:\; \mathrm{e}^{t\Omega} \in \mathcal{G} \quad \text{for all } t \in \mathbb{R}\}\,,$$

where $\mathfrak{u}(\mathbb{H})$ denotes the Lie algebra of all bounded skew-adjoint operators.

Note: So far, $\mathfrak{g}$ could collapse to $\{0\}$.

First results in infinite dimensions: some preparations

Proposition A (M. Keyl)

Let $H_0, \dots H_m$ satisfy assumption **A**.

(a) The dynamical group $\mathcal{G}$ coincides with the smallest strongly closed subgroup of $\mathrm{U}(\mathbb{H})$ generated by the one-parameter groups $\mathrm{e}^{\mathrm{i}tH_j}$, $t \in \mathbb{R}$ and $j = 0, \dots, m$.

(b) The dynamical Lie algebra $\mathfrak{g}$ is a strongly closed Lie subalgebra of $\mathfrak{u}(\mathbb{H})$.

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(b) The dynamical Lie algebra $\mathfrak{g}$ is a strongly closed Lie subalgebra of $\mathfrak{u}(\mathbb{H})$.

Comments on the proof:

Both results follow – similar to the finite dimensional case – by:

- continuity of the exponential map (with respect to the strong topology)
- the Trotter formula (RS for A being unbounded)

$$\mathrm{e}^{A+B} = \lim_{n\to\infty} \left(\mathrm{e}^{\frac{A}{n}} \mathrm{e}^{\frac{B}{n}}\right)^n \quad \text{and} \quad \mathrm{e}^{[A,B]} = \lim_{n\to\infty} \left(\mathrm{e}^{\frac{A}{n}} \mathrm{e}^{\frac{B}{n}} \mathrm{e}^{\frac{-A}{n}} \mathrm{e}^{\frac{-B}{n}}\right)^{n^2}$$

- and the trivial observations $H_j = (H_0 + H_j) - H_0$ for $j = 1, \dots, m$

First results in infinite dimensions: some preparations

Spectral assumption PS:

The drift Hamiltonian H_0 has pure point spectrum (not necessarily isolated).

Proposition B (M. Keyl)

Let $H_0, \ldots H_m$ satisfy assumption **A** and **PS**.

(a) The reachable set $\mathcal{R}(\mathrm{I}_{\mathbb{H}})$ contains the backward evolution one parameter group generated by $\mathrm{i}H_0$, i.e. $\mathrm{e}^{-\mathrm{i}tH_0} \in \mathcal{R}(\mathrm{I}_{\mathbb{H}})$ for all $t \geq 0$.

(b) and thus

$$\mathcal{R}(\mathrm{I}_{\mathbb{H}}) = \mathcal{G}\,.$$

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(b) and thus

$$\mathcal{R}(\mathrm{I}_{\mathbb{H}}) = \mathcal{G} \,.$$

Comments on the proof:

1. Part (a) follows by a straightforward truncation argument combined with the corresponding finite dimensional result.
2. Part (b) is an immediate consequence of part (a).

First results in infinite dimensions: some preparations

Next goal: improve part (a) of Prop. B.

Definition: Let H_0 satisfy assumption **A** and **PS**. Then H_0 allows the spectral decomposition

$$H_0 = \sum_{n=1}^{\infty} \lambda_n E_n$$

and let $\mathcal{E}(H_0)$ denote the smallest strongly closed complex subspace which contains all spectral projections E_n. Then

- $\mathcal{T}(H_0) := \mathcal{E}(H_0) \cap \mathrm{U}(\mathbb{H})$ is called the maximal torus of H_0.
- and $\mathfrak{t}(H_0) := \mathcal{E}(H_0) \cap \mathfrak{u}(\mathbb{H})$ its maximal torus algebra.

Remark:

- $\mathcal{E}(H_0)$ is an abelian von Neumann algebra with alternative description:

$$\mathcal{E}(H_0) = \{\mathrm{e}^{\mathrm{i}tH_0} \mid t \in \mathbb{R}\}''.$$

First results in infinite dimensions: Theorem A

Non-rationality assumption NR:

The eigenvalues of the drift Hamiltonian H_0 are rationally independent.

Theorem A (M. Keyl)

Let $H_0, \dots, H_m$ satisfy assumptions **A**, **PS**, and **NR**. Then the strong closure of $\{\mathrm{e}^{\mathrm{i}tH_0} \mid t \in \mathbb{R}\}$ in $U(\mathbb{H})$ coincides with $\mathcal{T}(H_0)$.

Comments on the proof:

- The inclusion $\{\mathrm{e}^{\mathrm{i}tH_0} \mid t \in \mathbb{R}\} \subset \mathcal{T}(H_0)$ is straightforward.
- The converse inclusion is based on the classification of abelian von Neumann algebras and the non-rationality condition.
- In principle, there are three different types of abelian von Neumann.
- Here we are faced with the case $l_\infty(\mathbb{N})$.

First results in infinite dimensions: Theorem A

What is Theorem A good for?

Formally, the finite dimensional LARC suggests to consider the “Lie algebra”

$$\langle \mathrm{i}H_0, \mathrm{i}H_1, \dots, \mathrm{i}H_n \rangle_{\mathrm{Lie}} \tag{$*$}$$

BUT $(*)$ is in general not well-defined when $\mathrm{i}H_0,$ is unbounded.

Theorem A tells us that we can replace H_0 in $(*)$ by its spectral projections E_n and consider instead

$$\langle \mathrm{i}H_1, \dots, \mathrm{i}H_m, \mathrm{i}E_1, \mathrm{i}E_2, \dots \rangle_{\mathrm{Lie}} \tag{$**$}$$

First results in infinite dimensions: Theorem A

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$$\langle \mathrm{i}H_1, \dots, \mathrm{i}H_m, \mathrm{i}E_1, \mathrm{i}E_2, \dots \rangle_{\mathrm{Lie}} \qquad (**)$$

Thus, checking controllability boils down to verifying

$$\langle \mathrm{i}H_1, \dots, \mathrm{i}H_m, \mathrm{i}E_1, \mathrm{i}E_2, \dots \rangle_{\mathrm{Lie}} = \mathfrak{u}(\mathbb{H})\,.$$

First results in infinite dimensions: Theorem B

Non-degeneracy and connectivity assumptions ND + C:

- All eigenvalues of H_0 are non-degenerate.
- The connectivity graph $\Gamma(H_0, H_1, \ldots, H_m)$ defined below is connected.

Connectivity graph:

$$(k, l) \text{ is an edge of } \Gamma(H_0, \ldots, H_m) \;:\Longleftrightarrow\; \exists\, j \in \{1, \ldots, m\} \text{ s.t. } \langle b_k, H_j b_l \rangle \neq 0$$

where b_k, $k \in \mathbb{N}$ denotes “the” complete eigenbasis of H_0.

First results in infinite dimensions: Theorem B

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Theorem B (M. Keyl)

Let $H_0, \dots, H_m$ satisfy assumptions **A**, **PS**, **NR** and **ND+C**. Then (S_∞) is strongly approximately controllable.

First results in infinite dimensions: Theorem B

Theorem B (M. Keyl)

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Comments on the proof:

- Combining Theorem A and Condition **ND** we conclude that $\mathfrak{g}$ contains all rank-one projections
$$\mathrm{i}\,|b_n\rangle\langle b_n|$$
where b_n is a complete eigenbasis of H_0.
- Now taking commutators (and allowing for complex linear combinations) yields
$$\langle b_k, H_j b_l\rangle\,|b_k\rangle\langle b_l| \in \mathfrak{g}^{\mathbb{C}}$$
- Finally, Condition C guarantees that all $|b_k\rangle\langle b_l|$ belong to $\mathfrak{g}^{\mathbb{C}}$ and thus the strong closure of $\mathfrak{g}^{\mathbb{C}}$ coincides with the set of all bounded operators on $\mathbb{H}$.
- Hence $\mathfrak{g} = \mathfrak{u}(\mathbb{H})$ which implies strong approximate controllability of (S_∞).

Genericity for ∞-dimensional systems: Theorem C

Spectral assumption DPS:

The Hamiltonian H_0 has discrete pure point spectrum (no accumulation points).

Theorem C (Dirr/Keyl in preparation)

Let H_0 satisfy assumptions **A**, **ND** and **DPS**. Then for $m \geq 1$ Eq. (S_∞) is generically strongly approximately controllable, i.e. the set of $(\mathrm{i}H_1, \ldots, \mathrm{i}H_m)$ such that (S_∞) is strongly approximately controllable is a generic subset of $\mathfrak{u}(\mathbb{H}) \times \cdots \times \mathfrak{u}(\mathbb{H})$.

Sketch of proof: ...

Open Problems

- Extend Thm. C to Hamiltonians H_0 with degenerate eigenvalues;
- Extend Thm. C to Hamiltonians H_0 with finitely many accumulation points;
- Extend Thm. B to Hamiltonian H_0 with continuous spectrum;
- Pass to more than one unbounded Hamiltonian;

Literature

- M. Keyl, *Quantum control in infinite dimensions and Banach-Lie algebras: Pure point spectrum*, arXiv: 1812.09211.
- More on finite dimensional control theory and QC: Sussmann, Jurdjevic, Brockett, Kupka, **Khaneja**, Altafini, Albertini, D'Alessandro, Schulte-Herbrüggen, Schirmer, DH, ...
- In the literature, there are several quite similar results available, e.g. by U. Boscain, M. Sigalotti, M. Caponigro, T. Chambrion, P. Rouchon, etc.
- Some recent findings by M. Sigalotti and M. Caponigro go beyond the results presented in this talk.
- Applications to the Jaynes-Cummings model by M. Keyl.
- Further application see the group of U.Boscain and his collaborators.
- ...

That's it!

Thanks for your attention and patience!