Various types of divisibility and the role they play in statistical mechanics

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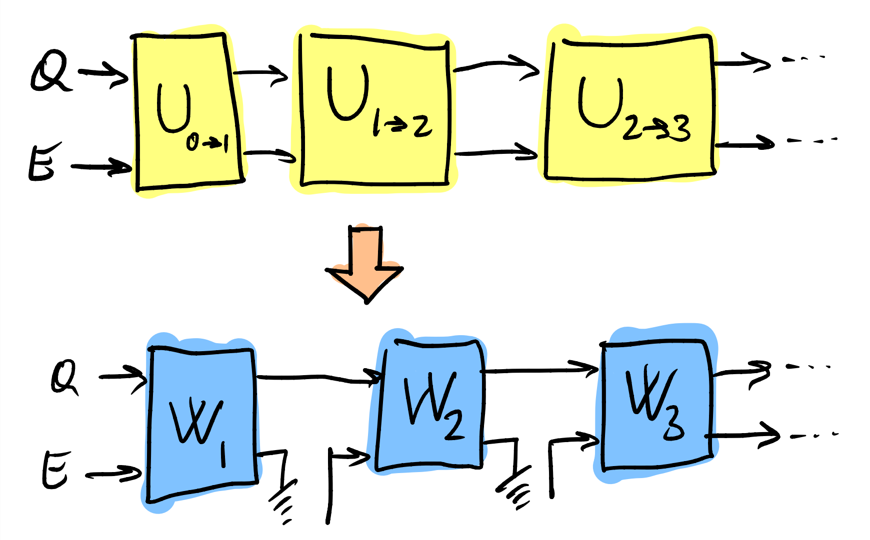
Outline

- divisibility as "inferential locality"
- inferential locality and system-environment correlations
- inferential locality and entropy production

Conventional divisibility



The system's state at time t_i can be predicted (inferred) from that at time t_{i-1}.

☞ In this sense, divisibility is a condition of "inferential locality": the information needed for inference remains localized.

Testing divisibility for a process

The problem. Given two channels $\mathcal{E}_1 : Q_0 \to Q_1$ and $\mathcal{E}_2 : Q_0 \to Q_2$, when does there exist another channel $\mathcal{F} : Q_1 \to Q_2$ such that $\mathcal{E}_2 = \mathcal{F} \circ \mathcal{E}_1$?

Formulated as such, this is a problem of statistical comparison.

Postprocessing preorder. Given two channels $\mathcal{E}_1 : Q_0 \to Q_1$ and $\mathcal{E}_2 : Q_0 \to Q_2$, we write that $\mathcal{E}_1 \succeq \mathcal{E}_2$ whenever there exists another channel $\mathcal{F} : Q_1 \to Q_2$ such that $\mathcal{E}_2 = \mathcal{F} \circ \mathcal{E}_1$.

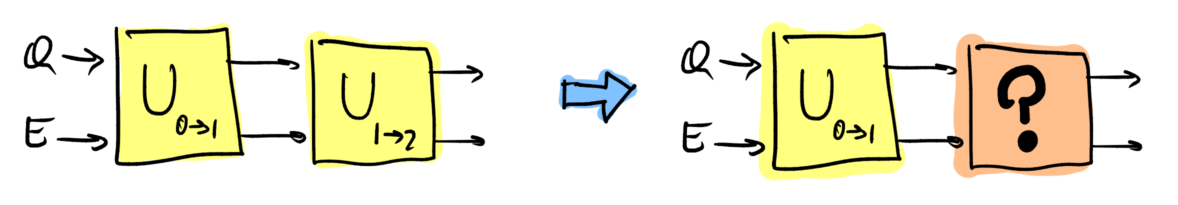
Reverse data-processing (or "Blackwell-type") theorem

$\mathcal{E}_1 \succeq \mathcal{E}_2$ if and only if $\$(\mathcal{E}_1) \geq \$(\mathcal{E}_2)$, for all functions $\$$ in a suitable family of real-valued payoffs/monotones.

Monotones (example). Channels can be compared using, e.g., guessing probabilities (i.e., min-conditional entropies) and the "completely information-non-increasing" property (i.e., "less noisy" preorder).

Inferential locality and system-environment correlations

Another form of inferential locality



When does inferential locality hold for all subsequent evolutions?

☞ This is now a property of the set of system-environment states emerging from $U_{0\to1}$.

Formalization

Let $\mathcal{S}(\mathcal{H})$ denote the set of all density matrices on $\mathcal{H}$.

Datum : an initial set of possible system-environment states $\mathcal{S}_{QE} = \{\rho_{QE} : \rho_{QE} \in \mathcal{S}_{QE}\} \subset \mathcal{S}(\mathcal{H}_Q \otimes \mathcal{H}_E)$.

Problem

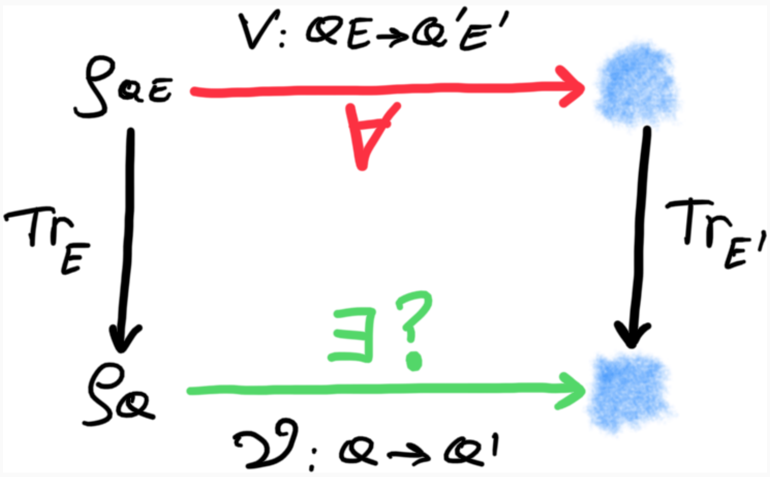
Find conditions on $\mathcal{S}_{QE}$ guaranteeing that,
$\forall$ isometric evolution $V : QE \to Q'E'$,
$\exists$ a CPTP linear map $\mathcal{V} : Q \to Q'$ such that

$$\mathcal{V}(\mathrm{Tr}_E\{\rho_{QE}\}) = \mathrm{Tr}_{E'}\{V\rho_{QE}V^\dagger\} \ ,$$

for all $\rho_{QE} \in \mathcal{S}_{QE}$.

Remark. When the above property holds, we say that the set $\mathcal{S}_{QE}$ is "CPTP reducible".

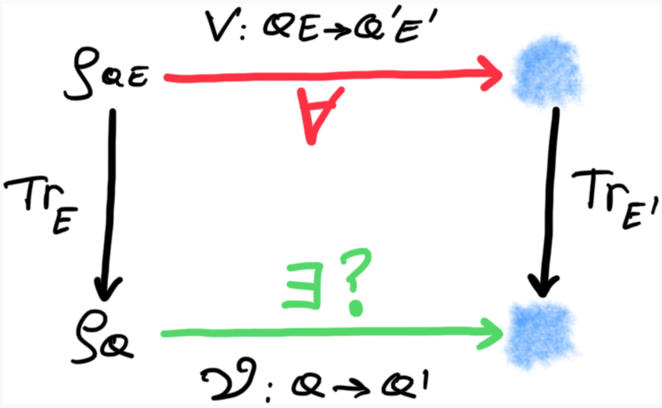
As a diagram



For later convenience, let us define the set $\mathcal{S}_Q := \mathrm{Tr}_E\{\mathcal{S}_{QE}\}$.

The problem is to check whether a CPTP linear map (the green arrow) closing the square always exists or not.

Pechukas–Alicki (1994)



Starting point

One requires the existence of an *assignment map* $\Phi : \mathcal{S}_Q \to \mathcal{S}_{QE}$, possibly satisfying some "natural" properties such as linearity, (complete) positivity, or consistency (i.e., $\mathrm{Tr}_E \circ \Phi = \mathrm{id}$ on $\mathcal{S}_Q$).

In this context, the above is a rather limiting assumption.

Example. Simple initial conditions like $\rho_{QE} = \bar{\rho}_Q \otimes \sigma_E$, for fixed $\bar{\rho}_Q$ and varying σ_E, cannot be treated in this approach.

An alternative approach

We require that the set $\mathcal{S}_{QE}$ be "preparable".

Definition

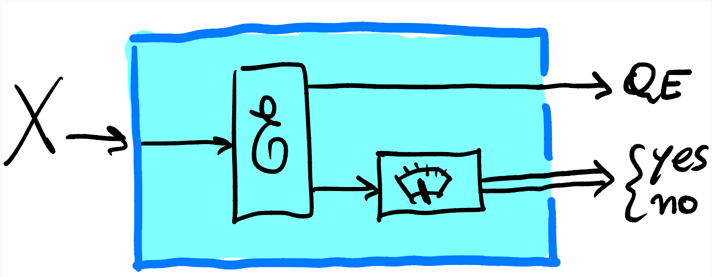
The set $\mathcal{S}_{QE}$ is said to be preparable if and only if there exists an input system X and a CP (not necessarily TP) map $\mathcal{E} : X \to QE$ such that $\mathcal{S}_{QE}$ is the image of $\mathcal{S}(\mathcal{H}_X)$ under $\mathcal{E}$, that is,

$$\mathcal{S}_{QE} = \left\{ \rho_{QE} = \frac{\mathcal{E}(\rho_X)}{\mathrm{Tr}\{\mathcal{E}(\rho_X)\}} : \rho_X \in \mathcal{S}(\mathcal{H}_X) \ \wedge \ \mathrm{Tr}\{\mathcal{E}(\rho_X)\} > 0 \right\} .$$

Remark. The preparation $\rho_X \hookrightarrow \rho_{QE}$ is CP but not necessarily linear nor consistent.

Remark. All $\mathcal{S}_{QE}$ which are polytopes, are preparable (from a classical X).

The operational meaning of preparability



The set of correlated states $\mathcal{S}_{QE}$ is preparable whenever there exists a physical process such that:

1. at each use, it says "yes" or "no"
2. if process says "no": discard and repeat
3. if it says "yes", we know that a state in $\mathcal{S}_{QE}$ was emitted, and any state in $\mathcal{S}_{QE}$ has a nonzero probability of being emitted

Example. Imagine of "freezing" a strongly coupled open system dynamics at some arbitrary time, and add some filtering operation.

Preparability as steerability

We also introduce the following:

Steerability

A set $\mathcal{S}_{QE}$ is said to be steerable if and only if there exists a reference system R and a tripartite density operator ω_{RQE} such that

$$\forall \rho_{QE} \in \mathcal{S}_{QE}\ ,\ \exists \pi_R \geq 0\ :\ \rho_{QE} = \frac{\mathrm{Tr}_R\{\omega_{RQE}\ (\pi_R \otimes \mathbb{1}_{QE})]\}}{\mathrm{Tr}\{\omega_{RQE}\ (\pi_R \otimes \mathbb{1}_{QE})\}}\ .$$

Example. The set of states $\rho_{QE} = \bar{\rho}_Q \otimes \sigma_E$ (where $\bar{\rho}_Q$ is fixed and σ_E varies) does not have an assignment map, but it can be steered from $\omega_{RQE} = \Psi^+_{RE} \otimes \bar{\rho}_Q$.

Consequences of this formulation

Fact

Let the set $\mathcal{S}_{QE}$ be preparable. T.f.a.e.:

1. $\mathcal{S}_{QE}$ is **CPTP reducible**;
2. $\mathcal{S}_{QE}$ is **steerable from a Markov state** ω_{RQE}, i.e., such that $I(R;E|Q)_\omega = 0$.

Remark. The above condition is robust against small deviations (using approximate recoverability theory).

☞ The Markov property is, again, a property of inferential locality.

The above proposition gives a condition, which, if satisfied, guarantees that the system can be "localized" (in an inferential sense), even if it is correlated with the surrounding environment.

Let's see some examples...

Example 1: no correlations

This is the "textbook" situation of initial factorization. In our formalism:

- fix one environment state σ_E
- $\mathcal{S}_{QE} = \{\rho_Q \otimes \sigma_E : \rho_Q \in \mathcal{S}(\mathcal{H}_Q)\}$
- in this case, $\omega_{RQE} = \Psi^+_{RQ} \otimes \sigma_E$
- $\implies I(R;E|Q)_\omega = 0 \implies$ CPTP reducible

Pechukas (PRL, 1994) argued for the importance of going *beyond* the factorization assumption and considering non-CPTP reduced dynamics.

Complementary problem: can CPTP reduced dynamics arise also with initial system-environment correlations?

Example 2: zero-discord correlations

This counterexample was discussed by Rodriguez-Rosario, Modi, Kuah, Shaji, and Sudarshan in 2008 (proof by direct inspection of the dynamical matrix). Our formalism provides an alternative proof.

- fix N environment states $\sigma_E^{(i)}$ $(i = 1, \dots, N)$
- $\mathcal{S}_{QE} = \left\{ \rho_{QE}^{\boldsymbol{p}} = \sum_{i=1}^N p_i |i\rangle\langle i|_Q \otimes \sigma_E^{(i)} : \forall \boldsymbol{p} \text{ prob. dist.} \right\}$
- $\omega_{RQE} = N^{-1} \sum_{i=1}^N |i\rangle\langle i|_R \otimes |i\rangle\langle i|_Q \otimes \sigma_E^{(i)}$
- $\implies I(R;E|Q)_\omega = 0 \implies$ CPTP reducible

Question. Are there other possibilities?

Example 3: discordant correlations

No! Shabani and Lidar (2009) published a paper claiming that the condition of null discord would be, not only sufficient, but also necessary for CPTP reducibility.

Yes! The above claim was disproved by the following counterexample (Brodutch, Datta, Modi, Rivas, Rodriguez-Rosario, 2013). In our formalism:

- fix three distinct environment states $\sigma_E^{(0)}$, $\sigma_E^{(1)}$, and $\sigma_E^{(2)}$
- fix two system-environment states, α and β as follows:
  - $\alpha_{QE} = \frac{1}{2}|0\rangle\langle 0|_Q \otimes \sigma_E^{(0)} + \frac{1}{2}|+\rangle\langle +|_Q \otimes \sigma_E^{(1)}$ (possibly discordant)
  - $\beta_{QE} = |2\rangle\langle 2|_Q \otimes \sigma_E^{(2)}$
- $\mathcal{S}_{QE} = \left\{ \rho_{QE}^p = p\alpha_{QE} + (1-p)\beta_{QE} : \forall p \in [0,1] \right\}$
- $\omega_{RQE} = \frac{1}{2}|0\rangle\langle 0|_R \otimes \alpha_{QE} + \frac{1}{2}|1\rangle\langle 1| \otimes \beta_{QE}$
- $\implies I(R;E|Q)_\omega = 0 \implies$ CPTP reducible

More general examples

All counterexamples to the factorization condition involve separable states.

Question: can we have CPTP reducible sets containing entangled states?

Answer: yes! Starting from tripartite states with $I(R;E|Q)_\omega = 0$, it is easy to construct a lot of counterexamples.

☞ However, if we require that $\mathcal{S}_Q = \mathcal{S}(\mathcal{H}_Q)$, then the factorization condition is the only one that works.

Inferential locality and entropy production

Entropy production in stochastic thermodynamics

- system's initial states $\mathcal{X} = \{x\}$ and final states $\mathcal{Y} = \{y\}$
- forward process: joint distribution $\Phi_F(x, y)$
- reverse process: joint distribution $\Phi_R(y, x)$
- stochastic entropy production: $s(x \to y) = \ln \frac{\Phi_F(x,y)}{\Phi_R(y,x)}$
- a lot of freedom in defining $\Phi_R(y, x)$!

But...

Requiring locality for entropy production

- ...if we impose that entropy production is "local", in the sense that $s(x \to y) = \ln \frac{\Phi_F(x,y)}{\Phi_R(x,y)} \overset{!}{=} g(y) - f(x)$
- $\implies \frac{\Phi_F(y|x)}{\Phi_R(x|y)} = \frac{\tilde{g}(y)}{\tilde{f}(x)} \implies \tilde{f}(x)\Phi_F(y|x) = \tilde{g}(y)\Phi_R(x|y)$
- sum over $x \implies \sum_x \tilde{f}(x)\Phi_F(y|x) = \tilde{g}(y)$

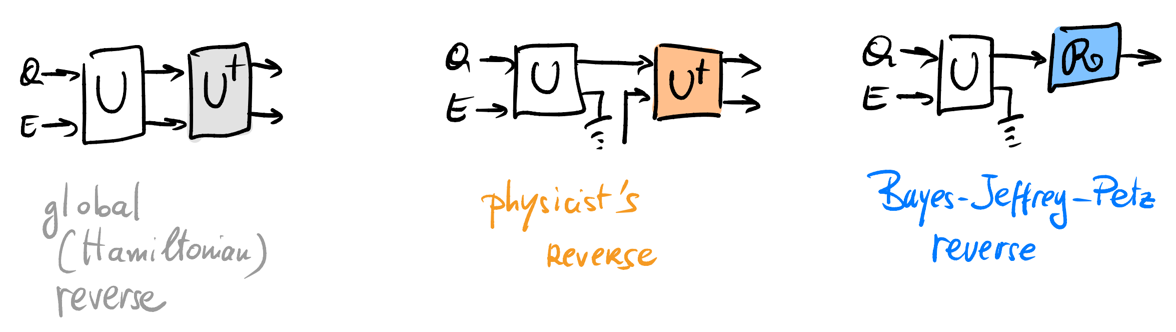
$$\implies \Phi_R(x|y) = \frac{1}{\sum_x \tilde{f}(x)\Phi_F(y|x)} \tilde{f}(x)\Phi_F(y|x)$$

Hence, a "Bayes-like" form for the reverse process is **equivalent** to a locality requirement for the stochastic entropy production.

☞ The Bayesian form for the reverse is, again, a divisibility condition in disguise. Why?

Three recipes to cook the reverse of an open evolution

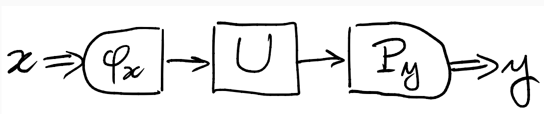
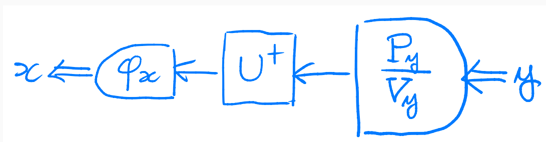
Three constructions:



1. full Hamiltonian reversal: non-local and non-operational
2. physicist's reverse: intuitive but often unjustified
3. Bayes–Jeffrey–Petz inferential reverse: local and operational

BJP reverse in action: an example

Let us consider the case of a two-point measurement process with initial orthonormal basis $\{\varphi_x\}_x$, unitary evolution U, and final POVM $\mathbf{P} = \{P_y\}$.

forward process:	BJP reverse w.r.t. uniform distribution:
$\Phi_F(y\|x) = \mathrm{Tr}\{U\|\varphi_x\rangle\langle\varphi_x\|U^\dagger \; P_y\}$	$\Phi_R^u(x\|y) = \mathrm{Tr}\left\{U^\dagger \frac{P_y}{\mathrm{Tr}\{P_y\}} U \; \|\varphi_x\rangle\langle\varphi_x\|\right\}$
	

With $\rho = \sum_x \lambda_x |\varphi_x\rangle\langle\varphi_x|$ (forward prior) and $\mu_y = \mathrm{Tr}\{U\rho U^\dagger P_y\}$ (backward prior), the average entropy production becomes:

$$\langle s(x \to y)\rangle_{\Phi_F} = D\Big(\{\Phi_F(y|x)\lambda_x\} \Big\| \{\Phi_R^u(x|y)\mu_y\}\Big) = S_{\mathbf{P}}(U\rho U^\dagger) - S(\rho) \;,$$

where $S_{\mathbf{P}}(\bullet) := -\sum_y \mathrm{Tr}\{\bullet \; P_y\} \ln \frac{\mathrm{Tr}\{\bullet \; P_y\}}{\mathrm{Tr}\{P_y\}}$ is the observational entropy w.r.t measurement $\mathbf{P}$.

A note in margin

If any "nice" (i.e., local and operational) definition of entropy production necessarily involves a reverse that is inferential (i.e., Bayesian), then

"the phenomenological onewayness of temporal developments in physics is due to irretrodictability, not irreversibility"

Satosi Watanabe

So we understand why the second law of thermo is so "special" among the laws of physics (A. Eddington): precisely because it is not a law **of** physics, but a law of consistent (i.e., Bayesian) reasoning **about** physics.

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Take-home ideas

☞ inferential locality appears to be a recurring idea in statistical mechanics, explicitly or implicitly:

① explicitly: divisibility and Markovianity
② less explicitly: separations such as system-environment
③ implicitly: entropy production

☞ inferential locality and Bayesian inference appear to be closely related

☞ how would a proper formalism for "(fully) quantum retrodiction" look like?

grazie e buon appetito