Quantum processes: divisibility, Markovianity and classicality

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Andrzej Kossakowski (1938-2021)

AK-gabinet.jpg

$$\Phi_1 \quad : \quad \mathbf{x} \rightarrow (x_1', x_2', x_3') = (x_1, -x_2, x_3) \tag{1}$$

$$\Phi_2 \quad : \quad \mathbf{x} \rightarrow (x_1', x_2', x_3') = -(x_1, x_2, x_3) \tag{2}$$

$$\Phi_3 \quad : \quad \mathbf{x} \rightarrow (x_1', x_2', x_3') = (-x_1, x_2, -x_3) \tag{3}$$

Preliminaries

- positive trace-preserving (PTP)
- completely positive trace-preserving (CPTP)
- unital: $\Phi(\mathbb{1}) = \mathbb{1}$
- dual map $\Phi^\ddagger$

$$(X, \Phi[Y])_{\mathrm{HS}} = (\Phi^\ddagger[X], Y)_{\mathrm{HS}}$$

Φ is PTP $\iff$ $\Phi^\ddagger$ is positive unital

Preliminaries: contractions

Φ is positive trace-preserving:

$$\|\Phi[X]\|_1 \leq \|X\|_1$$

Φ is positive unital:

$$\|\Phi[X]\|_\infty \leq \|X\|_\infty$$

Preliminaries: Schwarz maps

Let Φ be unital.

$$\Phi \text{ is a Schwarz map} \iff \Phi[X^\dagger X] \geq \Phi[X]^\dagger \Phi[X]$$

$$\Phi \text{ is a } k\text{-Schwarz map} \iff \mathrm{id}_k \otimes \Phi \text{ is a Schwarz map}$$

$$\mathrm{P}_k^U = \{\text{unital } k\text{-positive}\}$$

$$\mathrm{S}_k = \{k\text{-Schwarz}\}$$

$$\mathrm{P}_{k+1}^U \subseteq \mathrm{S}_k \subseteq \mathrm{P}_k^U$$

Preliminaries: relative entropy

Lindblad (1975)

$$\Phi \text{ is CPTP } \implies S(\Phi[\rho_1]\|\Phi[\rho_2]) \leq S(\rho_1\|\rho_2)$$

- 2-positive trace-preserving (Uhlmann)
- Φ is PTP and $\Phi^\ddagger$ is a Schwarz map (Petz)
- positive trace-preserving (Müller-Hermes and Reeb)

Dynamics

Quantum dynamical map

$$\rho_{t_0} \to \rho_t = \Lambda_{t,t_0}(\rho_{t_0})$$

$$\Lambda_{t,t_0} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \ ; \ t \geq t_0$$

- Λ_{t,t_0} is CPTP for $t \geq t_0$
- $\Lambda_{t_0,t_0} = \mathrm{id}$
- $t \to \Lambda_{t,t_0}$ differentiable.

Classical dynamical map

$$\text{classical evolution} \quad \longleftrightarrow \quad \mathbf{p}(t) = T(t, t_0)\mathbf{p}$$

- $T(t, t_0)$ is a stochastic matrix $t \geq t_0$
- $T(t_0, t_0) = \mathbb{1}$
- $t \to T(t, t_0)$ differentiable.

Quantum vs. classical

$$|1\rangle, \ldots, |d\rangle \ ; \quad P_i = |i\rangle\langle i|$$

$$\rho \longrightarrow p_i = \mathrm{Tr}(P_i \rho) = \langle i|\rho|i\rangle$$

$$\Phi = \text{PTP map}$$

$$T_{ij} := \mathrm{Tr}(P_i \Phi[P_j]) \text{— stochastic matrix}$$

Positivity vs. complete positivity

$$\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

Positivity: $X \geq 0 \;\longrightarrow\; \Phi(X) \geq 0$

$$\Phi(X) = \sum_i a_i A_i X A_i^\dagger \quad ; \quad a_i \in \mathbb{R}$$

$$a_i > 0 \;\longrightarrow\; K_i = \sqrt{a_i} A_i$$

$$\boxed{\Phi(X) = \textstyle\sum_i K_i X K_i^\dagger}$$

Positivity vs. complete positivity

$\rho \geq 0$; spectral property

Φ posititive ; not a spectral property

Φ CP ; spectral property

Why complete positivity?

Composite systems

$$\mathcal{H}_1 \otimes \mathcal{H}_2$$

Φ_1 , Φ_2 positive

$\Phi_1 \otimes \Phi_2$ needs not be positive

Qantum physics needs complete positivity

Open quantum system

$$\mathcal{H} \otimes \mathcal{H}_E$$

$$\Lambda_{t,t_0}(\rho) := \mathrm{Tr}_E \left(e^{-iH(t-t_0)} \, \rho \otimes \rho_E \, e^{iH(t-t_0)} \right)$$

$$\Lambda_{t,t_0}(\rho) \;=\; \text{reduced evolution}$$

$$t_0 = 0 \;\longrightarrow\; \Lambda_t := \Lambda_{t,0}$$

Semigroups of positive maps

$$\dot{\Lambda}_t = \mathcal{L}\Lambda_t \; ; \quad \Lambda_{t=0} = \mathrm{id}$$

Λ_t is trace-preserving $\iff \mathcal{L}^{\ddagger}(\mathbb{1}) = 0$

$\Lambda_t = e^{t\mathcal{L}}$ is positive for $t \geq 0$ if and only if

$$P_{\perp}\mathcal{L}[P]P_{\perp} \geq 0$$

for all rank-1 projectors $P = |\psi\rangle\langle\psi|$.

$\Lambda_t^{\ddagger} = e^{t\mathcal{L}^{\ddagger}}$ is a Schwarz map for $t \geq 0$ if and only if

$$\mathcal{L}^{\ddagger}(X^{\dagger}X) \geq X^{\dagger}\mathcal{L}^{\ddagger}(X) + \mathcal{L}^{\ddagger}(X)^{\dagger}X$$

Semigroups of positive maps

$\Lambda_t = e^{t\mathcal{L}}$ is positive for $t \geq 0$ if and only if

$$P_\perp \mathcal{L}[P] P_\perp \geq 0$$

for all rank-1 projectors $P = |\psi\rangle\langle\psi|$.

$\Lambda_t^\ddagger = e^{t\mathcal{L}^\ddagger}$ is a Schwarz map for $t \geq 0$ if and only if

$$\mathcal{L}^\ddagger(X^\dagger X) \geq X^\dagger \mathcal{L}^\ddagger(X) + \mathcal{L}^\ddagger(X^\dagger) X$$

- positive $\longrightarrow$ k-positive
- Schwarz $\longrightarrow$ k-Schwarz

Classical semigroup

$$\frac{d}{dt}T(t) = KT(t)$$

Theorem

$T(t) = e^{tK}$ is a stochastic matrix for $t \geq 0$ if and only if

1. $K_{ij} \geq 0, \quad i \neq j$
2. $\sum_i K_{ij} = 0$ *for all j*

Classical vs. quantum

$\mathcal{L}$ generates PTP semigroup iff for any ONB in $\mathcal{H}$

$$K_{ij} := \mathrm{Tr}(P_i \mathcal{L}[P_j])$$

generates classical semigroup of stochastic matrices.

A. Kossakowski 1972

Semigroups of completely positive maps

$$\Lambda_t \text{ is CP} \iff \Lambda_t^\ddagger \text{ is CP} \iff \Lambda_t^\ddagger \text{ is } d\text{-Schwarz}$$

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{t\mathcal{L}}$ is CPTP for $t \geq 0$ if and only if there exist

1. $H = H^\dagger$
2. *CP map* Φ

$$\mathcal{L}[\rho] = -i[H, \rho] + \left(\Phi[\rho] - \frac{1}{2}\{\Phi^\ddagger[\mathbb{1}], \rho\}\right)$$

$$\mathcal{L}^\ddagger[X] = i[H, X] + \left(\Phi^\ddagger[X] - \frac{1}{2}\{\Phi^\ddagger[\mathbb{1}], X\}\right)$$

GNS symmetry

$$\mathcal{L}[\rho] = -i[H,\rho] + \left(\Phi[\rho] - \frac{1}{2}\{\Phi^\ddagger[\mathbb{1}],\rho\}\right)$$

$$\mathcal{L}^\ddagger[X] = i[H,X] + \left(\Phi^\ddagger[X] - \frac{1}{2}\{\Phi^\ddagger[\mathbb{1}],X\}\right)$$

$$\mathcal{L}(\sigma) = 0$$

$$(X,Y)_\sigma := \mathrm{Tr}(\sigma X^\dagger Y)$$

$$(\Phi^\ddagger[X],Y)_\sigma = (X,\Phi^\ddagger[Y])_\sigma$$

Infinite dimensions

There are CP semigroups which are not of the standard Lindblad form

- A. Holevo
- W. Arveson
- A. Holevo and R. Werner

DIVISIBILITY

Divisibility

- Divisibility of quantum channels
- Divisibility of dynamical maps

Divisibility: quantum channels

$$\Phi = \Phi_1 \circ \Phi_2 \quad (\Phi_k \text{ are not unitary channels})$$

M. Wolf, I. Cirac (CMP, 2008)

$$\Phi \text{ is Markovian} \;\longleftrightarrow\; \Phi = e^{\mathcal{L}}$$

M. Wolf, I. Cirac, T. Cubitt, J. Eisert (PRL, 2008)

$$\Phi = e^{\mathcal{L}} \circ \Phi_{\text{boundary}}$$

M. Ziman, D. Davalos (PRL, 2023)

Divisibility – dynamical maps

$$\boxed{\Lambda_t = V_{t,s} \circ \Lambda_s \quad ; \; t \geq s}$$

$$V_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

If Λ_t is invertible for all $t \geq 0$, then

$$V_{t,s} = \Lambda_t \Lambda_s^{-1}$$

If Λ_t is not invertible for some $t > 0$, then divisibility is not guaranteed.

What are the properties of $V_{t,s}$?

Divisibility

- P-divisible iff $V_{t,s}$ is PTP
- CP-divisible iff $V_{t,s}$ is CPTP $\longleftrightarrow$ **Markovianity**

A. Rivas, S. Huelga, M. Plenio (PRL 2009)

$$\text{CP-divisible} \implies \text{P-divisible}$$

k-divisible, if $V_{t,s}$ is k-positive and trace-preserving.

$$k\text{-divisibility} \implies \ell\text{-divisibility} \quad ; \quad k > \ell$$

Divisibility

$\mathrm{D}_k = k$-divisible dynamical maps

$\mathrm{D}_d =$ CP-divisible $=$ Markovian

$\mathrm{D}_1 =$ P-divisible

Markovian $= \mathrm{D}_d \subseteq \mathrm{D}_{d-1} \subseteq \ldots \subseteq \mathrm{D}_1 \subset$ all dynamics

Divisibility vs. information backflow

Breuer, Laine, Piilo (BLP)

$$\textbf{Markovianity} \longleftrightarrow \frac{d}{dt}\|\Lambda_t[\rho_1 - \rho_2]\|_1 \leq 0$$

$$\text{P-divisibility} \implies \frac{d}{dt}\|\Lambda_t[p_1\rho_1 - p_2\rho_2]\|_1 \leq 0$$

CP-divisibility $\implies$ P-divisibility $\implies$ no information backflow

Divisibility: invertible maps

Λ_t is k-divisible if and only if

$$\frac{d}{dt}\|\mathrm{id}_k \otimes \Lambda_t(X)\|_1 \leq 0$$

for all

$$X^\dagger = X \in M_k(\mathcal{B}(\mathcal{H})).$$

DC, A. Kossakowski, A. Rivas (PRA 2011)

Λ_t is CP-divisible iff $\Lambda_t \otimes \Lambda_t$ is P-divisible

F. Benatti, DC, S. Filipov (PRA 2017)

Divisibility: invertible maps

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Λ_t is CP-divisible iff $\Lambda_t \otimes \Lambda_t$ is P-divisible

F. Benatti, DC, S. Filipov (PRA 2017)

Divisibility: non-invertible maps

F. Buscemi and N. Datta, PRA 2016

Distingushability of quantum states

Wait for Francesco talk.

Divisibility: non-invertible maps

$\Lambda_t = V_{t,s}\Lambda_s$ only if

$$\mathrm{Ker}\,\Lambda_s \subseteq \mathrm{Ker}\,\Lambda_t \ ; \quad t > s$$

but $V_{t,s}$ is uniquely defined only on the image of Λ_s.

$\widetilde{V}_{t,s} := \Lambda_t \Lambda_s^- \ ; \ (\Lambda_s^- =$ generalized inverse (non unique!))

$$\frac{d}{dt}\|\mathrm{id}_k \otimes \Lambda_t(X)\|_1 \leq 0 \implies$$

$V_{t,s}$ is k-positive and trace-preserving on the image of Λ_s.

Could we find k-positive trace-preserving extension $\widetilde{V}_{t,s}$?

Arveson extension theorem

$$M \subset \mathcal{B}(\mathcal{H}) \quad \text{– operator system}$$

- $\mathbb{1} \in M$
- $X \in M \implies X^\dagger \in M$

Theorem (Arveson)

Let M be an operator system and $\Phi : M \to \mathcal{B}(\mathcal{H})$ a unital CP map. Then there exists unital CP extension $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

It is not true for positive maps

Generalizing Arveson theorem

$M \subset \mathcal{B}(\mathcal{H})$ – spanned by positive operators

Theorem (Jencova)

If $\Phi : M \to \mathcal{B}(\mathcal{H})$ *is a CP map, then there exists CP extension* $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

It is not true for positive maps

Theorem (DC,Rivas,Størmer)

$$\text{If} \quad \frac{d}{dt} \| (\mathrm{id} \otimes \Lambda_t)[X] \|_1 \leq 0$$

then $V_{t,s}$ *is CPTP on* $\mathrm{Im}\,\Lambda_s$, *and it can be extended to CP map* $\widetilde{V}_{t,s}$.

$\widetilde{V}_{t,s}$ is always trace-preserving on $\mathrm{Im}\,\Lambda_s$

$\widetilde{V}_{t,s}$ need not be trace-preserving on $\mathcal{B}(\mathcal{H})$

Could we have both CP and trace-preservation ?

Λ_t is "image decreasing" $\iff \mathrm{Im}\Lambda_t \subseteq \mathrm{Im}\Lambda_s \ (t > s)$

Theorem (DC,Rivas,Størmer)

If Λ_t is image decreasing, then it is CP-divisible iff

$$\frac{d}{dt} \| (\mathrm{id} \otimes \Lambda_t)[X] \|_1 \leq 0$$

- $[\Lambda_{t_1}, \Lambda_{t_2}] = 0$ for all $t_1, t_2 \geq 0$,
- Λ_t is diagonalizable (generic case)

Qubit is special

Theorem (DC, S. Chakraborty)

Λ_t *is CP-divisible iff*

$$\frac{d}{dt} \| (\mathrm{id}_2 \otimes \Lambda_t)[X] \|_1 \leq 0$$

$$\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$$

- $\dim \mathrm{Im}\, \Lambda_t = 4$ (invertible)
- $\dim \mathrm{Im}\, \Lambda_t = 2$
- $\dim \mathrm{Im}\, \Lambda_t = 1 \quad \longrightarrow \quad \Lambda_t(\rho) = \omega_t \mathrm{Tr} \rho$

Qubit is special

Theorem (DC, S. Chakraborty)

Λ_t *is CP-divisible iff*

$$\frac{d}{dt} \| (\mathrm{id}_2 \otimes \Lambda_t)[X] \|_1 \leq 0$$

$$\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$$

- $\mathrm{dim} \mathrm{Im}\, \Lambda_t = 4$ (invertible)
- $\mathrm{dim} \mathrm{Im}\, \Lambda_t = 2$
- $\mathrm{dim} \mathrm{Im}\, \Lambda_t = 1 \quad \longrightarrow \quad \Lambda_t(\rho) = \omega_t \mathrm{Tr} \rho$

Qubit

$$\{\rho_1, \rho_2\} \quad ; \quad \{\sigma_1, \sigma_2\}$$

Does there exists a quantum chanel Φ such that $\sigma_k = \Phi(\rho_k)$?

Theorem (Alberti and Uhlmann)

$$\| \rho_1 - t\rho_2\|_1 \geq \| \sigma_1 - t\sigma_2\|_1$$

for all $t > 0$*. Equivalently*

$$\| p_1\rho_1 - p_2\rho_2\|_1 \geq \| p_1\sigma_1 - p_2\sigma_2\|_1$$

for all $p_1 + p_2 = 1$*.*

Qubit

$$\{\rho_1, \rho_2\} \quad ; \quad \{\sigma_1, \sigma_2\}$$

Does there exists a quantum chanel Φ such that $\sigma_k = \Phi(\rho_k)$?

Theorem (Alberti and Uhlmann)

$$\| \rho_1 - t\rho_2 \|_1 \geq \| \sigma_1 - t\sigma_2 \|_1$$

for all $t > 0$. *Equivalently*

$$\| p_1\rho_1 - p_2\rho_2 \|_1 \geq \| p_1\sigma_1 - p_2\sigma_2 \|_1$$

for all $p_1 + p_2 = 1$.

Divisibility vs. time-local generator

$$\dot{\Lambda}_t = \mathcal{L}_t \Lambda_t \; ; \quad \Lambda_{t=0} = \mathrm{id}$$

How to characterize divisibility in terms of $\mathcal{L}_t$?

Divisibility vs. time-local generator

Let us assume that Λ_t is CPTP

P-divisibility

$$P_\perp \mathcal{L}_t(P) P_\perp \geq 0$$

CP-divisibility

$\mathcal{L}_t$ is time-dependent Lindbladian (GKLS)

Divisibility: qubit dynamics

$$\mathcal{L}_t[\rho] = \sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho \sigma_k - \rho)$$

CP-divisibility

$$\gamma_1(t) \geq 0\ , \quad \gamma_2(t) \geq 0\ , \quad \gamma_3(t) \geq 0.$$

P-divisibility $\Longleftrightarrow$ BLP condition

$$\gamma_1(t) + \gamma_2(t) \geq 0\ , \quad \gamma_2(t) + \gamma_3(t) \geq 0\ , \quad \gamma_3(t) + \gamma_1(t) \geq 0$$

Qubit: eternally non-Markovian dynamics

$$\mathcal{L}_t[\rho] = \sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho \sigma_k - \rho)$$

$$\gamma_1 = \gamma_2 = 1 \ , \quad \gamma_3(t) = -\tanh t$$

- not CP-divisible $\longleftrightarrow$ non-Markovian
- P-divisible

$$\gamma_1(t) + \gamma_2(t) \geq 0 \ , \quad \gamma_2(t) + \gamma_3(t) \geq 0 \ , \quad \gamma_3(t) + \gamma_1(t) \geq 0$$

M. J. W. Hall, J. D. Cresser, Li. Li, and E. Andersson (PRA 2014)

Divisibility: qubit dynamics

$$\mathcal{L}_t[\rho] = \sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho \sigma_k - \rho)$$

$$\gamma_1 = \gamma_2 = 1 \ , \quad \gamma_3(t) = -\tanh t \quad (***)$$

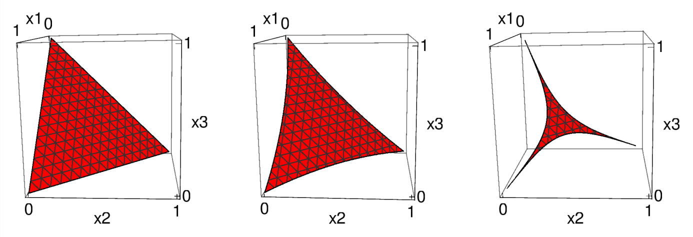
$$\mathcal{L}_k[\rho] = \sigma_k \rho \sigma_k - \rho \ ; \quad k = 1, 2, 3$$

$$\Lambda_t = \frac{1}{2}\left(e^{t\mathcal{L}_1} + e^{t\mathcal{L}_2}\right)$$

$$\mathcal{L}_t = \dot{\Lambda}_t \Lambda_t^{-1} \ \longrightarrow \ (***)$$

$$\Lambda_t = x_1 e^{t\mathcal{L}_1} + x_2 e^{t\mathcal{L}_2} + x_3 e^{t\mathcal{L}_3}$$

- Λ_t is a Markovian semi-group if only one $x_k = 1$
- Λ_t is P-divisible for all x_k ($\equiv$ BLP condition)
- Λ_t is CP-divisible only for special (x_1, x_2, x_3)



N. Megier, DC, J. Piilo, W. Strunz, (Sc. Rep. 2017)

Divisibility: qubit dynamics

$$\mathcal{L}_t = -i\omega(t)[\sigma_z, \rho] + \gamma_+(t)\mathcal{L}_+ + \gamma_-(t)\mathcal{L}_-(\rho) + \gamma_z(t)\mathcal{L}_z(\rho)$$

$$\begin{aligned} \mathcal{L}_+[\rho] &= \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\ ; \quad \mathcal{L}_z[\rho] = \sigma_z\rho\sigma_z - \rho \\ \mathcal{L}_-[\rho] &= \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\} \end{aligned}$$

- CP-divisible iff $\gamma_\pm(t) \geq 0$ and $\gamma_z(t) \geq 0$,
- P-divisible iff $\gamma_\pm(t) \geq 0, \quad \sqrt{\gamma_+(t)\gamma_-(t)} + 2\gamma_z(t) \geq 0$
- satisfies BLP condition iff

$$\gamma_+(t) + \gamma_-(t) \geq 0, \quad \gamma_+(t) + \gamma_-(t) + 4\gamma_z(t) \geq 0,$$

Markovian quantum processes

Classical stochastic process

$$X(t) \in \mathcal{X} = \{x_1, x_2, \ldots\}$$

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = \mathrm{Prob}\left\{X(t_1) = x_1, \ldots, X(t_n) = x_n\right\}$$

Kolmogorov consistency conditions

$$\mathbb{P}_{n-1}\left(x_n, t_n; \ldots; \cancel{x_j, t_j}; \ldots; x_1, t_1\right) = \sum_{x_j \in \mathcal{X}} \mathbb{P}_n(x_n, t_n; \ldots; x_j, t_j; \ldots; x_1, t_1)$$

Classical stochastic process

The process is Markovian

$$\mathbb{P}_n\left(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1\right) = \mathbb{P}\left(x_n, t_n | x_{n-1}, t_{n-1}\right)$$

$$\mathbb{P}_n(x_n, \ldots, x_1) = \mathbb{P}(x_n | x_{n_1}) \ldots \mathbb{P}(x_2 | x_1) \mathbb{P}_1(x_1)$$

Classical stochastic process

If the process is Markovian then the infinite hierarchy of **Kolmogorov Consistency Conditions** (KCC) reduce to

$$
\begin{aligned}
\mathbb{P}_1(x) &= \sum_y \mathbb{P}(x|y)\mathbb{P}_1(y) \\
\mathbb{P}(x|z) &= \sum_y \mathbb{P}(x|y)\mathbb{P}(y|z)
\end{aligned}
$$

Quantum stochastic process

Reduced evolution of a composite system in $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_\mathrm{B}$

$$\Lambda_{t,t_0}(\rho_{t_0}) = \mathrm{Tr}_\mathrm{B}\, \boldsymbol{\mathcal{U}}_{t,t_0}(\rho_{t_0} \otimes \varrho_\mathrm{B}), \qquad \boldsymbol{\mathcal{U}}_{t,t_0} = \mathsf{U}_{t,t_0}(\cdot)\mathsf{U}^\dagger_{t,t_0},$$

Fix an orthonormal basis in $\mathcal{H}_\mathrm{S}$: $|x\rangle$; $\;x = 1, \ldots, d = \dim \mathcal{H}_\mathrm{S}$

$$\mathcal{P}_x(\rho) = P_x \rho P_x \;; \quad P_x = |x\rangle\langle x|$$

$$\begin{aligned}&\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) := \\ &\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_n,t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_1,t_0}(\rho_{t_0} \otimes \varrho_\mathrm{B})\right],\end{aligned}$$

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_n,t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_1,t_0}(\rho_{t_0} \otimes \varrho_\mathrm{B})\right],$$

In general KCC does not hold

$$\mathbb{P}_{n-1}\left(x_n, t_n; \ldots; \cancel{x_j, t_j}; \ldots; x_1, t_1\right) =$$
$$\sum_{x_j=1}^{d} \mathbb{P}_n(x_n, t_n; \ldots; x_j, t_j; \ldots; x_1, t_1)$$

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_n,t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_1,t_0}(\rho_{t_0} \otimes \varrho_\mathrm{B})\right],$$

In general KCC does not hold

$$\mathbb{P}_{n-1}\left(x_n, t_n; \ldots; \cancel{x_j, t_j}; \ldots; x_1, t_1\right) =$$
$$\sum_{x_j=1}^{d} \mathbb{P}_n(x_n, t_n; \ldots; x_j, t_j; \ldots; x_1, t_1)$$

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_n, t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_1, t_0}(\rho_{t_0} \otimes \varrho_{\mathrm{B}})\right],$$

The quantum process is

- *N-classical*, if KCC is satisfied for all $n = 1, \ldots, N$.
- *classical*, if the process is N-classical for all N.

(w.r.t. ONB $|x\rangle$).

S. Milz, D. Egloff, P. Taranto, T. Theurer, M. Plenio, A. Smirne, and S. Huelga (PRX 2020)

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_n, t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_\mathrm{B})\boldsymbol{\mathcal{U}}_{t_1, t_0}(\rho_{t_0} \otimes \varrho_\mathrm{B})\right],$$

$$\mathbb{P}\left(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1\right) = \mathbb{P}\left(x_n, t_n | x_{n-1}, t_{n-1}\right)$$

The quantum process is

- *N-Markovian*, if MC is satisfied for all $n = 1, \ldots, N$.
- *Markovian*, if the process is N-Markovian for all N.

(w.r.t. ONB $|x\rangle$).

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_n,t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_1,t_0}(\rho_{t_0} \otimes \varrho_{\mathrm{B}})\right],$$

The quantum process is Markovian iff

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) =$$
$$\mathrm{Tr}\left[\mathcal{P}_{x_n}\Lambda_{t_n,t_{n-1}} \cdots \mathcal{P}_{x_1}\Lambda_{t_1,t_0}(\rho_{t_0})\right].$$

(Quantum regression formula)

Process tensor

$$\mathcal{T}[\mathcal{P}_{x_n}, \ldots, \mathcal{P}_{x_1}] := \mathrm{Tr}_B \left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_{\mathrm{B}}) \boldsymbol{\mathcal{U}}_{t_n, t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_{\mathrm{B}}) \boldsymbol{\mathcal{U}}_{t_1, t_0} (\rho_{t_0} \otimes \varrho_{\mathrm{B}}) \right]$$

$$\mathbb{P}_n(x_n, t_n; \ldots; x_1, t_1) = \mathrm{Tr}_S \mathcal{T}[\mathcal{P}_{x_n}, \ldots, \mathcal{P}_{x_1}]$$

The quantum process is Markovian iff

$$\mathcal{T}[\mathcal{P}_{x_n}, \ldots, \mathcal{P}_{x_1}] = \mathcal{P}_{x_n} \Lambda_{t_n, t_{n-1}} \cdots \mathcal{P}_{x_1} \Lambda_{t_1, t_0} (\rho_{t_0})$$

S. Milz and K. Modi (PRX Quantum 2021)

quantum comb

Quantum stochastic process

$$\mathbb{P}_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1) :=$$
$$\mathrm{Tr}\left[(\mathcal{P}_{x_n} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_n,t_{n-1}} \cdots (\mathcal{P}_{x_1} \otimes \mathcal{I}_{\mathrm{B}})\boldsymbol{\mathcal{U}}_{t_1,t_0}(\rho_{t_0} \otimes \varrho_{\mathrm{B}})\right],$$

- neither classical nor Markovian
- classical and non-Markovian
- non-classical and Markovian
- classical and Markovian

Markovianity and classicality – examples

- amplitude damping dynamics
- dephasing dynamics

Spin-boson model

$$\mathbf{H} = \omega_0 |1\rangle\langle 1| \otimes \mathbb{1}_B + \int \mathrm{d}\omega \, \omega \, b_\omega^\dagger b_\omega + H_{\text{int}}$$

$$H_{\text{int}} = \int \mathrm{d}\omega f(\omega) |0\rangle\langle 1| \otimes b_\omega^\dagger + \text{h.c.}$$

$$\varrho_B = |\text{vac}\rangle\langle\text{vac}| \;\longrightarrow\; \Lambda_t(\rho) = \begin{pmatrix} |a(t)|^2 \rho_{11} & a(t)\rho_{10} \\ a(t)^* \rho_{01} & \rho_{00} - |a(t)|^2 \rho_{11} \end{pmatrix}$$

$$i\dot{a}(t) = \omega_0 a(t) + \int_0^t G(t-s) a(s) \, \mathrm{d}s \; ; \quad a(0) = 1$$

$$G(t) := -i \int \mathrm{d}\omega \, |f(\omega)|^2 e^{-i\omega t}$$

Spin-boson model

Λ_t is CP-divisible (iff it is P-divisible) iff

$$\frac{d}{dt}|a(t)| \leq 0$$

The quantum process $\mathbb{P}_n(x_n, t_n; \ldots; x_1, t_1)$ is Markovian for any choice of $\{P_0, P_1\}$ iff $f(\omega) = \text{const.}$ In this case the reduced dynamics Λ_t defines a Markovian semigroup.

If $\mathbb{P}_n(x_n, t_n; \ldots; x_1, t_1)$ is Markovian, then it is classical iff

$$P_0 = |0\rangle\langle 0| \ , \quad P_1 = |1\rangle\langle 1|$$

Dephasing dynamics

Fixed ONB $|j\rangle$ in $\mathcal{H}_\mathrm{S}$ (dephasing basis)

$$\mathbf{H} = \sum_j P_j \otimes H_j$$

$$P_j = |j\rangle\langle j| \; ; \quad H_j = H_j^\dagger \in \mathcal{B}(\mathcal{H}_\mathrm{B})$$

$$d = 2 \;\longrightarrow\; \Lambda_t(\rho) = \begin{pmatrix} \rho_{11} & \varphi(t)\rho_{10} \\ \varphi(t)^*\rho_{01} & \rho_{00} \end{pmatrix}$$

$$\varphi(t) = \mathrm{Tr}\left(e^{-iH_0 t}\varrho_B e^{iH_1 t}\right)$$

Dephasing dynamics

Λ_t is CP-divisible (iff it is P-divisible) iff

$$\frac{d}{dt}|\varphi(t)| \leq 0$$

Dephasing dynamics – quantum regression

$$\Lambda_{t,t_0}(\rho) = \sum_{j,\ell} P_j \rho P_\ell \, \mathrm{Tr}\left[\mathcal{U}^{j,\ell}_{t,t_0}(\rho_\mathrm{B})\right]$$

$$\mathcal{U}^{j,\ell}_{t,t_0}(\rho_\mathrm{B}) := e^{-iH_j(t-t_0)}\rho_\mathrm{B} e^{iH_\ell(t-t_0)}$$

$$u^{j,\ell}_{t,t_0} = \mathrm{Tr}\left[\mathcal{U}^{j,\ell}_{t,t_0}(\rho_\mathrm{B})\right]$$

$$\mathrm{Tr}\left[\mathcal{U}^{j_2,\ell_2}_{t_2,t_1} \circ \mathcal{U}^{j_1,\ell_1}_{t_1,t_0}(\rho_\mathrm{B})\right] = u^{j_2,\ell_2}_{t_2,t_1}\, u^{j_1,\ell_1}_{t_1,t_0}$$

$$\mathrm{Tr}\left[\mathcal{U}^{j_3,\ell_3}_{t_3,t_2} \circ \mathcal{U}^{j_2,\ell_2}_{t_2,t_1} \circ \mathcal{U}^{j_1,\ell_1}_{t_1,t_0}(\rho_\mathrm{B})\right] = u^{j_3,\ell_3}_{t_3,t_2}\, u^{j_2,\ell_2}_{t_2,t_1}\, u^{j_1,\ell_1}_{t_1,t_0}$$

Dephasing dynamics — classicality ($d = 2$)

If the measurement basis $\{e_k\}$ coincides with the dephasing one $\{|k\rangle\}$, then the quantum process is trivially classical

$$\mathbb{P}_n(x_n, t_n; \ldots, x_1, t_1) = \delta_{x_n, x_{n-1}} \delta_{x_{n-1}, x_{n-2}} \ldots \delta_{x_2 x_1} \mathbb{P}_1(x_1, t_1)$$

If $\{e_k\}$ and $\{|k\rangle\}$ are MUB, i.e. $|\langle e_k|l\rangle|^2 = 1/2$, and

$$\rho_{t_0} = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$$

then the Markovian process is classical iff

$$\varphi_{t,s} = \mathrm{Tr}\left(e^{-iH_0 t} \varrho_B e^{iH_1 s}\right)$$

is real.

Amplitude damping and pure dephasing dynamics can be generalized for arbitrary for arbitrary d.

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