

Koszulity and

Stirling representations

- a preliminary report

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Various Guises of Reflection Arrangements

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1. Stirling numbers $c(n, k)$ (1st kind), $S(n, k)$ (2nd kind)
2. (Familiar) algebras of the 1st kind
3. Koszulity review
4. Supersolvability
5. (Koszul dual) algebras of the 2nd kind
6. Properties and Questions

1. Stirling numbers $c(n,k)$, $S(n,k)$

(Signless) Stirling number of the 1st kind

$c(n,k) := \#$ permutations in \mathfrak{S}_n with k cycles
for $1 \leq k \leq n$

$c(4,4)$	$c(4,3)$	$c(4,2)$	$c(4,1)$
\parallel	\parallel	\parallel	\parallel
1	6	11	6
$e = (1)(2)(3)(4)$	(12) (13) (14) (23) (24) (34)	(123) (132) (124) (142) (134) (143) (234) (243) $(12)(34)$ $(13)(24)$ $(14)(23)$	(1234) (1243) (1324) (1342) (1423) (1432)

Generating function definition:

$$\sum_{k=1}^n c(n,k) t^{n-k} = (1+t)(1+2t)(1+3t) \cdots (1+(n-1)t)$$

$n=4$:

$$(1+t)(1+2t)(1+3t) = 1 + 6t + 11t^2 + 6t^3$$

$c(4,4) \quad c(4,3) \quad c(4,2) \quad c(4,1)$

(Signless) Stirling number of the 2nd kind

$S(n, k) := \#$ partitions of $\{1, 2, \dots, n\}$ with k blocks
for $1 \leq k \leq n$

$$S(4, 4) \\ \parallel \\ 1$$

1|2|3|4

$$S(4, 3) \\ \parallel \\ 6$$

12|3|4
13|2|4
14|2|3
23|1|4
24|1|3
34|1|2

$$S(4, 2) \\ \parallel \\ 7$$

123|4
124|3
134|2
234|1
12|34
13|24
14|23

$$S(4, 1) \\ \parallel \\ 1$$

1234

Generating function definition:

$$\sum_{n=k}^{\infty} S(n,k) t^n = \frac{t^k}{(1-t)(1-2t)(1-3t) \cdots (1-k \cdot t)}$$

Rewritten for later purposes:

$$\sum_{i=0}^{\infty} S(n-1+i, n-1) t^i = \frac{1}{(1-t)(1-2t) \cdots (1-(n-1)t)}$$

$$\left(\text{cf. } \sum_{k=1}^n c(n,k) t^{n-k} = (1+t)(1+2t)(1+3t) \cdots (1+(n-1)t) \right)$$

Triangles and recursions

$c(n,k)$

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
5	24	50	35	10	1

$$c(n,k) = (n-1) \cdot c(n-1,k) + c(n-1,k-1)$$

$S(n,k)$

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$

2. (Familiar) algebras of the 1st kind

$$A(n) := H^{\bullet} \text{Conf}(n, \mathbb{R}^d)$$

(ordered)
configuration
space of n distinct
points (p_1, p_2, \dots, p_n) in \mathbb{R}^d
 $p_i \neq p_j$

skew-commutative
exterior algebra

$$A_{\text{OS}}(n) = \underbrace{\Lambda[e_{ij}]}_{\text{Arnol'd 1968}} / (e_{ij}e_{ik} - e_{ij}e_{jk} + e_{ik}e_{jk}) \quad d \text{ even}$$

for $1 \leq i < j < k \leq n$

\uparrow
||2

rescale
cohomological
grading
by $d-1$

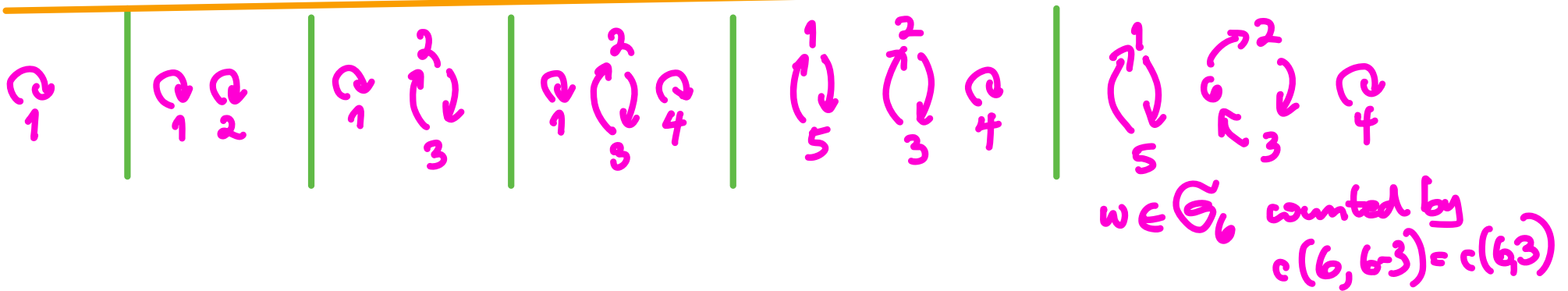
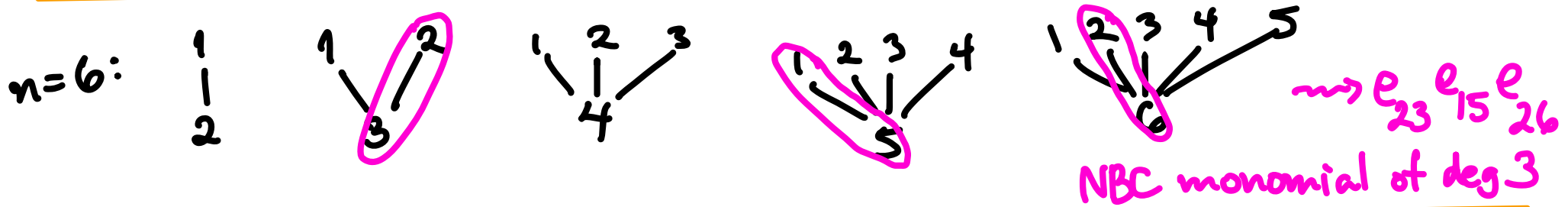
$$A_{\text{VG}}(n) = \underbrace{\mathbb{K}[x_{ij}]}_{\text{commutative polynomial ring}} / (x_{ij}^2 x_{ik} - x_{ij} x_{jk} + x_{ik} x_{jk}) \quad d \text{ odd}$$

F. Cohen 1972

Both $A(n) = A_{OS}(n), A_{VG}(n)$ have

$$\text{Hilb}(A(n), t) = (1+t)(1+2t)(1+3t) \dots (1+(n-1)t)$$

since those were Gröbner bases for the ideals,
 with initial terms $e_{ik}e_{jk}$ and $x_{ik}x_{jk}, x_{ij}^2$, leading to
 standard monomial NBC bases =
 "at most one finger from each hand" " ← Barcelo 1988



As \mathfrak{G}_n -representations, both $A(n) = A_{\text{OS}}(n), A_{\text{VG}}(n)$ are well-studied, but not completely understood.

$n=4$	1 A_0	$+$ $6t$ A_1	$+$ $11t^2$ A_2	$+$ $6t^3$ A_3	total rep'n (ungraded)
$A_{\text{VG}}(4)$		 	 	 	$\text{rk}[\mathfrak{G}_4]$ = regular rep.
$A_{\text{OS}}(4)$		 	 	 	2 copies of $\text{rk}[\mathfrak{G}_4 / (\mathfrak{G}_2 \times \mathfrak{G}_1 \times \mathfrak{G}_1)]$

THEOREM
(Sundaram-
Welker
1997)

As G_n -representations,

$$\sum_{n=0}^{\infty} \sum_{k=1}^n \text{ch } A(n)_{n-k} t^k =$$

$$\sum_{\lambda=1^{m_1} 2^{m_2} \dots} t^{l(\lambda)} \cdot \prod_{j=1}^{\infty} h_{m_j}[\text{Lie}_j] = \prod_{m=1}^{\infty} (1 - p_m)^{-a_m(t)} \quad \text{for VG}$$

plethysm formulas vs. product generating functions

$$\sum_{\lambda=1^{m_1} 2^{m_2} \dots} t^{l(\lambda)} \prod_{\substack{j \\ \text{odd}}} h_{m_j}[\pi_j] \cdot \prod_{\substack{j \\ \text{even}}} e_{m_j}[\pi_j] = \prod_{m=1}^{\infty} (1 + (-1)^m p_m)^{a_m(-t)} \quad \text{for OS}$$

where $a_m(t) = \frac{1}{m} \sum_{d|m} \mu(d) t^{m/d}$

Many results by

Lehrer-Solomon, Whitehouse, Douglass-Pfeiffer-Röhrlé, ...

Branching for $A(n)$

The recurrence

$$c(n,k) = (n-1) \cdot c(n-1,k) + c(n-1,k-1)$$

lifts easily (via the generating functions):

$c(n,k)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=1$	1				
$n=2$	1	1			
$n=3$	2	3	1		
$n=4$	6	11	6	1	
$n=5$	24	50	35	10	1

PROPOSITION: Both $A(n) = A_{OS}(n), A_{VG}(n)$
 (Sundaram 1994, 2020) have these **branching rules** for restriction \mathfrak{S}_n to \mathfrak{S}_{n-1} :

$$A(n)_i \downarrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \cong \chi_{\text{def}}^{(n-1)} \otimes A(n-1)_{i-1} \oplus A(n-1)_i$$

defining rep. of \mathfrak{S}_{n-1}
 as permutation matrices

Better phrasing:

PROPOSITION: $A(n) = \begin{Bmatrix} A_{OS}(n) \\ A_{VG}(n) \end{Bmatrix}$ have \mathfrak{S}_{n-1} -equivariant s.e.s.

$$0 \rightarrow A(n-1) \rightarrow A(n) \begin{array}{c} \downarrow \mathfrak{S}_n \\ \mathfrak{S}_{n-1} \end{array} \rightarrow \left[\chi_{\text{def}}^{(n-1)} \otimes A(n-1) \right](-1) \rightarrow 0$$

This generalizes to $\left\{ \begin{array}{l} \text{Orlik-Solomon} \\ \text{Varchenko-Gelfand} \end{array} \right\}$ algebras

$$A_{OS}(\mathcal{A}) := \mathbb{K}\{e_H\}_{H \in \mathcal{A}} / \left(\partial e_C : \text{circuits } C \right)$$

$$A_{VG}(\mathcal{A}) := \mathbb{K}[x_H]_{H \in \mathcal{A}} / \left(\partial x_C : \text{circuits } C \right)$$

where for a circuit $C = \{H_1, \dots, H_p\}$ with $\sum_{j=1}^p c_j \alpha_j = \underline{0}$ if $H_j = \ker(\alpha_j)$

$$\partial e_C := \sum_{j=1}^p (-1)^j e_{H_1} \wedge \dots \wedge \widehat{e_{H_j}} \wedge \dots \wedge e_{H_p}$$

$$\partial x_C := \sum_{j=1}^p \text{sgn}(c_j) x_{H_1} \dots \widehat{x_{H_j}} \dots x_{H_p}$$

PROPOSITION: Assume the arrangement \mathcal{A} has symmetries W and X a modular atom (=line) with W -stabilizer N_X .

(Orlik-Terao 1992, ARS 2023)

if $H, H' \not\supset X$
then $\exists H'' \supset X$
with $H'' \supset H \cap H'$

Then both algebras $A(\mathcal{A}) = A_{\text{os}}(\mathcal{A}), A_{\text{vc}}(\mathcal{A})$ have an N_X -equivariant s.e.s.

$$0 \rightarrow A(\mathcal{A}_X) \rightarrow A(\mathcal{A}) \begin{matrix} \downarrow^W \\ \downarrow_{N_X} \end{matrix} \xrightarrow{\bigoplus_{H: H \not\supset X} j_H} \left[\bigoplus_{H: H \not\supset X} A(\mathcal{A}^H) \right] (-1) \rightarrow 0$$

The maps j_H come from addition-deletion sequences

$$0 \rightarrow A(\mathcal{A} - \{H\}) \rightarrow A(\mathcal{A}) \xrightarrow{j_H} A(\mathcal{A}^H) \rightarrow 0$$

$$e_{H_1} \wedge \dots \wedge e_{H_p} \longmapsto 0 \text{ if } H \notin \{H_1, \dots, H_p\}$$

$$e_H \wedge e_{H_1} \wedge \dots \wedge e_{H_p} \longmapsto e_{H \cap H_1} \wedge \dots \wedge e_{H \cap H_p}$$

3. Koszulity review

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a **standard graded associative k -algebra**,
generated by $A_1 =: V$

$$= k \oplus A_1 \oplus A_2 \oplus \dots$$

$\underbrace{\hspace{2cm}}_{k\text{-basis}}$
 x_1, \dots, x_n for $V = A_1$

$$\cong \underbrace{k\langle x_1, \dots, x_n \rangle}_{T(V)} / I$$

where the 2-sided ideal I
is **homogeneous**:

$$I = \bigoplus_{d=0}^{\infty} I_d \text{ where } I_d = I \cap A_d$$

DEFINITION: A is Koszul if \exists an A -free resolution of $k = A/A_+$ which is linear:

$$\begin{array}{ccccccc}
 0 \leftarrow k \leftarrow A & \xleftarrow{d_1} & A(-1)^n & \xleftarrow{d_2} & A(-2)^{\beta_2} & \xleftarrow{d_3} & A(-3)^{\beta_3} \leftarrow \dots \\
 0 \leftarrow 1 & \xleftarrow{x_1} & [x_1 \dots x_n] & & \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} & & \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} \\
 0 \leftarrow & \xleftarrow{x_n} & & & & &
 \end{array}$$

all entries lie in A_+

Equivalently, $\text{Tor}_i^A(k, k)_j = 0$ unless $i=j$

(or same for $\text{Ext}_A^i(k, k)_j$)

When A is Koszul, one can write down a beautiful explicit resolution of k called the **Priddy complex** ...

A is necessarily a **quadratic algebra**

i.e. $A = k\langle x_1, \dots, x_n \rangle / I$ with $I = \left(\underset{V \otimes V}{I_2} \right)$

so one can define its **quadratic dual algebra**

$A^! := k\langle \underbrace{y_1, \dots, y_n}_{V^* \text{ basis dual to } x_1, \dots, x_n} \rangle / J$ where $J := \left(\underset{\substack{\text{perp with respect to} \\ (x \otimes x', y \otimes y') := (x, y) \cdot (x', y')}}{I_2^\perp} \right)$

The **Priddy complex** is $A \otimes (A^!)^*$, linearly resolving k :

$$0 \leftarrow k \leftarrow A \otimes (A^!)^* \xleftarrow{d_1} A \otimes (A^!)^* \xleftarrow{d_2} A \otimes (A^!)^* \xleftarrow{d_3} \dots$$

with $d_i = \text{mult. by } \sum_{j=1}^n x_j \otimes (y_j)^*$

Exactness of the Priddy complex shows

$$\text{Hilb}((A^!)^*, -t) \cdot \text{Hilb}(A, t) = 1 \quad [= \text{Hilb}(k, t)]$$

so

$$\text{Hilb}(A^!, t) = \frac{1}{\text{Hilb}(A, -t)}$$

When W acts on A via graded k -algebra automorphisms, exactness in degree d shows **virtual character identities**

$$(A^!_d)^* - A_1 \otimes (A^!_{d-1})^* + A_2 \otimes (A^!_{d-2})^* - \dots \pm (A^!_1)^* \otimes A_{d-1} \mp A_d = 0$$

expressing $(A^!_d)^*$ (and hence also $A^!_d$)

recursively in terms of A_0, A_1, \dots, A_d

A useful sufficient condition for Koszulity of A :

PROPOSITION:

When $A = \Lambda(e_1, \dots, e_n)/I$
or
 $k[x_1, \dots, x_n]/I$

and \exists a monomial order \prec on $\Lambda(e_1, \dots, e_n)$
 $k[x_1, \dots, x_n]$

for which I has a **quadratic Gröbner basis**
i.e. $\text{in}_{\prec}(I)$ is a **quadratic (monomial) ideal**,

then **A is Koszul.**

4. Supersolvability

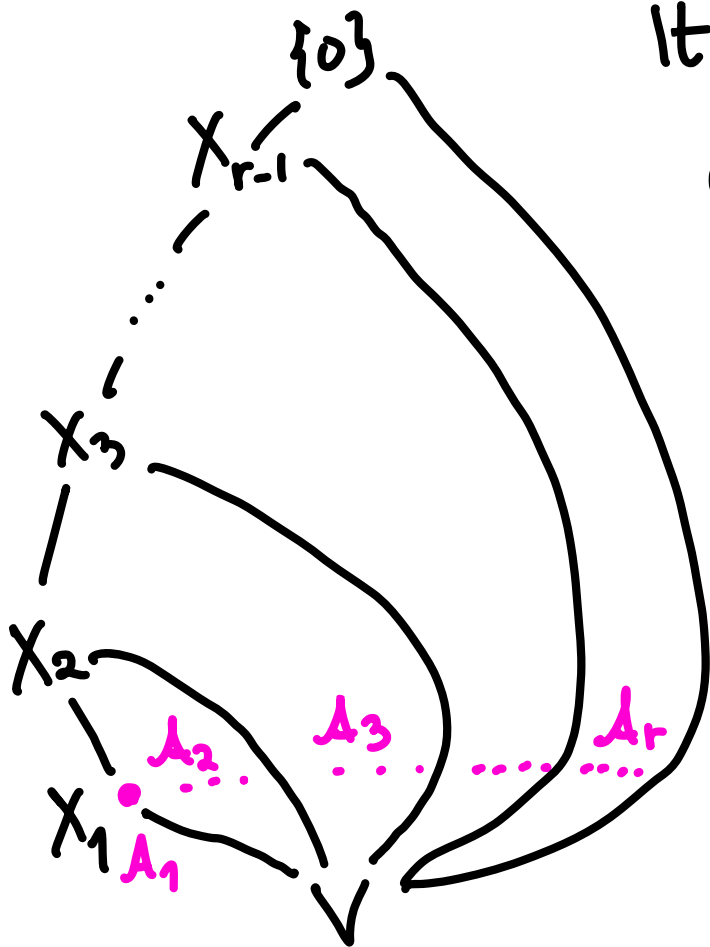
DEFINITION: A central arrangement \mathcal{A} is **supersolvable** if it has an **M-chain** := maximal flag of modular flats $V \supset X_1 \supset X_2 \supset \dots \supset X_{r-1} \supset \{0\}$

It partitions $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \dots \sqcup \mathcal{A}_r$

where $\mathcal{A}_i = \mathcal{A}_{X_i} - \mathcal{A}_{X_{i-1}} = \{H \in \mathcal{A} : H \supset X_i, H \not\supset X_{i-1}\}$

and defines exponents e_1, e_2, \dots, e_r by

$$e_i := |\mathcal{A}_i|$$



THEOREM

(Björner 1990

Björner-Ziegler 1991

Shelton-Yuzvinsky 1997

Peeva 2003

Dorpalen-Barry 2021)

Either of the rings $A(t) = A_{\text{os}}(A)$, $A_{\text{rg}}(A)$
has a term order $<$ on $\Lambda(e_1, \dots, e_n)$ or $k[x_1, \dots, x_n]$
for which its defining ideal has a
quadratic Gröbner basis

$\Leftrightarrow A$ is supersolvable

In this case, the term orders $<$ which work are
those having $x_{H_i} < x_{H_j}$ if $\begin{cases} H_i \in A_i \\ H_j \in A_j \end{cases}$ with $i < j$.

Furthermore, the initial terms look like $x_H x_{H'}$ for $H, H' \in A_i$
so the standard monomial NBC-basis is the set of
monomials with at most one "finger" x_H from each "hand" A_i

$$\Rightarrow \text{Hilb}(A(A), t) = (1 + e_1 t)(1 + e_2 t) \dots (1 + e_r t)$$

COROLLARY: When A is supersolvable,

both $A(A) = A_{OS}(A)$, $A_{VG}(A)$ are Koszul,

$$\text{with } \text{Hilb}(A', t) = \frac{1}{\text{Hilb}(A, -t)} = \frac{1}{(1-e_1 t)(1-e_2 t) \dots (1-e_r t)}$$

COROLLARY: For A supersolvable, $A(A)' = \begin{cases} A_{OS}(A)' \\ A_{VG}(A)' \end{cases}$
(ARS 2023?)

have (noncommutative) quadratic Gröbner-basis

with initial terms $y_{H_j} y_{H_i}$ for $\begin{cases} H_i \in A_i \\ H_j \in A_j \end{cases}$ with $i < j$

and standard monomial basis

$\left\{ m^{(1)} \cdot m^{(2)} \cdot \dots \cdot m^{(r)} : \begin{array}{l} m^{(i)} \text{ noncommutative} \\ \text{monomial in } \{y_H\}_{H \in A_i} \end{array} \right\}$

i.e. "no revisiting an earlier hand".

(or "all-you-can-eat salad bar, but keep it moving!")

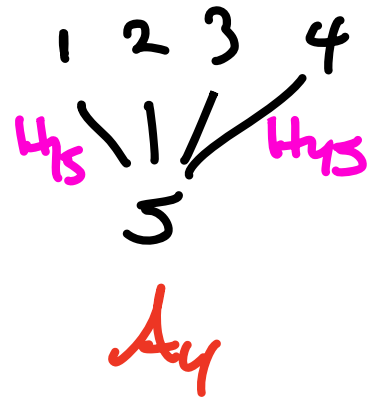
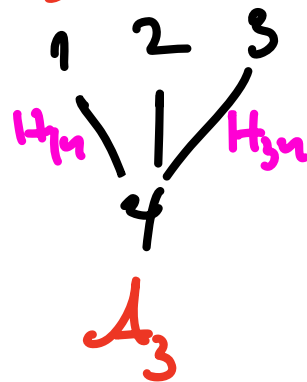
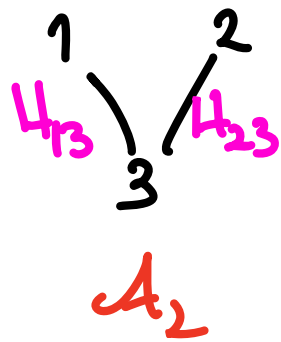
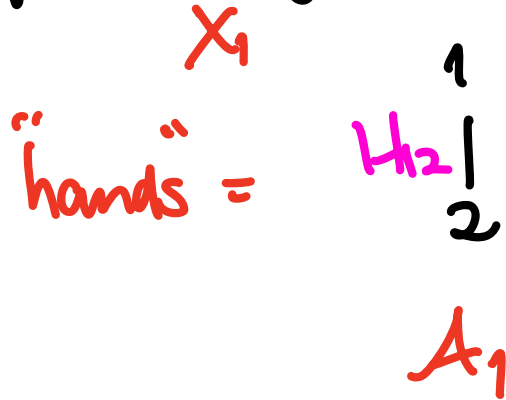
5. (Koszul dual) algebras of the 2nd kind

EXAMPLE Type A reflection arrangement in $V = \mathbb{R}^n / \mathbb{R} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

for $W = \mathfrak{S}_n$ has hyperplanes $\{H_{ij} = \{x_i = x_j\} : 1 \leq i < j \leq n\}$,

supersolvable, M-chain:

$$V \supset \{x_1 = x_2\} \supset \{x_1 = x_2 = x_3\} \supset \dots \supset \{x_1 = x_2 = \dots = x_{n-1}\} \supset \{x_1 = x_2 = \dots = x_n\} = \{0\}$$

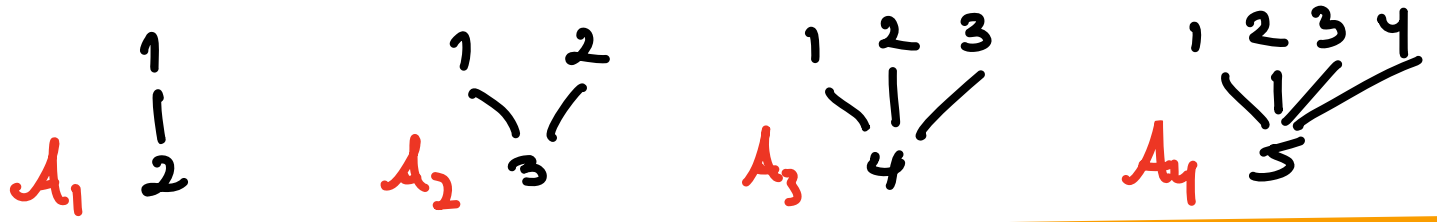


COROLLARY:

$$\text{Hilb}(A(n)!, t) = \frac{1}{(1-t)(1-2t)\dots(1-(n-1)t)} = \sum_{i=0}^{\infty} S(n+i, n-1) t^i$$

EXAMPLE

$A(\mathfrak{S}_5)!$ has standard monomials $m^{(1)} m^{(2)} m^{(3)} m^{(4)}$ where $m^{(i)}$ picks noncommutative monomials in i^{th} hand A_i :



e.g. $m^{(1)} \cdot m^{(2)} \cdot m^{(3)} \cdot m^{(4)}$
 $= y_{12} y_{12} \cdot y_{13} y_{23} y_{23} y_{13} \cdot y_{34} y_{14} \cdot y_{45} y_{45}$

$\in A(\mathfrak{S}_5)!_{10}$
 $\text{dim} = S(14, 4)$
 $5-1+10$

to get a set partition } insert spacers
 1 — 2 — 3 — 4 — 5
 and record 1st indices i in y_{ij}
 to get restricted growth function

1 1 1 2 1 2 2 1 3 3 1 4 4 2 5
 1 2 3 4 5 6 7 8 9 10 11 12 13 14

\mapsto block 1 {1, 2, 3, 5, 8, 11}, block 2 {4, 6, 7, 14}, block 3 {9, 10}, block 4 {12, 13}

$S(n,k)$

	k=				
n=	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

OS

	1	2	3	4	5
1	田				
2	田	田			
3	田	田	田		
4	田	田	田	田	
5	田	田	田	田	田

$A_{OS}(2)!$ $A_{OS}(3)!$ $A_{OS}(4)!$ $A_{OS}(5)!$ $A_{OS}(6)!$

VG

	1	2	3	4	5
1	田				
2	田	田			
3	田	田	田		
4	田	田	田	田	
5	田	田	田	田	田

$A_{VG}(2)!$ $A_{VG}(3)!$ $A_{VG}(4)!$ $A_{VG}(5)!$ $A_{VG}(6)!$

6. Properties and Questions

- Branching rule

$$S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$$

generalizes to...

PROPOSITION: Both $A(n)_i^!$ and $A_{OS}(n)_i^!$, $A_{VG}(n)_i^!$
(Sundaram)
1994,
2020
have these branching rules
for restriction \mathfrak{G}_n to \mathfrak{G}_{n-1} :

$$A(n)_i^! \downarrow_{\mathfrak{G}_{n-1}}^{\mathfrak{G}_n} \cong \chi_{\text{def}}^{(n-1)} \otimes A(n)_{i-1}^! \downarrow_{\mathfrak{G}_{n-1}}^{\mathfrak{G}_n} \oplus A(n-1)_i^!$$

Better phrasing:

PROPOSITION: $A^{(n)} = \left\{ \begin{matrix} AOS^{(n)} \\ AVG^{(n)} \end{matrix} \right\}$ have \mathfrak{S}_{n-1} -equivariant s.e.s.
 (ARS 2023)

$$0 \leftarrow A^{(n-1)} \leftarrow A^{(n)} \begin{matrix} \downarrow \mathfrak{S}_n \\ \mathfrak{S}_{n-1} \end{matrix} \leftarrow \left[\chi_{\text{def}}^{(n-1)} \otimes A^{(n)} \right](-1) \leftarrow 0$$

and more generally ...

PROPOSITION: For \mathcal{A} supersolvable, χ a modular datum,
 (ARS 2023) one has an N_χ -equivariant s.e.s.

$$0 \leftarrow A(\mathcal{A}_\chi) \leftarrow A(\mathcal{A}) \begin{matrix} \downarrow N_\chi \\ \mathfrak{S}_H \end{matrix} \begin{matrix} \oplus i_H \\ H: H \not\cong \chi \end{matrix} \left[\begin{matrix} \oplus A(\mathcal{A}) \\ H: H \not\cong \chi \end{matrix} \right](-1) \leftarrow 0$$

$$0 \leftarrow \gamma_H \begin{matrix} \longleftarrow \\ \text{if } H \not\cong \chi \end{matrix} \leftarrow \alpha_H \leftarrow \gamma_H \leftarrow a$$

• Representation stability

DEFINITION (Church-Farb 2005) A sequence $\{V_n\}_{n=1,2,\dots}$ of G_n -reps is **representation-stable**

if $\exists \lambda^{(1)}, \dots, \lambda^{(t)}$
and constants $c_1, \dots, c_t \in \mathbb{N}$

such that $\forall n \gg 0$, the G_n -irreducible decomposition looks like

$$V_n \cong \bigoplus_{i=1}^t \left[\lambda^{(i)} \right]^{\oplus c_i}$$

The diagram shows a direct sum of irreducible representations. For each i from 1 to t , there is a box containing a Young diagram for the partition $\lambda^{(i)}$ (shaded green) and a pink bar above it representing the multiplicity $n - h^{(i)}$. The entire direct sum is labeled $\oplus c_i$.

THEOREM: For fixed $i = 0, 1, 2, \dots$
(Church-Farb)

$$\text{both } A(n) = \begin{cases} A_{OS}(n) \\ A_{VG}(n) \end{cases}$$

have $\{A(n)_i\}$ representation - stable

... and more generally,

$\{H^i \text{ Conf}(n, X)\}$ is rep-stable

for certain kinds of manifolds X .

COROLLARY For fixed $i=0,1,2,\dots$

(ARS 2023)

$\{A(n)_i\}$

are also representation-stable

proof: Induct on i . Virtually we have

$$A(n)_i = \sum_{j=1}^i A(n)_j \otimes A(n)_{i-j}$$

rep-stable by Church-Farb rep-stable by induction

rep-stable by Murnaghan's Stability Thm:

$$\chi^{\square_a} \otimes \chi^{\square_{\mu}} \text{ stabilizes}$$

for large n .



QUESTION:

Can we use more FI-module theory to bound the stable range for $n \gg 0$?

This could help approach ...

QUESTION:
















Can we find exact formulas for $A(n)_i$?

e.g., generating functions for $\sum_{n,i} \text{ch } A(n)_i \cdot t^i$
















involving plethysm

or infinite products?

Boundary cases

OS	1	2	3	4	5
1					
2					
3					
4					
5					

$A_{OS}(2)!$ $A_{OS}(3)!$ $A_{OS}(4)!$ $A_{OS}(5)!$ $A_{OS}(6)!$

VG	1	2	3	4	5
1					
2					
3					
4					
5					

$A_{VG}(2)!$ $A_{VG}(3)!$ $A_{VG}(4)!$ $A_{VG}(5)!$ $A_{VG}(6)!$

$$S(n-1, n-1) = 1 \rightsquigarrow \text{ch } A(n)_{n-1}! = S_{\text{tower}}$$

$$S(m, 1) = 1 \rightsquigarrow \text{ch } A_{OS}(2)_i! = S_{\text{tower}}$$

$$\text{ch } A_{VG}(2)_i! = \begin{cases} S_{\text{tower}} & i \text{ even} \\ S_{\text{tower}} & i \text{ odd} \end{cases}$$

OS	1	2	3	4	5
1					
2					
3					
4					
5					

$A_{OS}(2)!$ $A_{OS}(3)!$ $A_{OS}(4)!$ $A_{OS}(5)!$ $A_{OS}(6)!$

VG	1	2	3	4	5
1					
2					
3					
4					
5					

$A_{VG}(2)!$ $A_{VG}(3)!$ $A_{VG}(4)!$ $A_{VG}(5)!$ $A_{VG}(6)!$

$$S(n, n-1) = \binom{n}{2} \rightsquigarrow \text{ch } A(n)_{n-1}! = \begin{cases} s_{\square} s_{\underbrace{\text{---}}_{n-2}} & \text{for OS} \\ s_{\text{---}} s_{\underbrace{\text{---}}_{n-2}} & \text{for VG} \end{cases}$$

$$S(n, 2) = 2^{n-1} = 1 + 2 + 2^2 + \dots + 2^{n-2} \rightsquigarrow A_{OS}(3)_2! = 1 + \chi^{\square} + \chi^{\square} \otimes \chi^{\square} + \chi^{\square} \otimes \chi^{\square} \otimes \chi^{\square} + \dots + (\chi^{\square})^{\otimes i}$$

Most mysterious ...

CONJECTURE: $A_{OS}(n)_i!$ is always

an \mathfrak{S}_n -permutation representation (!)

$i=0$

$i=1$

OS	1	2	3	4	5
1					
2					
3					
4					
5					

$A_{OS}(2)!$ $A_{OS}(3)!$ $A_{OS}(4)!$ $A_{OS}(5)!$ $A_{OS}(6)!$

Verified for $n = 2, 3, 4$
 $i = 0, 1$

(The \mathfrak{S}_n -orbit stabilizers are
not all parabolic subgroups
 $\mathfrak{S}_{\alpha_1} \times \dots \times \mathfrak{S}_{\alpha_l}$)

• Homotopy Lie algebras (see A. Susi's talk?)

\exists a graded Lie (super)algebra $\mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d$ with

$$A(n)! = \text{Ext}^{A(n)}(\mathbb{k}, \mathbb{k}) \cong \mathcal{U}(\mathcal{L})$$

universal enveloping algebra

$$\cong \text{Sym}^{\pm}(\mathcal{L})$$

PBW Thm.

graded polynomial algebra

$$\cong \left\{ \begin{array}{l} \bigoplus_{\lambda=1}^{\infty} \bigotimes_{m_i} \text{Sym}^{m_i}(\mathcal{L}_i) \quad \text{for } A_{OS}(n) \\ \bigoplus_{\lambda=1}^{\infty} \bigotimes_{i \text{ odd}} \wedge^{m_i}(\mathcal{L}_i) \otimes \bigotimes_{i \text{ even}} \text{Sym}^{m_i}(\mathcal{L}_i) \quad \text{for } A_{VG}(n) \end{array} \right.$$

So if we understood the \mathfrak{S}_n -representations on

$$\mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d$$

it would help us understand those on $A(n)$;

Sadly, calculations for $A_{OS}(n) = \text{Sym}(\mathcal{L})$

show that these \mathcal{L}_d

are **not** \mathfrak{S}_n -permutation reps!

Thanks ICMS,
and thank you for
your attention!