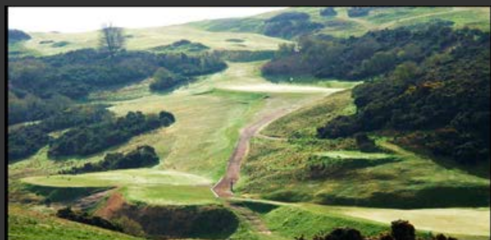
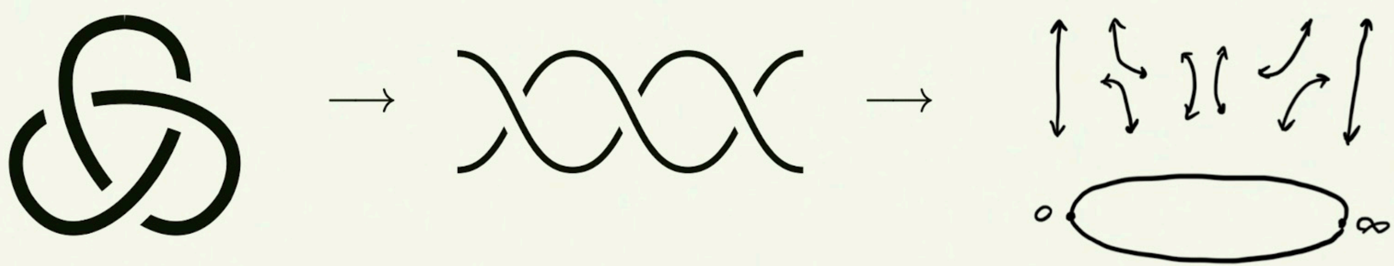
Combinatorics & Braid Varieties



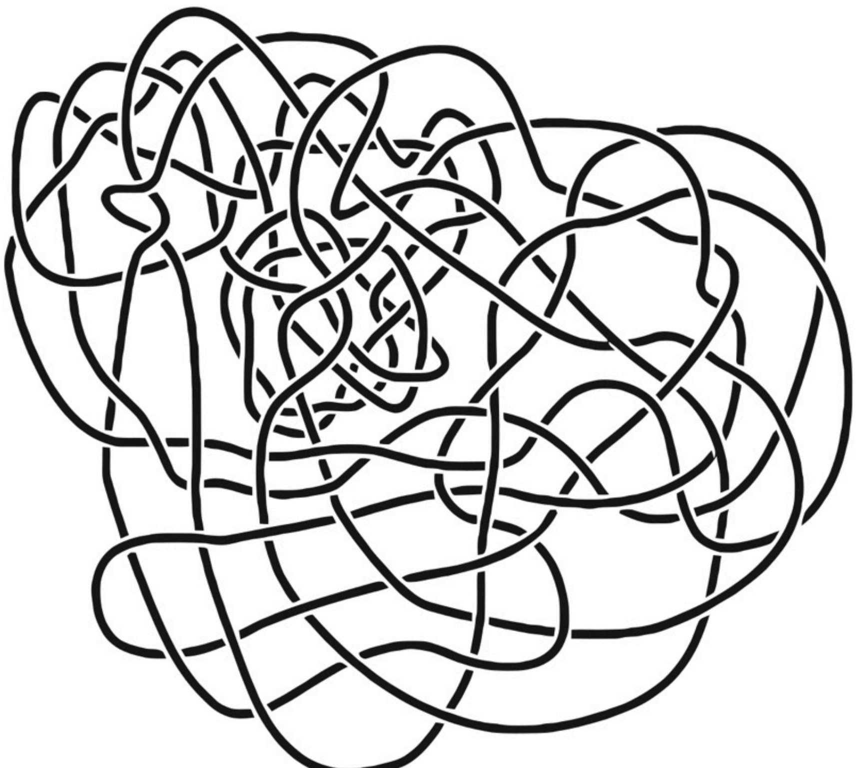
Nathan Williams
UTD
personal.utdallas.edu/~nxw170830



ORIENTED LINKS

THE GORDIAN KNOT

A *knot* is an embedding of S^1 into $\mathbb{R}^3$.



$= \bigcirc$?

REIDEMEISTER MOVES

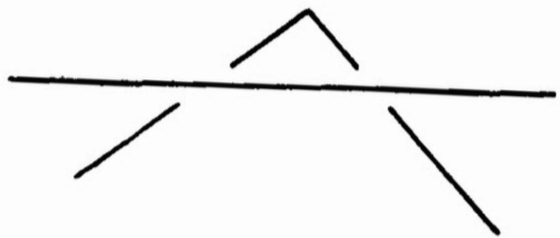


Fig. 1.

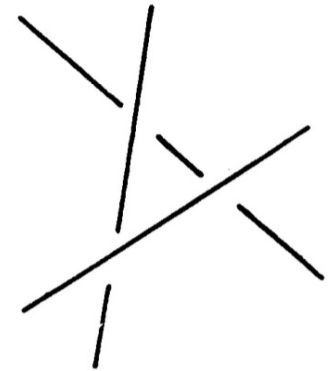


Fig. 3.

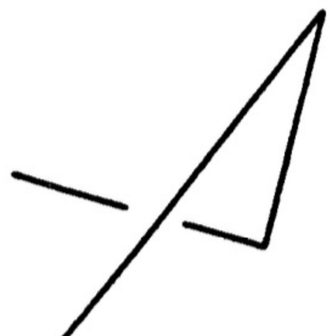


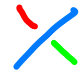
Fig. 2.

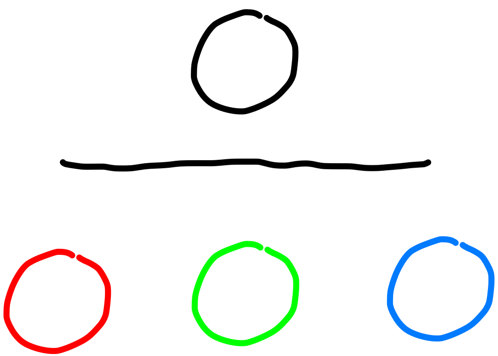
Elementare Begründung der Knotentheorie[1]).

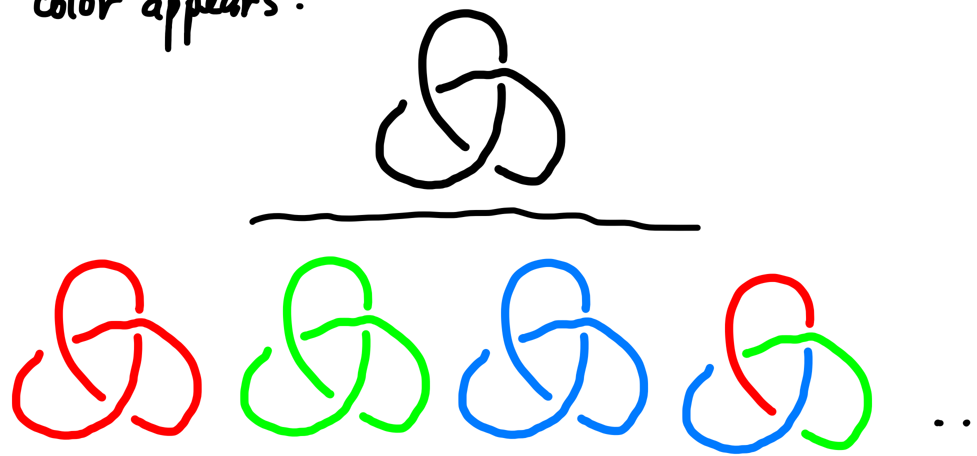
Von KURT REIDEMEISTER in Königsberg.

INVARIANTS

 # tricolorings : at each crossing  either

- all three colors appear  or
- only one color appears.





HOMFLYPT Polynomial

A *knot* is an embedding of S^1 into $\mathbb{R}^3$. A *link* is a disjoint collection of knots.

Definition (Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki, Traczyk)

Received by the editors January 14, 1985.
1980 *Mathematics Subject Classification*. Primary 57M25.

[1] *Editor's Note*. The editors received, virtually within a period of a few days in late September and early October 1984, four research announcements, each describing the same result—the existence and properties of a new polynomial invariant for knots and links. There was variation in the approaches taken by the four groups and variation in corollaries and elaboration. These were: *A new invariant for knots and links* by Peter Freyd and David Yetter; *A polynomial invariant of knots and links* by Jim Hoste; *Topological invariants of knots and links*, by W. B. R. Lickorish and Kenneth C. Millett, and *A polynomial invariant for knots: A combinatorial and an algebraic approach*, by A. Ocneanu.

It was evident from the circumstances that the four groups arrived at their results completely independently of each other, although all were inspired by the work of Jones (cf. [**10**], and also [**8**, **9**]). The degree of simultaneity was such that, by common consent, it was unproductive to try to assess priority. Indeed it would seem that there is enough credit for all to share in.

Each of these papers was refereed, and we would have happily published any one of them, had it been the only one under consideration. Because the alternatives of publication of all four or of none were both unsatisfying, all have agreed to the compromise embodied here of a paper carrying all six names as coauthors, consisting of an introductory section describing the basics written by a disinterested party, and followed by four sections, one written by each of the four groups, briefly describing the highlights of their own approach and elaboration.

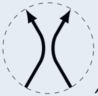
HOMFLYPT Polynomial

A *knot* is an embedding of S^1 into $\mathbb{R}^3$. A *link* is a disjoint collection of knots.

Definition (Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki, Traczyk)

The *HOMFLYPT polynomial* $P(L)$ associated to a link L is defined by skein relations:

$$P\left(\bigcirc\right) = 1 \qquad \text{and}$$

aP () $- a^{-1}P$ () $= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})P$ () .

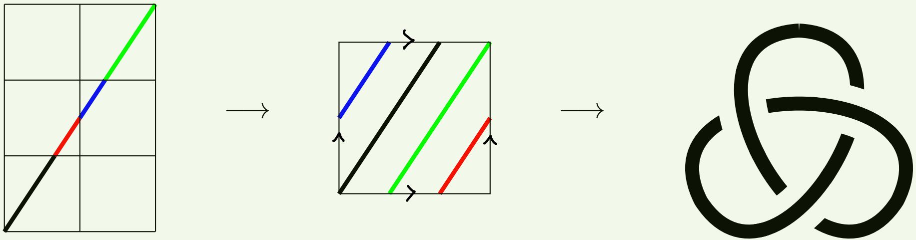
Specializations of a and q recover the Jones and Alexander polynomials.

Torus Knots

Definition

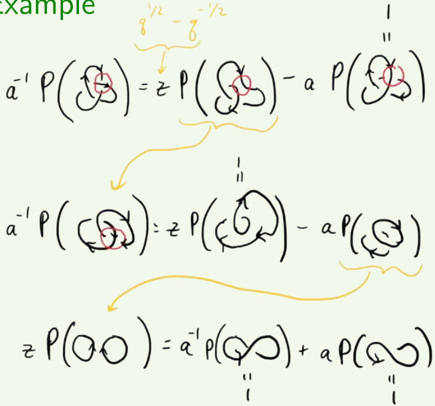
Fix n and p coprime. Thinking of a torus as $S^1 \times S^1$, the *(n,p)-torus knot* $T_{n,p}$ winds n times around one S_1 and p times around the other S^1.

Example



HOMFLYPT of $T_{1,2}$ and $T_{2,3}$

Example



$P($$) = aza\left[z - az^{-1}\left[a^{-1}+a\right]\right] - a$

$= -a^{-4} + [z^2+2]\,a^{-2}$

$= -a^{-4} + [q^2+1]\,q^{-1}a^{-2}$

$2!$

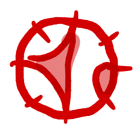
WHAT ABOUT $T_{3,4}, T_{4,5}, \dots, T_{n,n+1}$?

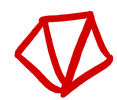
DEF The **Catalan numbers** are the integers

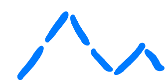
Pak credits Riordan for the name

$$Cat(n) = \frac{1}{2n+1}\binom{2n+1}{n}$$

EX 1, 2, 5, 14, 42, 132, ...







Cat(n) counts: noncrossing partitions, triangulations, Dyck paths,

etc, etc, etc, etc, etc, ...

REF Pak. "History of Catalan Numbers"
Stanley. "Catalan Numbers"

"<u>THM</u>" (Folklore)

Just about every combinatorial object is Catalan.

HOMFLYPT of a Torus Knot

for $p=n+1$, these are
Kirman numbers – f-vector of associahedron
(see Reiner-Shepler-Sommers!)

Theorem (Jones, see also Gorsky)

Up to (predictable) sign and a power of q,

$$[\text{coeff of } a^{-(p-1)(n-1)-2k}]P(T_{n,p}) = \overbrace{\frac{1}{[p]_q}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n+p-k-1 \\ n \end{bmatrix}_q}.$$

Also from Theo's talk

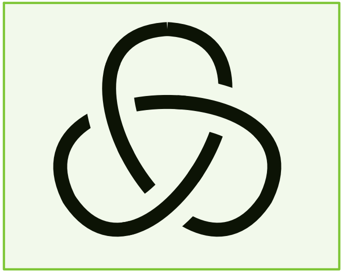
As a very special case,

$$[\text{top coeff of } a]P(T_{n,n+1})\big|_{@q=1} = \frac{1}{n+1}\binom{2n}{n} = \mathrm{Cat}(n).$$

Combinatorial Motivation

Somehow, torus knots $T_{n,n+1}$ seem to "know" about Catalan numbers.

$$[\text{top coefficient of } a]P(T_{n,n+1})\Big|_{@q=1} = \mathrm{Cat}(n).$$



- Find Catalan objects hidden in the knot $T_{n,n+1}$.
- Find "rational" Catalan objects hidden in the knot $T_{n,p}$.
- Generalize. (other types, other braids, etc.)

BRAIDS

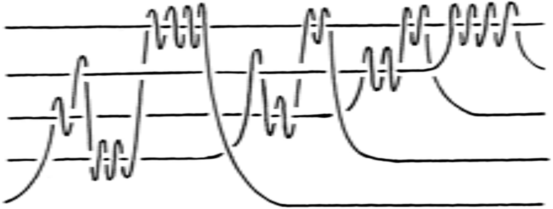
Braid Groups

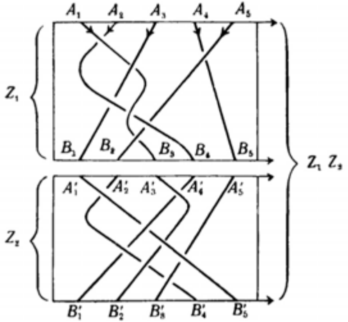
Definition (Artin)

The *braid group* B_n has presentation

$$B_n = \left\langle \mathsf{s}_1, \ldots, \mathsf{s}_{n-1} : \mathsf{s}_i \mathsf{s}_{i+1} \mathsf{s}_i = \mathsf{s}_{i+1} \mathsf{s}_i \mathsf{s}_{i+1}, \mathsf{s}_i \mathsf{s}_j = \mathsf{s}_j \mathsf{s}_i \right\rangle.$$

sorry theo!





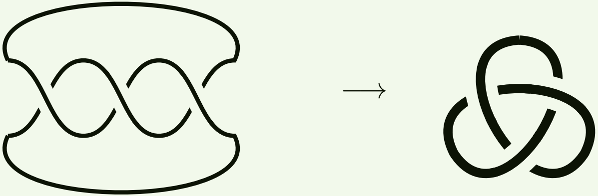
Theorie der Zöpfe.

Von EMIL ARTIN in Hamburg.

Oriented Links from Braids

Observe that a braid α closes to an oriented link $\widehat{\alpha}$.

Example



- Alexander: *every* oriented link arises as a braid closure. $T_{n,p} \sim (\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1})^p$.
- *Markov moves*: any two braid closures representing the same oriented link are connected by moves of the form

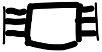
$$\widehat{\boldsymbol{\alpha}\boldsymbol{\beta}} \sim \widehat{\boldsymbol{\beta}\boldsymbol{\alpha}} \qquad \text{and} \qquad \widehat{\boldsymbol{\alpha}\mathbf{s}_n^{\pm}} \sim \widehat{\boldsymbol{\alpha}} \text{ for } \boldsymbol{\alpha} \in B_n.$$

Traces

Write $K = \mathbb{Q}(q^{\pm 1/2})$.

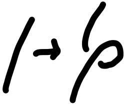
- Any K-linear link invariant $\mathrm{tr} : K[B_n] \to K(a)$ must be a *trace*:

$$\mathrm{tr}(\boldsymbol{\alpha}\boldsymbol{\beta}) = \mathrm{tr}(\boldsymbol{\beta}\boldsymbol{\alpha}).$$



- A trace is called *Markov* if also

$$\mathrm{tr}(\boldsymbol{\alpha}\mathbf{s}_n) = a\,\mathrm{tr}(\boldsymbol{\alpha}) \text{ for } \boldsymbol{\alpha} \in B_n.$$



Idea: look for trace invariants that factor through the *Hecke algebra*.

Hecke Algebras

Definition

Write $K = \mathbb{Q}(q^{\pm 1/2})$. The *Hecke algebra* H_n:

- is a quotient of the group algebra $K[B_n]$ with presentation

$$H_n = K[B_n]/\langle \mathbf{s}_i^2 = (q-1)\mathbf{s}_i + q \rangle,$$

  (write T_i for the image of $\mathbf{s}_i$ under this quotient);
- has a basis T_w as an K-algebra indexed by elements of the symmetric group S_n;
- is a deformation of the group algebra of S_n ($q \to 1$); and
- has "the same" representation theory as S_n ($\chi \in \mathrm{Irr}_{S_n} \leftrightarrow \chi_q \in \mathrm{Irr}_{H_n}$).

Markov Traces

Theorem (Ocneanu, Jones)

HOMFLYPT is the unique Markov trace from H_n to $K(a)$.

Theorem

Up to (predictable) sign and power of q, for any positive braid $\beta \in B_n$

$$\frac{1}{(q-1)^{n-1}}[\textit{top coeff in } a]\mathrm{HOMFLY}(\widehat{\beta}) = [T_1]T_\beta^{-1} =: \mathrm{tr}(T_\beta).$$

Not Markov

coeff of identity

Markov Traces

Theorem (Trinh, W. (2023+))
Up to (predictable) sign and power of q, for any positive braid $\beta \in B_n$

$$\frac{1}{(q-1)^{n-1}}[\textit{coeff of } a^{-\mathrm{top}-2k}]P(T_{n,p}) = \sum_{\substack{w \in S_n \\ \mathrm{des}_R(w)=\{s_1,s_2,\ldots,s_{k-1}\}}} q^{-\ell(w)}\mathrm{tr}(T_w T_{w^{-1}} T_\beta^{-1}).$$

The trace can be computed directly using the relations in the Hecke algebra, or by using the fact that H_n decomposes as a (weighted) direct sum over irreps:

$$\mathrm{tr} = \sum_{\chi_q \in \mathrm{Irr}(H_n)} \frac{1}{s(\chi_q)} \chi_q.$$

Example

Fix $n = 2$. Recall that $T_s^2 = (q-1)T_s + q$ and $T_s^{-1} = q^{-1}(T_s - (q-1))$.

Example

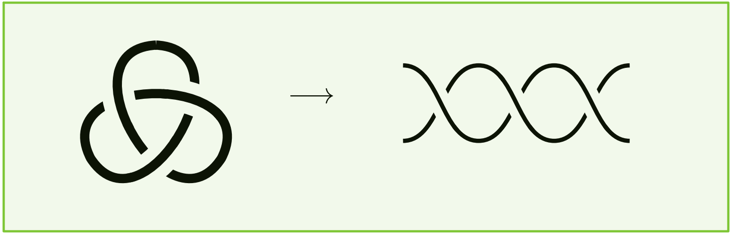
Compute the top coefficient in a of $\mathrm{HOMFLY}(\widehat{\mathbf{sss}}) = -a^{-4} + a^{-2}q^{-1}(q^2+1)$ as

$$
\begin{aligned}
&[T_1]T_{\mathbf{sss}}^{-1} = [T_1]T_s^{-3} = [T_1]q^{-3}(T_s - (q-1))^3 \\
&= q^{-3}[T_1]\left(T_s^3 - 3(q-1)T_s^2 + 3(q-1)^2T_s - (q-1)^3\right) \\
&= q^{-3}[T_1]\left((q-1)T_s^2 + qT_s - 3(q-1)((q-1)T_s + q) + 3(q-1)^2T_s - (q-1)^3\right) \\
&= q^{-3}[T_1]\left((q-1)((q-1)T_s + q) + qT_s - [3q(q-1) + (q-1)^3]\right) \\
&= q^{-3}[T_1]\left((q + (q-1)^2)T_s - [3q(q-1) + (q-1)^3 - q(q-1)]\right) \\
&= -q^{-3}\left(3(q-1)q + (q-1)^3 - q(q-1)\right) \\
&= -q^{-3}(q-1)(q^2+1).
\end{aligned}
$$

Combinatorial Motivation

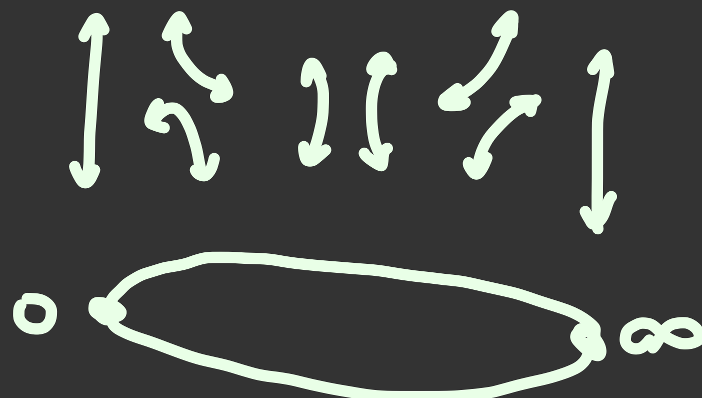
Write $\mathbf{c} = \mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-1} \in B_n$ so that the torus knot $T_{n,p} \sim \widehat{\mathbf{c}^p}$.
The braid $\mathbf{c}^{n+1}$ seems to "know" about Catalan numbers:

$$\left(\frac{1}{(q-1)^{n-1}}\mathrm{tr}(T_{\mathbf{c}^{n+1}})\right)\Bigg|_{@q=1} = \mathrm{Cat}(n).$$



- Find Catalan objects hidden in the *braid* $\mathbf{c}^{n+1}$.
- Find "rational" Catalan objects hidden in the braid $\mathbf{c}^p$.
- Generalize.

BRAID VARIETIES



PHILOSOPHY (Tits): "$\mathfrak{S}_n$ is $SL_n(\mathbb{F}_q)$ at $q=1$"

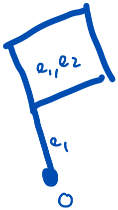
$$|SL_n(\mathbb{F}_q)| = (q-1)^{n-1} q^{\binom{n}{2}} \prod_{i=1}^{n-1} [i+1]_q$$

- Lie group $\qquad SL_n(\mathbb{F}_q)$
- Braid group $\qquad B_n$
- Hecke algebra $\qquad \mathcal{H}_n$
- Affine symmetric group $\qquad \widetilde{\mathfrak{S}}_n$

Flags

- the Borel subgroup $B = B_+ = B_+(\mathbb{F}_q)$ of upper triangular matrices:

$$B \simeq \left[0 \subseteq \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \mathbb{F}_q^n\right];$$



- the flag variety:

$$G/B \simeq \left\{ [V_0 \subset V_1 \subset \cdots \subset V_n] \text{ with } \dim(V_i) = i \right\}.$$

> **Definition**
> For $B', B'' \in G/B$, we say B' is in *relative position* s_i to B'' (written $B' \xrightarrow{s_i} B''$) if B' and B'' differ exactly in their ith and $(i+1)$st subspaces.

Relative position

Definition

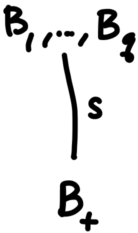
For $B', B'' \in G/B$, we say B' is in *relative position* s_i to B'' (written $B' \xrightarrow{s_i} B''$) if B' and B'' differ exactly in their ith and $(i+1)$st subspaces.

Example

For $G = \mathrm{SL}_2(\mathbb{F}_q)$ we have $|G/B| = q + 1$ and $S_2 = \{1, s\}$:

$$G/B = \{B_0 = B_+, B_1, B_2, \ldots, B_q = B_-\}$$

with relative positions given by $B_i \xrightarrow{e} B_i$ and $B_i \xrightarrow{s} B_j$ for $i \neq j$.



Braid Varieties

Definition

Let $\boldsymbol{\beta} = \boldsymbol{\beta}_2\boldsymbol{\beta}_2 \cdots \boldsymbol{\beta}_m \in B_n$ be a positive braid (with each $\boldsymbol{\beta}_i = \mathbf{s}_k$ for some k).
The *braid variety* corresponding to $\boldsymbol{\beta}$ is the closed subvariety of $(G/B)^{m+1}$:

$$R_{\boldsymbol{\beta}}(\mathbb{F}_q) = \left\{ B = B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} B_m \xleftarrow{w_\circ} B_- : B_i \in G/B \right\}$$

We can also "twist" braid varieties by an element $w \in W$:

$$R^{(w)}_{\boldsymbol{\beta}}(\mathbb{F}_q) = \left\{ w \cdot B = B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} B_m \xleftarrow{ww_\circ} B_- : B_i \in G/B \right\}.$$

For $G = \mathrm{SL}_2(\mathbb{F}_q)$ we have $|G/B| = q + 1$:

$$G/B = \{B_0 = B_+, B_1, B_2, \dots, B_q = B_-\}$$

with relative positions given by $B_i \xrightarrow{e} B_i$ and $B_i \xrightarrow{s} B_j$ for $i \neq j$.

$$\begin{tikzcd} B_1, \dots, B_q \arrow[d, dash, "s"] \\ B_+ \end{tikzcd}$$

Example

$$R_{\mathrm{sss}}(\mathbb{F}_q) = \left\{ \begin{array}{ll} (B \xrightarrow{s} B_i \xrightarrow{s} B_j \xrightarrow{s} B_k \xleftarrow{s} B_-) & \text{for } 1 \leq i \leq q-1 \text{ and } \substack{0 \leq j,k \leq q-1 \text{ with} \\ i \neq j \neq k}, \\ (B \xrightarrow{s} B_- \xrightarrow{s} B_i \xrightarrow{s} B_j \xleftarrow{s} B_-) & \text{for } 0 \leq i, j \leq q-1 \text{ with } i \neq j, \\ (B \xrightarrow{s} B_i \xrightarrow{s} B_- \xrightarrow{s} B_j \xleftarrow{s} B_-) & \text{for } 1 \leq i \leq q-1 \text{ and } 0 \leq j \leq q-1 \end{array} \right\}.$$

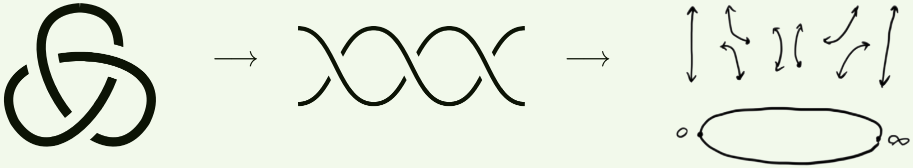
$$|R_{\mathrm{sss}}(\mathbb{F}_q)| = (q-1)^3 + 2q(q-1) = (q-1)(q^2+1).$$

Point Counts

Theorem

Up to (predictable) sign and a power of q, for any positive braid $\beta \in B_n$

$$\frac{1}{(q-1)^{n-1}}[\textit{top coefficient in } a]\mathrm{HOMFLY}(\widehat{\beta}) = \mathrm{tr}(T_\beta) = |R_\beta(\mathbb{F}_q)|.$$

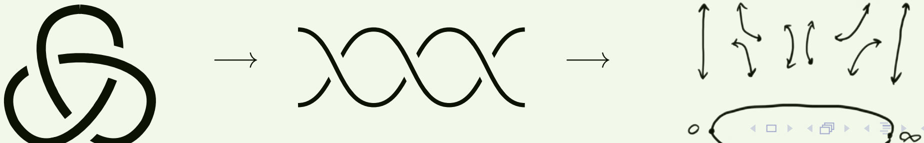


Point Counts

Theorem (Tran, W. (2023+))
Up to (predictable) sign and a power of q, for any positive braid $\beta \in B_n$

$$\frac{1}{(q-1)^{n-1}}[\textit{coeff of } a^{-\mathrm{top}-2k}]\mathrm{HOMFLY}(\widehat{\beta}) = \sum_{\substack{w \in S_n \\ \mathrm{des}_R(w)=\{s_1,s_2,\ldots,s_{k-1}\}}} q^{-\ell(w)}\mathrm{tr}(T_w T_{w^{-1}} T_\beta^{-1})$$

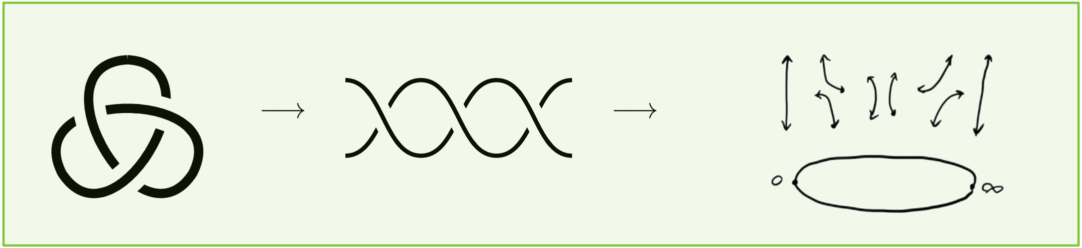
$$= \left| \bigsqcup_{\substack{w \in S_n \\ \mathrm{des}_R(w)=\{s_1,s_2,\ldots,s_{k-1}\}}} R_\beta^{(w)}(\mathbb{F}_q) \right| .$$



Combinatorial Motivation

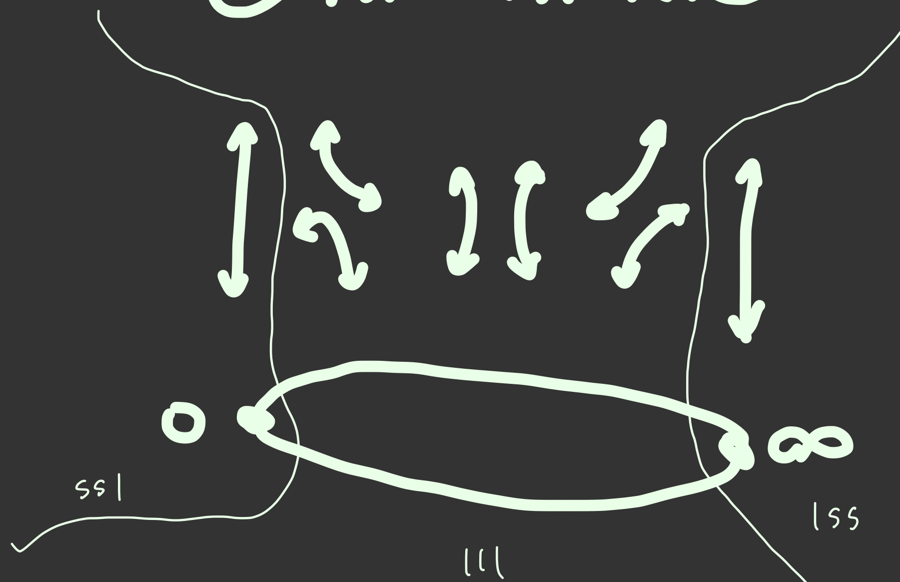
The braid variety $R_{\mathbf{c}^{n+1}}(\mathbb{F}_q)$ seems to "know" about Catalan numbers:

$$\left(\frac{1}{(q-1)^{n-1}}\left|R_{\mathbf{c}^{n+1}}(\mathbb{F}_q)\right|\right)\Big|_{@q=1} = \mathrm{Cat}(n).$$

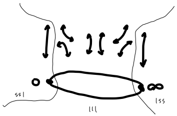


- Find Catalan objects hidden in the braid variety $R_{\mathbf{c}^{n+1}}(\mathbb{F}_q)$.
- Find "rational" Catalan objects hidden in the braid variety $R_{\mathbf{c}^{p}}(\mathbb{F}_q)$.
- Generalize.

COMBINATORICS



Deodhar decomposition



Definition (Deodhar)

Fix a positive braid $\boldsymbol{\beta} = \boldsymbol{\beta}_1 \boldsymbol{\beta}_2 \cdots \boldsymbol{\beta}_m$.

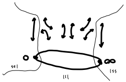
- A *subword* of $\boldsymbol{\beta}$ is a word $\mathbf{u} = \mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_m$ with
  - each $\mathbf{u}_i$ equal to either $\mathbf{1}$ or $\boldsymbol{\beta}_i$ (write $e_{\mathbf{u}} = \#\{\mathbf{u}_i = \mathbf{1}\}$);
  - $u_1 u_2 \cdots u_m = 1$ (the product is in S_n).
- $\mathbf{u}$ is *distinguished* if when $u_1 \cdots u_i \beta_{i+1} < u_i$, then $\mathbf{u}_{i+1} = \boldsymbol{\beta}_{i+1}$. (Set of: $D_{\boldsymbol{\beta}}$).
- $\mathbf{u}$ is *maximal distinguished* if it has as many $\mathbf{u}_i = \boldsymbol{\beta}_i$ as possible. (Set of: $M_{\boldsymbol{\beta}}$).

Example

$$D_{\mathbf{sss}} = \{\mathbf{111}, \mathbf{ss1}, \mathbf{1ss}\} \text{ and } M_{\mathbf{sss}} = \{\mathbf{ss1}, \mathbf{1ss}\}\,.$$

NOT s1s

Deodhar decomposition



Theorem

The braid variety decomposes as

$$R_\beta(\mathbb{F}_q) = \bigsqcup_{\mathbf{u}\in D_\beta} R_{\mathbf{u},\beta}(\mathbb{F}_q),$$

where $R_{\mathbf{u},\beta}(\mathbb{F}_q) = \left\{ B = B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} B_m : B_- \xrightarrow{u_{(i)}w_\circ} B_i \right\}$ *and each* $R_{\mathbf{u},\beta}(\mathbb{F}_q) \simeq (\mathbb{F}_q^\times)^{e_\mathbf{u}} \times \mathbb{F}_q^{d_\mathbf{u}}$.

Example

$$\begin{aligned} R_{\mathsf{sss}}(\mathbb{F}_q) &= R_{\mathbf{ss1},\mathsf{sss}}(\mathbb{F}_q) \sqcup R_{\mathbf{1ss},\mathsf{sss}}(\mathbb{F}_q) \sqcup R_{\mathbf{111},\mathsf{sss}}(\mathbb{F}_q) \\ &\simeq \left((\mathbb{F}_q^\times)^2 \times \mathbb{F}_q\right) \sqcup \left((\mathbb{F}_q^\times)^2 \times \mathbb{F}_q\right) \sqcup \left((\mathbb{F}_q^\times)^3\right). \\ |R_{\mathsf{sss}}(\mathbb{F}_q)| &= \underset{\text{ss1}}{(q-1)\,q} + \underset{\text{1ss}}{(q-1)\,q} + \underset{\text{111}}{(q-1)^3} = (q-1)(q^2+1). \end{aligned}$$

Fixed (thanks Etingof)

Combinatorial objects and the Deodhar decomposition

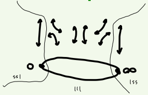
$$|R_\beta(\mathbb{F}_q)| = \sum_{\mathbf{u}\in D_\beta} |R_{\mathbf{u},\beta}(\mathbb{F}_q)| = \sum_{\mathbf{u}\in D_\beta} \left|(\mathbb{F}_q^\times)^{e_\mathbf{u}} \times \mathbb{F}_q^{d_\mathbf{u}}\right| = \sum_{\mathbf{u}\in D_\beta} (q-1)^{e_\mathbf{u}} q^{d_\mathbf{u}}.$$

At $q = 1$, M_β gives combinatorial objects:

$$\frac{1}{(q-1)^m}|R_\beta(\mathbb{F}_q)| = \sum_{\mathbf{u}\in D_\beta} (q-1)^{e_\mathbf{u}-m} q^{d_\mathbf{u}} \qquad = \sum_{\mathbf{u}\in M_\beta} q^{d_\mathbf{u}} + (q-1)\cdot\Big(\cdots\Big)$$

$$\lim_{q\to 1} \frac{1}{(q-1)^m}|R_\beta(\mathbb{F}_q)| \qquad = |M_\beta|.$$

Example



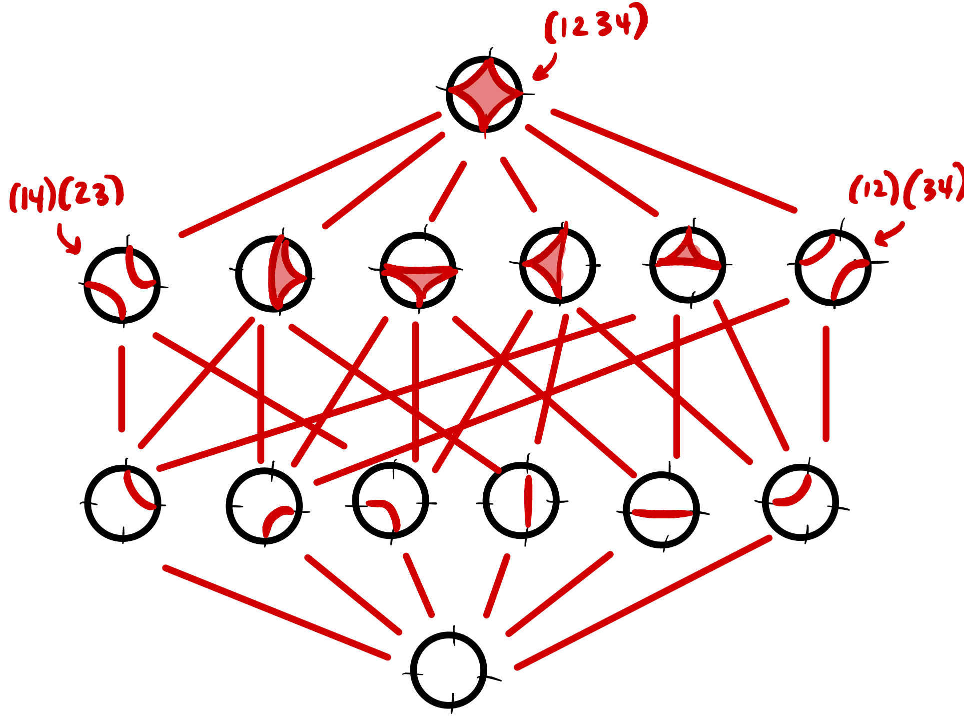
$$\lim_{q\to 1}\frac{1}{(q-1)}|R_{\mathsf{sss}}(\mathbb{F}_q)| = \lim_{q\to 1}\big(\overset{ssl}{q} + \overset{lss}{q} + \overset{lll}{(q-1)^2}\big) = 2 = \overset{\{ssl,\,lss\}}{|M_{\mathsf{sss}}|}.$$

also fixed.

NONCROSSING PARTITIONS

DEF $NC(n)$ = noncrossing (set) partitions ordered by refinement.

$NC(4) =$



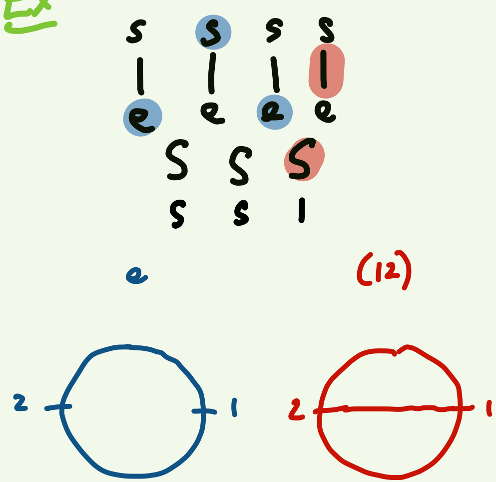
$\leq \mathfrak{S}_4$

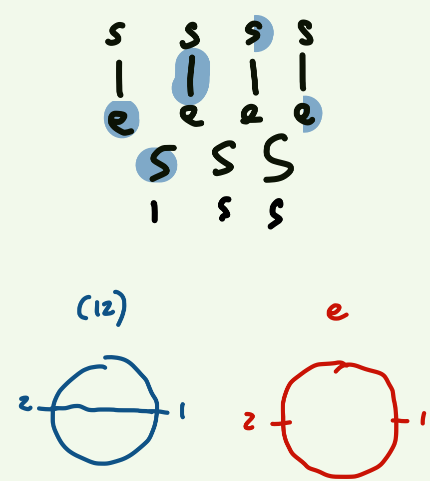
NONCROSSING PARTITIONS = $M_{c}^{?}$

THM (Galashin, Lam, Trinh, W.)

There is an "easy" bijection between $M_{c^{n+1}}$ and $NC(n)$.

Ex





NONCROSSING PARTITIONS = $M_{c'}$

(12) (24) (49) (23)

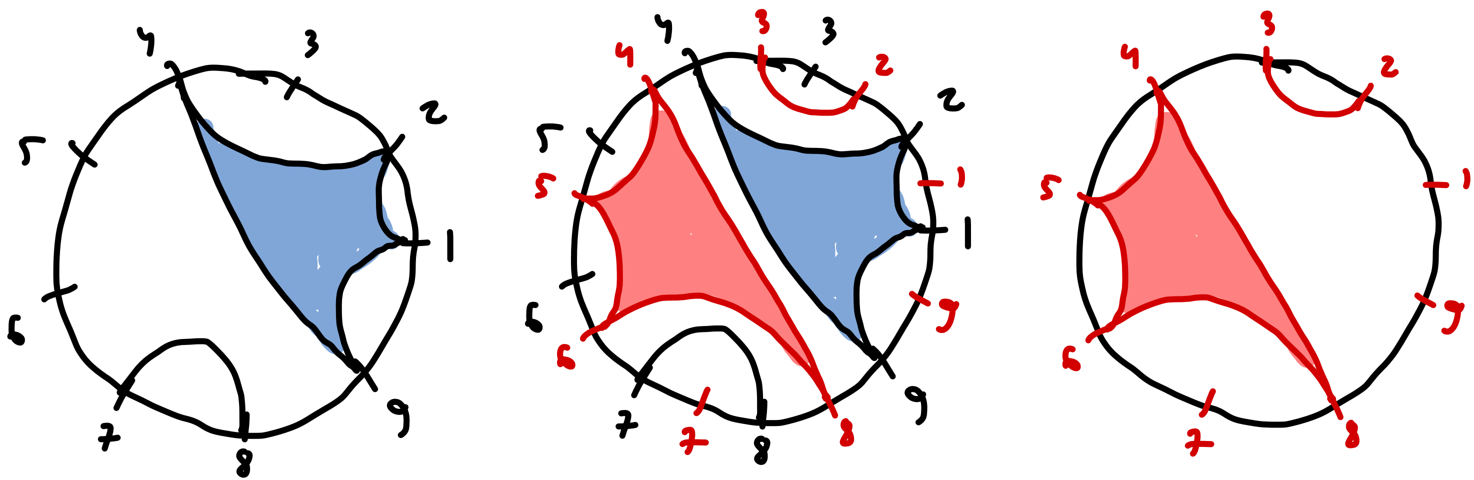
1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8

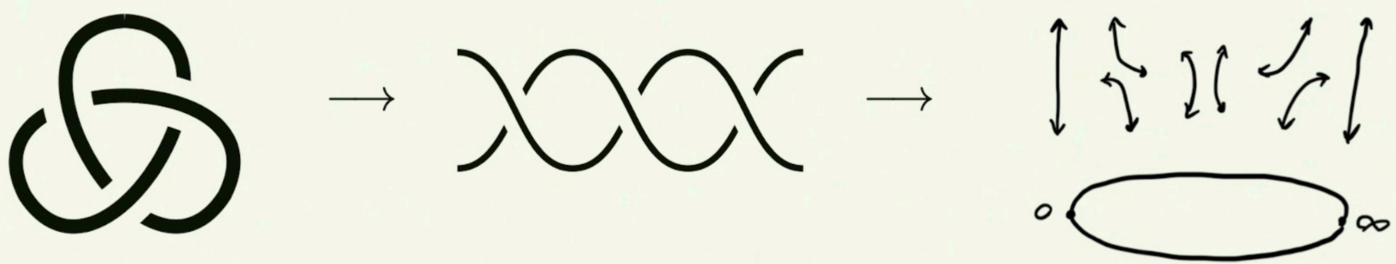
1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8

(78) (45)(56)(68)

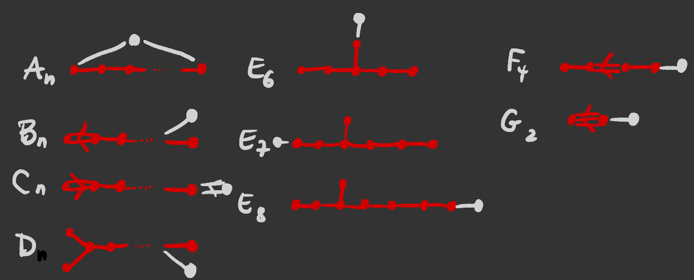
(12)(24)(49)(78)

(23)(45)(56)(68)




0
∞

EVANGELISM: TRUE REFLECTION GROUPS



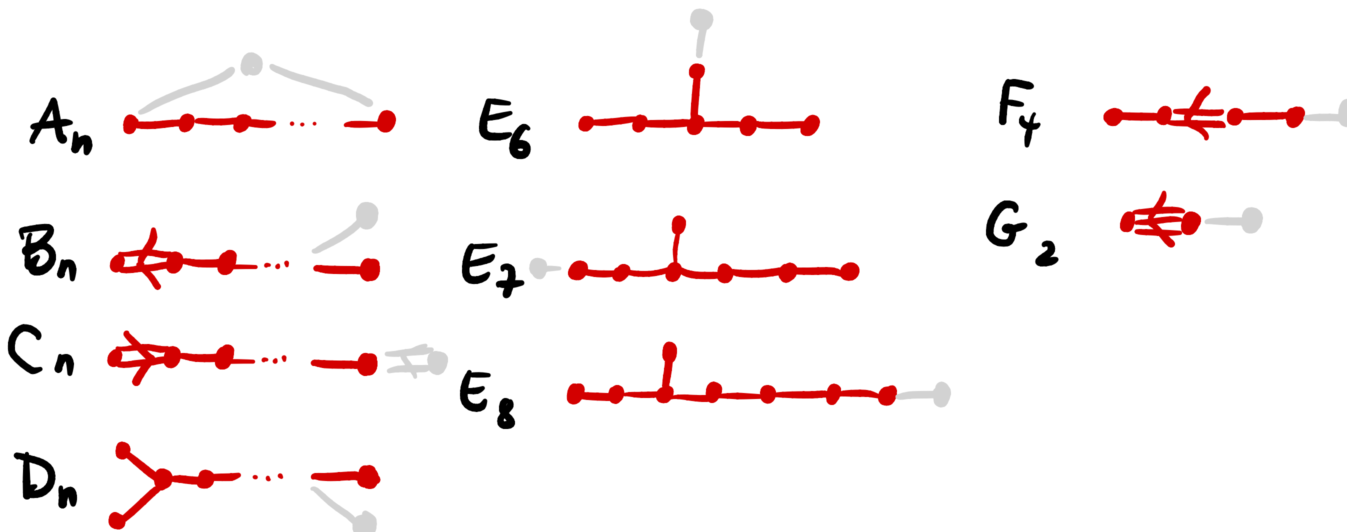
WEYL GROUPS ($\mathbb{Q}$)

- connected reductive group over $\overline{\mathbb{F}}_q$, Frobenius F — G
- Weyl group — $W = N_G(T)/T$ PHILOSOPHY (Tits): "W is G^F at $q=1$"
- Braid group — $B_W = \pi_1(V^{reg}/W)$
- Hecke algebra — $\mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$
- Affine Weyl group — $\widetilde{W} = W \ltimes Q^\vee$

CLASSIFICATION: WEYL GROUPS ($\mathbb{Q}$)

THM The list of irreducible Weyl groups is:

connected Dynkin diagram



REF Coxeter, "The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$." 1935

COXETER GROUPS ($\mathbb{R}$)

DEF A Coxeter system (W,S) is a group W with presentation $W = \langle s_1, s_2, \dots, s_n \mid (s_i s_j)^{m_{ij}} = id \rangle$

$m_{ij} \in \mathbb{N}$, $m_{ii} = 1$

S is the set of simple reflections.

(Coxeter groups act as reflection groups on $\mathbb{R}^n$ with corresponding hyperplane arrangement $\mathcal{H}_W$)

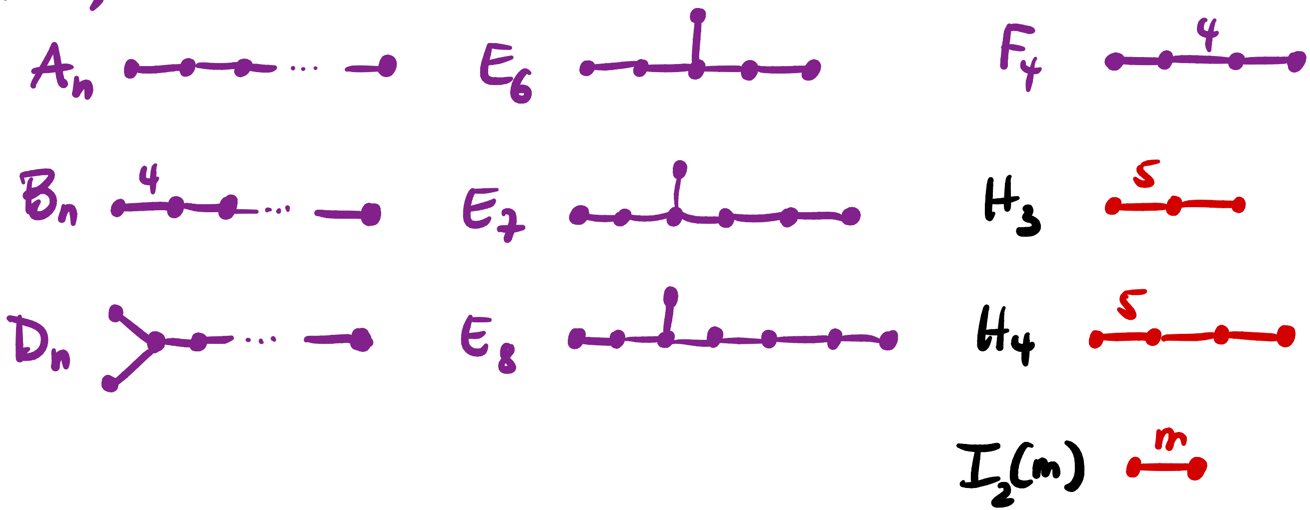
- Braid group $\quad B_W = \pi_1(V^{reg}/W)$
- Hecke algebra $\quad \mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$
- Affine Weyl group $\quad \tilde{W}$
- Lie group

REF Hiller. "Geometry of Coxeter groups"
Humphreys. "Reflection groups and Coxeter groups"
Björner & Brenti, "Combinatorics of Coxeter Groups"

CLASSIFICATION: COXETER GROUPS ($\mathbb{R}$)

THM (Coxeter) The list of finite irreducible Coxeter groups is:

connected Coxeter diagram



REF Coxeter, "The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$." 1935

COMPLEX REFLECTION GROUPS ($\mathbb{C}$)

DEF A <u>complex reflection group</u> is a group $W \subseteq GL_n(\mathbb{C})$ generated by complex reflections.

Braid group $\qquad B_W = \pi_1(V^{reg}/W)$

Simple reflections $\qquad S$

Hecke algebra $\qquad \mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$

Affine Weyl group $\qquad \tilde{W}$

Lie group (spetses)

<u>REF</u> Broué, Malle, Rouquier, "On Complex Reflection Groups and their Associated Braid Groups." 1994

Broué, Malle, Rouquier. "Complex reflection groups, braid groups, Hecke algebras". 1998

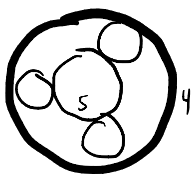
Shephard, Todd. "Finite Unitary Reflection Groups". 1953

CLASSIFICATION: COMPLEX REFLECTION GROUPS ($\mathbb{C}$)

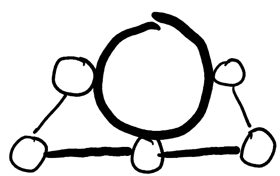
THM (Shephard, Todd) The list of finite irreducible complex reflection groups is:

- $G(m,p,n)$
- G_4
- G_5
- $\vdots$
- G_{37} $= E_8$

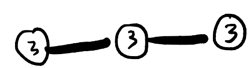
EX



G_{13}



G_{31}



G_{25}

REF Broué, Malle, Rouquier, "On Complex Reflection Groups and their Associated Braid Groups." 1994

Broué, Malle, Rouquier. "Complex reflection groups, braid groups, Hecke algebras". 1998

Shephard, Todd. "Finite Unitary Reflection Groups". 1953

INVARIANT THEORY AND NUMEROLOGY

W acts on $\mathbb{C}^n = \mathrm{span}_{\mathbb{C}}\{x_1, \dots, x_n\}$, hence on $\mathbb{C}[x_1, \dots, x_n]$.

THM (Chevalley) Let $W \subseteq GL_n(\mathbb{C})$. Then
W is a complex reflection group iff $\mathbb{C}[x_1, \dots, x_n]^W = \mathbb{C}[f_1, \dots, f_n]$.

DEF Let $\deg f_i = d_i$ with $\underbrace{d_1 \leq d_2 \leq \cdots \leq d_n}_{\text{degrees}}$.

($h = d_n$ is the Coxeter number. $e_i = d_i - 1$ are the exponents)

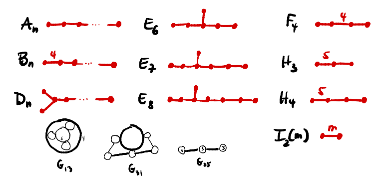
EX $\mathfrak{S}_n \circlearrowright \mathbb{C}^{n-1}$ has invariant polys power sum, elementary, homogeneous, Schur, monomial, forgotten, ...
but always $\deg f_i = i+1$.

REF Chevalley. Invariants of finite groups generated by reflections.

THE GOLD STANDARD

$\underbrace{\mathbb{Q}/\mathbb{R}/\mathbb{C}\text{-UNIFORM}}$ definitions & proofs for reflection groups.

" does not appeal to the $\mathbb{Q}/\mathbb{R}/\mathbb{C}$-classification "



<u>Ex</u> $\quad |W| = \prod_{i=1}^{n} d_i$

(i) $\mathrm{Hilb}\left(\mathbb{C}[x_1,\dots,x_n]^W\right) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}} = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1-tw)}$

(ii) multiply by $(1-t)^n$: $\prod_{i=1}^{n} \frac{1}{[d_i]} = \frac{1}{|W|}\left(1 + (1-t)*\right)$

identity

(iii) set $t \to 1$

EXAMPLE 1: NONCROSSING CATALAN COMBINATORICS

DEF The **Coxeter-Catalan numbers** are the integers

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h+1+e_i}{d_i}.$$

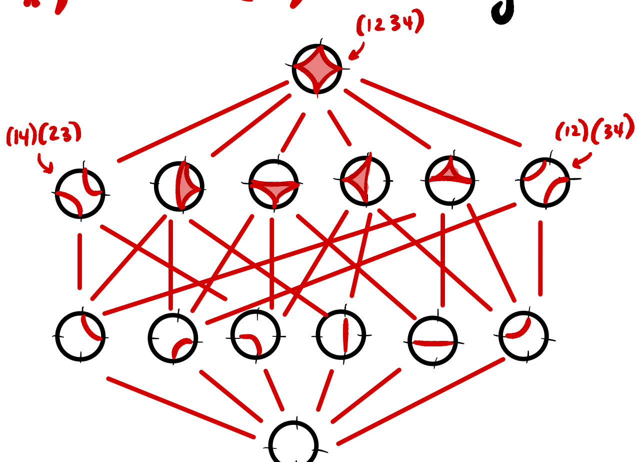
EX $\mathrm{Cat}(n) = \mathrm{Cat}(\mathfrak{S}_n) = \prod_{i=1}^{n-1} \frac{(n+1)+i}{i+1}$

$\mathbb{R}$-NONCROSSING PARTITIONS

DEF The noncrossing partition lattice is the interval

$$\underset{\substack{\uparrow \\ \text{Coxeter element}}}{NC_c}(W) = [e,c]_T \quad \text{in the } \underbrace{\text{oriented Cayley graph}}_{\text{called absolute order, denoted } \le_T} \text{ of } (W,T).$$

EX $NC_{(12\cdots n)}(\mathfrak{S}_n) \cong NC(n)$ via cycles.



Non-crossing partitions for classical reflection groups[1]

Victor Reiner*

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Received 9 March 1995; revised 2 April 1996

REF Bessis. The dual braid monoid

Reiner. Noncrossing partitions for classical reflection groups.

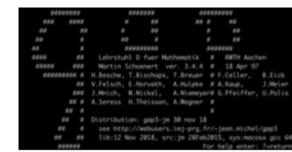
$\mathbb{R}$- NONCROSSING PARTITIONS

parameter

type

THM (Reiner, Bessis)

$$|NC_c(W)| = Cat(W) = \prod_{i=1}^{n} \frac{h+1+e_i}{d_i}.$$



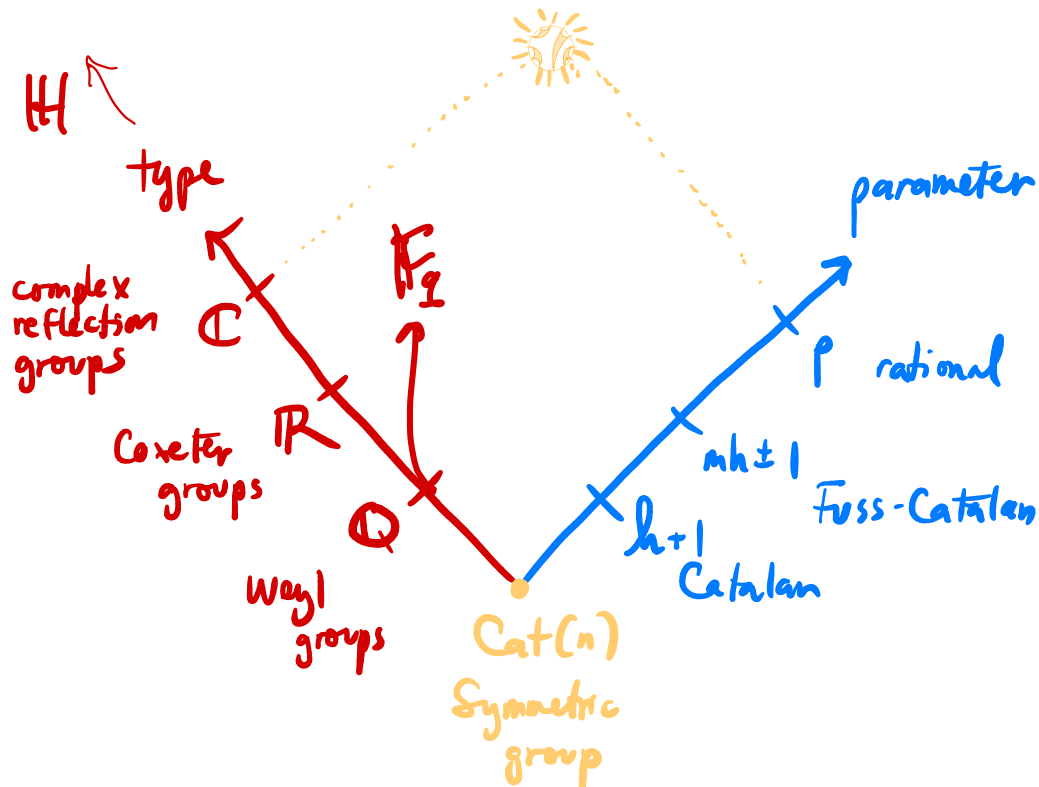
Proof was **NOT UNIFORM**: combinatorial models + computer checks

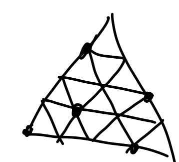
(classical types) (exceptional types)

Only TWO Coxeter-Catalan objects; NC-object

(In particular, the number of clusters in a cluster algebra of finite type was **NOT UNIFORMLY** proven to be counted by $Cat(W)$.)

Type vs. Parameter





ℝ-TYPE HISTORY OF NONCROSSING PARTITIONS

1971 – Kreweras. Sur les partitions non croisées d'un cycle.

1993 – Montenegro. The fixed point non-crossing partition lattices

1995 – Reiner. Non-crossing partitions for classical reflection groups

1997 – Birman, Ko, Lee. A new approach to the word problem in the braid groups

2002 – Brady, Watt. $K(\pi,1)$'s for Artin groups of finite type

2002 – Picantin. Explicit presentations for the dual braid monoids

2003 – Bessis. The dual braid monoid

$\mathbb{R}$-PARAMETER HISTORY OF NONCROSSING PARTITIONS

1971 – Kreweras. Sur les partitions non croisées d'un cycle.

1980 – Edelman. Chain enumeration and non-crossing partitions.

2007 – Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups

Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups

Drew Armstrong

Author address:

Department of Mathematics, Cornell University, Ithaca, New York 14853

Current address: School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

E-mail address: armstron@math.umn.edu

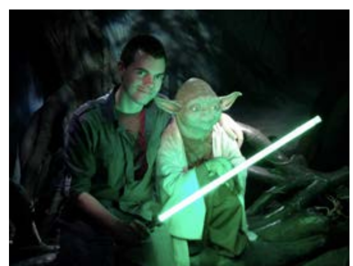
slight lie

BUT... $\displaystyle\prod_{i=1}^{n} \frac{p+e_i}{d_i}$ is ALWAYS an integer for $\gcd(p,h)=1$.

PROBLEM (D. Armstrong, ~2012):

WHAT NC OBJECT IS COUNTED BY $\displaystyle\prod_{i=1}^{n} \frac{p+e_i}{d_i}$???

fractional multichains?
support conditions?
subwords?



REF Rational Catalan Combinatorics: An Outline from the AIM workshop, Dec 2012

THE BIG PROBLEMS IN CATALAND

(circa 2022)

1. uniform enumeration of $NC_c(W)$
2. construction of rational noncrossing objects
3. bijection between NC & NN

Noncrossing braid varieties

1

Theorem (Galashin, Lam, Trinh, W. ($\mathbb{Q}$-uniform))

Fix p coprime to h. Then

$$|R_{\mathbf{c}^p}(\mathbb{F}_q)| = (q-1)^r \prod_{i=1}^{r} \frac{[p+e_i]_q}{[d_i]_q}.$$

2

Theorem (Galashin, Lam, Trinh, W. (uniform))

The Deodhar decomposition of $R_{\mathbf{c}^{h+1}}(\mathbb{F}_q)$ "gives" noncrossing partitions.

Easily generalizes to Armstrong's Fuss-Catalan noncrossing partitions.
So the maximal distinguished subwords of $\mathbf{c}^p$ are the long-desired construction of rational noncrossing partitions.

WHAT THE HECKE?

PROOF METHOD

(i) Hecke algebra character-theoretic method

+

Minh-Tâm Trinh (ii) Lusztig exotic Fourier transform

+

Gordon, Griffeth (iii) Connection to rational Cherednik algebra

similar to J. Michel's proof for the Chapuy-Stump formula

Stump, Thomas, W. (iv) noncrossing combinatorics why the Fuss?

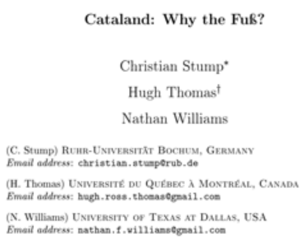
$$\sum_{\chi\in\mathrm{Irr}(W)} q^{-\frac{p}{h}\,\mathrm{c}(\chi)}\mathrm{Feg}_\chi(e^{\frac{2\pi i p}{h}})\mathrm{Deg}_\chi(q) = \sum_{\chi\in\mathrm{Irr}(W)} q^{-\frac{p}{h}\,\mathrm{c}(\chi)}\mathrm{Feg}_\chi(q)\mathrm{Deg}_\chi(e^{\frac{2\pi i p}{h}}),$$

REF Gordon, Griffeth. Catalan numbers for complex reflection groups.
Trinh. From the Hecke category to the Unipotent locus

2

WHY THE FUSS?

THM (Stump, Thomas, W.) The subwords of $c\,w_\circ^m(c)$ that start at e with n stays and end at $w_\circ^m = \begin{cases} w_\circ \\ e \end{cases}$ are in bijection with $NC_c^m(W)$.


Cataland: Why the Fuß?

Christian Stump*

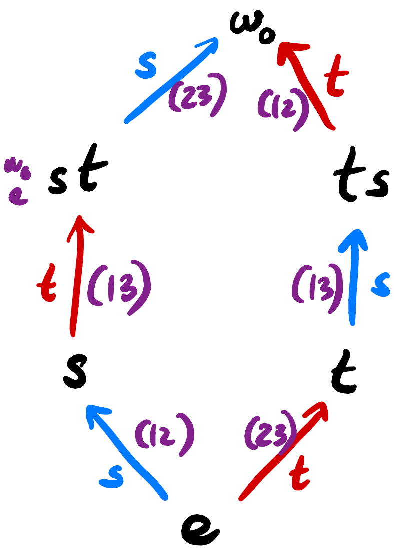
Hugh Thomas†

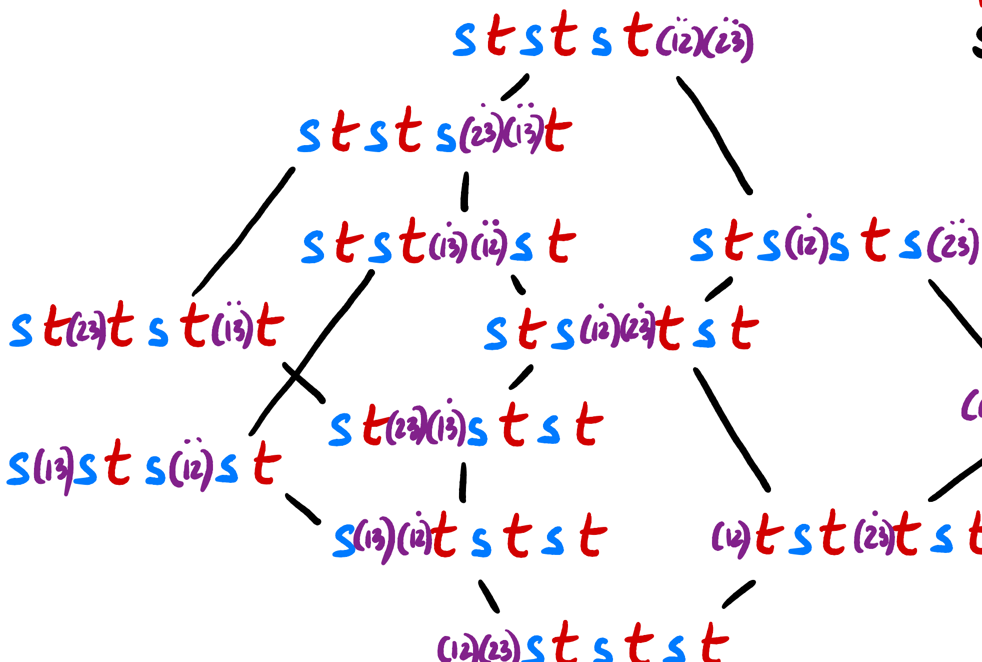
Nathan Williams

(C. Stump) Ruhr-Universität Bochum, Germany
Email address: christian.stump@rub.de

(H. Thomas) Université du Québec à Montréal, Canada
Email address: hugh.ross.thomas@gmail.com

(N. Williams) University of Texas at Dallas, USA
Email address: nathan.f.williams@gmail.com





REF Stump, Thomas, Williams. Cataland: Why the Fuss?

Rational noncrossing PARKING FUNCTIONS™

Theorem (Galashin, Lam, Trinh, W. (uniform))

$$\left| \bigsqcup_{w \in W} R_{\mathbf{c}^p}^{(w)}(\mathbb{F}_q) \right| = (q-1)^r [p]_q^r.$$

Theorem (Galashin, Lam, Trinh, W. (uniform))

The Deodhar decomposition of $\bigsqcup_{w \in W} R_{\mathbf{c}^{h+1}}^{(w)}(\mathbb{F}_q)$ gives Armstrong-Rhoades-Reiner's noncrossing parking functions.

Parking Functions™

v	s_1	s_2	s_1	s_2	s_1	s_2	s_1	s_2	π_1
e	(12)	(23)	s_1	s_2	s_1	s_2	s_1	s_2	(123)
e	s_1	(13)	s_1	s_2	s_1	$(\ddot{12})$	s_1	s_2	(13)
e	s_1	s_2	(23)	s_2	s_1	s_2	$(\ddot{13})$	s_2	(23)
e	(12)	s_2	s_1	s_2	s_1	s_2	s_1	$(\ddot{23})$	(12)
e	s_1	s_2	s_1	s_2	s_1	s_2	$(\ddot{12})$	$(\ddot{23})$	(e)
s_1	s_1	(13)	$(\ddot{12})$	s_2	s_1	s_2	s_1	s_2	(13)
s_1	s_1	s_2	(23)	s_2	s_1	s_2	$(\ddot{13})$	s_2	(23)
s_1	s_1	s_2	s_1	$(\ddot{12})$	s_1	s_2	s_1	$(\ddot{23})$	e
s_2	(12)	s_2	s_1	s_2	$(\ddot{23})$	s_2	s_1	s_2	(12)
s_2	s_1	(13)	s_1	s_2	s_1	$(\ddot{12})$	s_1	s_2	(13)
s_2	s_1	s_2	s_1	s_2	s_1	$(\ddot{23})$	$(\ddot{13})$	s_2	e
s_2s_1	s_1	s_2	(23)	$(\ddot{13})$	s_1	s_2	s_1	s_2	(23)
s_2s_1	s_1	s_2	s_1	$(\ddot{12})$	s_1	s_2	s_1	$(\ddot{23})$	e
s_1s_2	(12)	s_2	s_1	s_2	$(\ddot{23})$	s_2	s_1	s_2	(12)
s_1s_2	s_1	s_2	s_1	s_2	$(\ddot{13})$	$(\ddot{12})$	s_1	s_2	e
$s_1s_2s_1$	s_1	s_2	s_1	$(\ddot{12})$	$(\ddot{23})$	s_2	s_1	s_2	e

EXAMPLE 2 : AFFINE BRAID VARIETIES

Affine symmetric group

Theorem (Opdam)

Let $[k]_q = \frac{(q-1)^2}{q}\frac{q^k-q^{-k}}{q-q^{-1}}$. *For* $\lambda \in Q^+$,

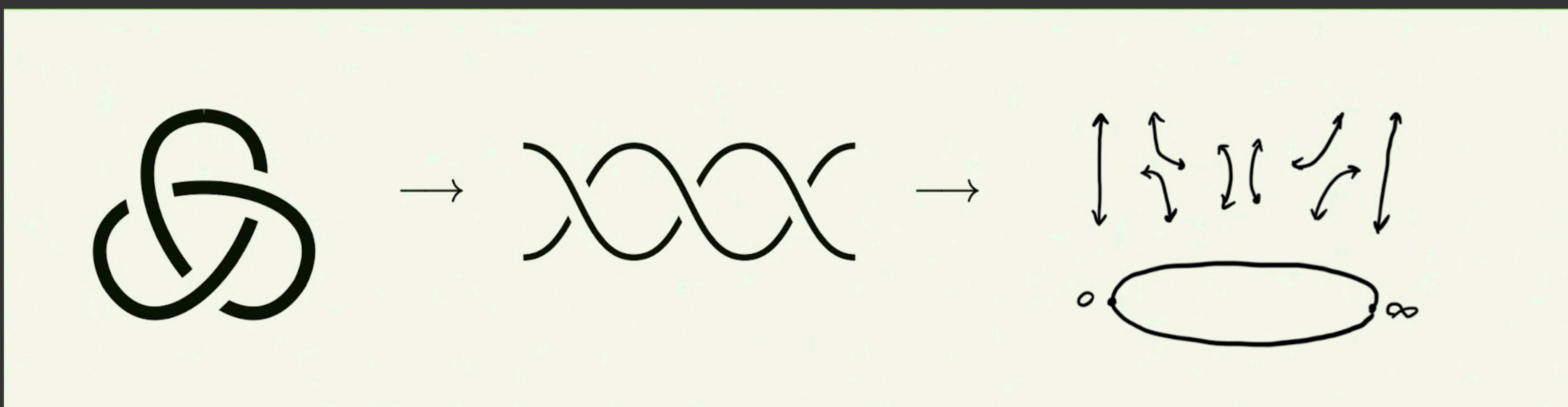
$$\mathrm{tr}(T_{t_{\lambda_-}} T_{t_{\lambda_+}}^{-1}) = q^{(\ell(t_{\lambda_-})-\ell(t_{\lambda_+}))/2} \sum_{(a_\alpha)\in K(\lambda)} \prod_{\substack{\alpha\in\Phi^+\\ a_\alpha>0}} [a_\alpha]_q.$$

Theorem (Galashin, Lam, W.)

Fix the extended affine Weyl group $\widehat{S}_n$, *and let* $v = t_{(m-1)\lambda_1}$ *and* $w = t_{(m(n-1)+1)\lambda_{n-1}}$. *Then the number of* $\mathbb{F}_q$*-points in the braid variety* $R_{v,w}(\mathbb{F}_q)$ *is given by*

$$|R_{v,\mathbf{w}}(\mathbb{F}_q)| = (q-1)^{2(n-1)} \left(\frac{q^{m(n-1)+1}-1}{q-1}\right)^{n-2}.$$

FUTURE DIRECTIONS



Bijections?

Very hard, general open problem in general:

Problem

Find bijections between maximal distinguished subwords and existing combinatorial objects.

Special cases:

- noncrossing vs. nonnesting;
- Galashin and Lam's positroid braid varieties and rational Dyck paths;
- affine braid variety and parking functions;
- etc.

Mixed Hodge?

Even harder open problem in general:

Problem

Compute the mixed Hodge decomposition of $R_{\mathbf{c}^p}(\mathbb{C})$.

Expect to get q, t-Catalan numbers, q, t-parking numbers, etc.

Thomas Lam

Re: Parking analogue of Galashin-Lam?

To: Nathan Williams, Cc: Pavel Galashin

Details

Hi Nathan,

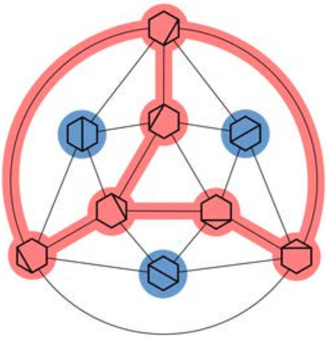
I think your new conjecture for parking functions is compatible with the mixed Hodge structure! For the 4^2 case, we are getting six separate varieties. It looks like they are of type E6, A4 (twice), A2 (twice), A0 as cluster varieties. Using the mixed Hodge polynomial conventions in [GL], we get
[illegible]
[illegible]
[illegible]
= 1
(See Section 6.2 and 6.3 of arXiv [illegible])
Summing gives the correct q,t analogue of 4^2. Is this a known decomposition of the q,t-parking function?

Best,
Thomas

Graphical models

Problem

Find reasonable graphical models for rational noncrossing Catalan objects.



Periodic elements?

There might be a uniform formula for braid varieties built from periodic elements, generalizing the usual Coxeter–Catalan numbers. What is the combinatorics?

Example

For type D_4 with $d = 4$ and $\mathbf{w} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_4$, we have that

$$\left|R_{e,\mathbf{w}^3}(\mathbb{F}_q)\right| = q^{-18}(q-1)^4(1+q^2+3q^4+4q^6+4q^8+3q^{10}+q^{12}+q^{14}),$$

At $q = 1$ and for p odd, we appear to have

$$\lim_{q\to 1}(q-1)^{-4}\left|R_{e,\mathbf{w}^p}(\mathbb{F}_q)\right| = \frac{((p+1)(p+3))^2}{32}$$

Note that the order d of w is 4, and that the eigenvalues of w in the ref rep are i^1 and i^3 (each with multiplicity 2).

Complex reflection groups?

Theorem (W. Miller (undergraduate!))

Let W be a spetsial imprimitive complex reflection group and p coprime to h. Then (up to a power of q)

$$\mathrm{tr}(T_{\mathbf{c}^p}) = (q-1)^r \prod_{i=1}^{r} \frac{[p+e_i(V^p)]}{[d_i]},$$

where the $e_i(V^p)$ are the fake degrees of the p-th Galois twist of the reflection representation and the trace is taken in the Hecke algebra H_W.

Example

The complex reflection group G_4 has rank $r=2$, Coxeter number $h=6$. Its reflection representation has fake degrees 3 and 5. We compute using GAP3 that

$$\mathrm{tr}(T_{\mathbf{c}^7}) = (q-1)^2(q^{12}+q^8+q^6+q^4+1) = (q-1)^2\frac{[7+3][7+5]}{[4][6]}.$$

Complex reflection groups?

The Deodhar decomposition gives a combinatorial model of braid varieties for general Coxeter groups—but we lose the obvious notion of distinguished for complex reflection groups.

Problem

Find a combinatorial description of the Deodhar decomposition for spetsial complex reflection groups.

For the case $p = h + 1$, this should recover noncrossing partitions.

WHY THE FUSS?

Ex $W = \mathfrak{S}_3$, $p = 4$ $\quad \underbrace{\text{elements of } D(c^4, e) \text{ with 2 stays}}_{\text{start \& end at } e,\ \text{no odd colors}}$

$c = st$