Various Guises of Reflection Arrangements

Edelman-Reiner Conjecture Revisted

15. Mar. 2023. ICMS, Edinburgh.

Masahiko Yoshinaga (Osaka U.)

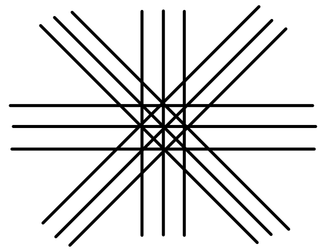
Edelman-Reiner Conj. ('96, Solved by Terao, Y. 2004)

$\mathbb{R}^\ell > \Phi > \Phi^+$: root sys. with exp: $e_1, \cdots, e_\ell$, Coxeter #: h.

$$H_{\alpha,k} := \{x \in \mathbb{R}^\ell \mid \alpha(x) = k\}$$

$$\mathrm{Cat}^m_\Phi := \{H_{\alpha,k} \mid \alpha \in \Phi^+,\ -m \le k \le m\}.$$

$$\mathrm{Shi}^m_\Phi := \{H_{\alpha,k} \mid \alpha \in \Phi^+,\ 1-m \le k \le m\}$$



$\mathrm{Cat}^1_{B_2}$

Then, (i) $c\mathrm{Cat}^m_\Phi$ is free with $\exp = (1, e_1+mh, \cdots, e_\ell+mh)$.

(ii) $c\mathrm{Shi}^m_\Phi$ is free with $\exp = (1, \underbrace{mh, mh, \cdots, mh}_{\ell})$

Cor. (i) $\chi(\mathrm{Cat}^m_\Phi, t) = \prod_{i=1}^{\ell} (t - e_i - mh)$,

$\#\mathrm{ch}(\mathrm{Cat}^m_\Phi, t) = \prod_i (1 + e_i + mh)$

(ii) $\chi(\mathrm{Shi}^m_\Phi, t) = (t - mh)^\ell$, $\#\mathrm{ch}(\mathrm{Shi}^m_\Phi) = (1 + mh)^\ell$.


Cataland
E-R
Conj.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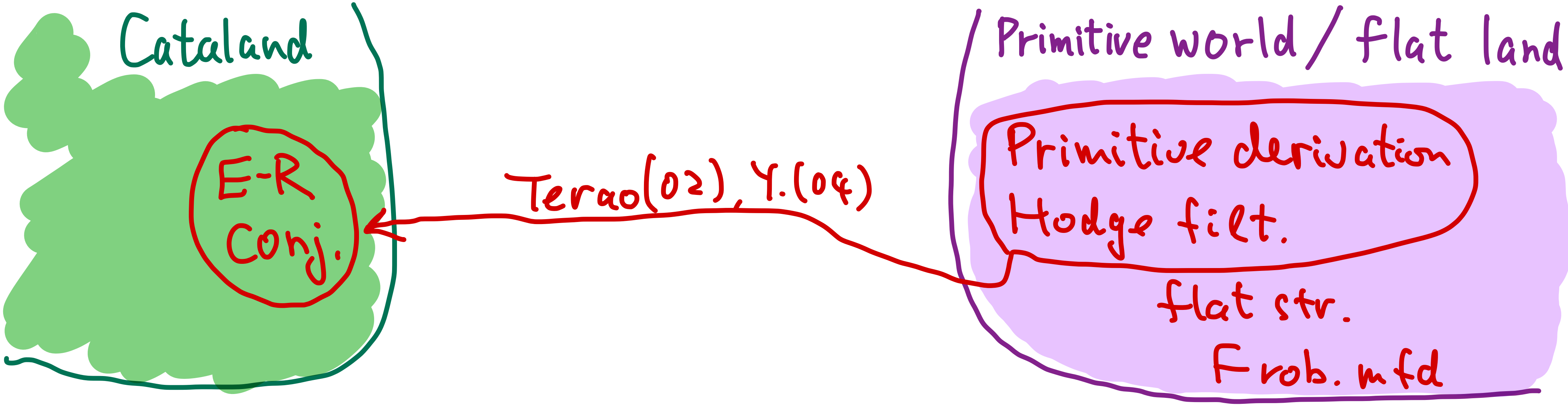
E-R
Conj.


Primitive world / flat land
Primitive derivation
Hodge filt.
flat str.
Frob. mfd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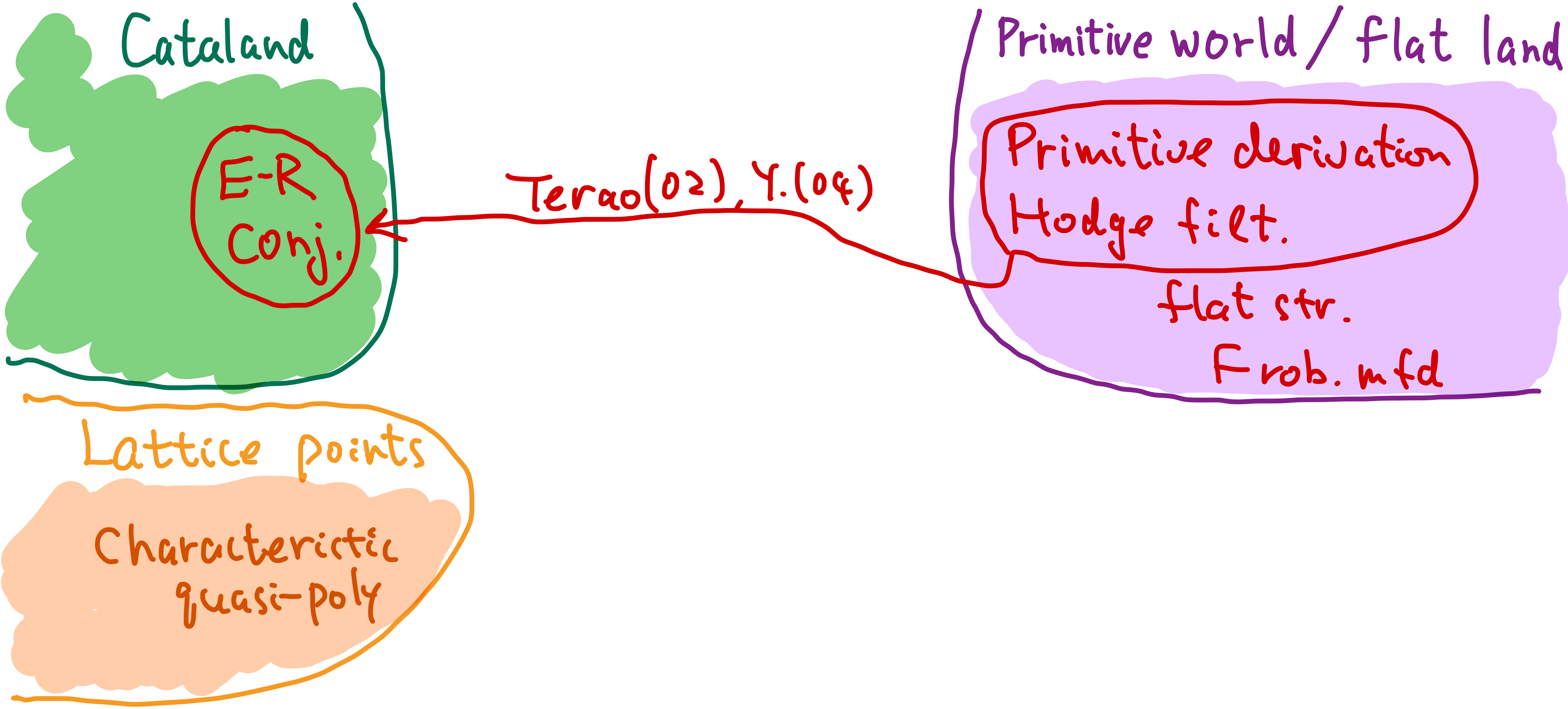
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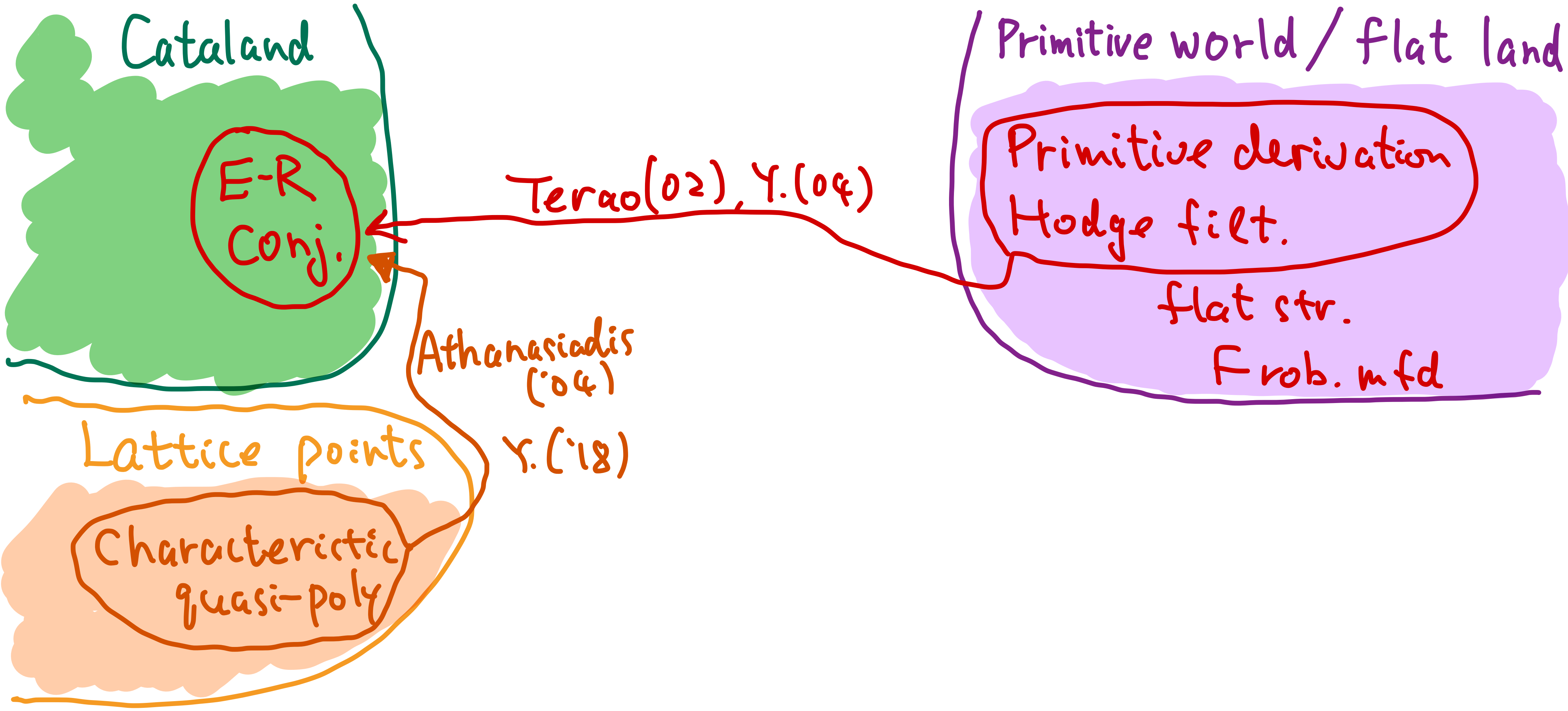
Lattice points
Characteristic
quasi-poly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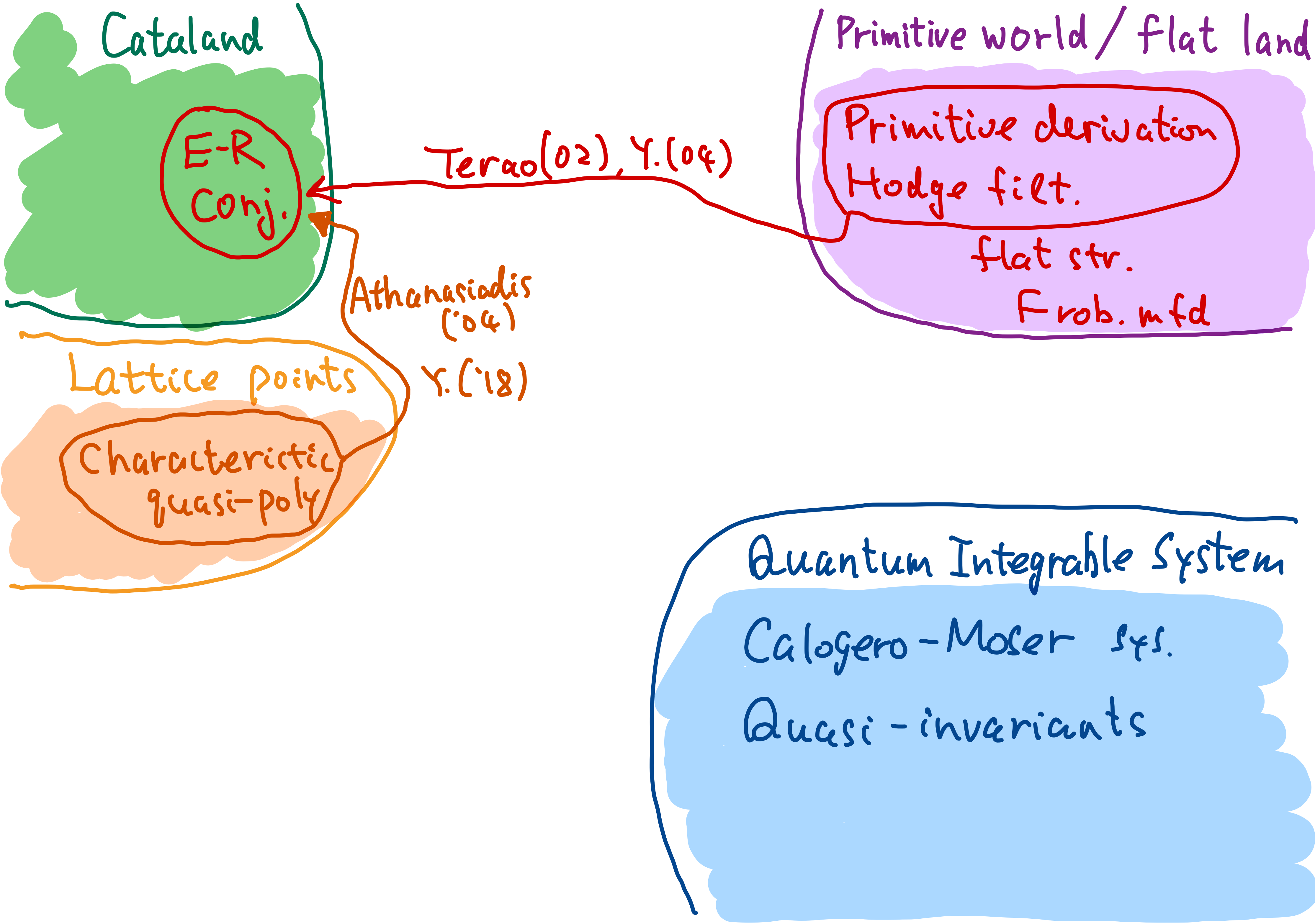
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Calogero-Moser sys.
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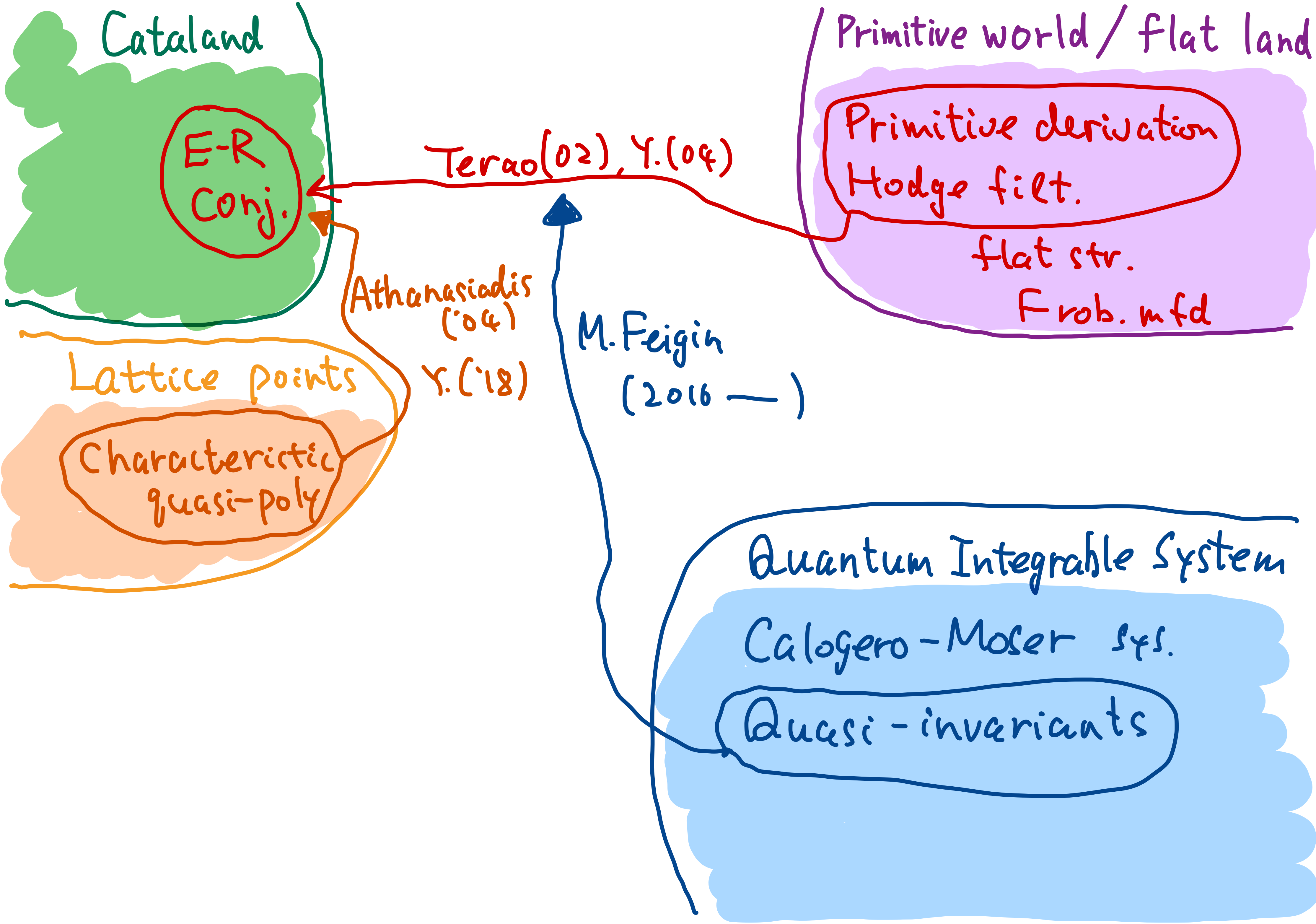

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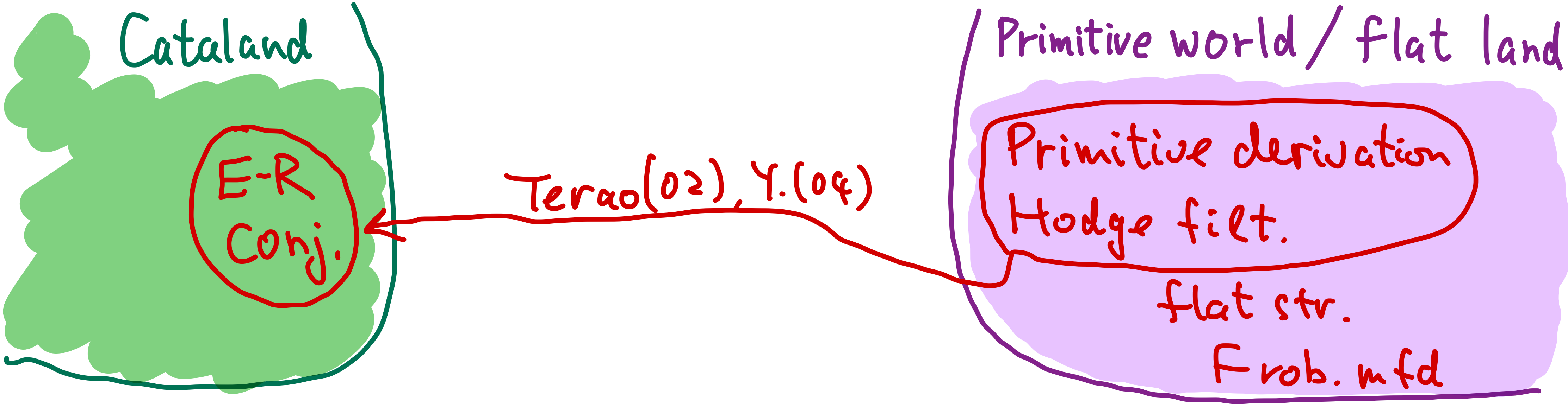
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1. Free arrangements

1. Free arrangements

$\mathcal{A} = \{H_1, \cdots, H_n\}$: a central hyp. arr. $H_i = \{\alpha_i = 0\} \subseteq \mathbb{R}^\ell$.

$H_i \ni 0$

$S = \mathbb{C}[x_1, \cdots, x_\ell]$. $\mathrm{Der}_S := S\partial_1 \oplus \cdots \oplus S\partial_\ell$, $\left(\partial_i = \dfrac{\partial}{\partial x_i}\right)$.

Def $D(\mathcal{A}) := \{\delta \in \mathrm{Der}_S \mid \delta\alpha_i \in (\alpha_i) \ \forall i = 1, \cdots, n\}$.

Def. $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$

$\overset{\text{def}}{\iff}$ $\exists \delta_1, \cdots, \delta_\ell \in D(\mathcal{A})$ homogeneous

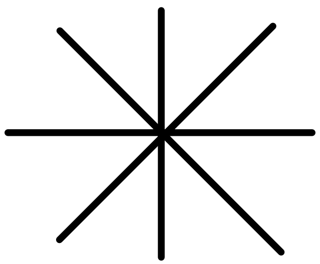
s.t. $\deg \delta_i = d_i$ and $D(\mathcal{A}) = S\delta_1 \oplus \cdots \oplus S\delta_\ell$.

$\delta_i = \sum_j f_{ij}\partial_j$

$\Updownarrow$

$\deg f_{i1} = \cdots = \deg f_{i\ell}$

Example $\mathcal{A}: xy(x-y)(x+y)$



$\delta_1 = x\partial_x + y\partial_y$

$\delta_2 = x^3\partial_x + y^3\partial_y$

$\in D(\mathcal{A})$. free with $\exp(\mathcal{A}) = (1, 3)$

Free arrangements

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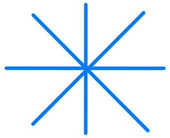


$\left.\begin{array}{l} \delta_1 = x\partial_x + y\partial_y \\ \delta_2 = x^3\partial_x + y^3\partial_y \end{array}\right. \in D(\mathcal{A})$. free with $\exp(\mathcal{A}) = (1, 3)$

Fact Let $\mathcal{A}$ be a free arr. with $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$.

Then

(1) (Terao)

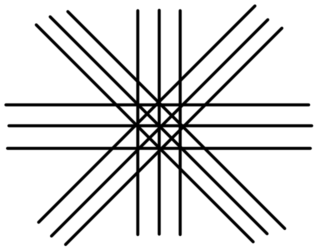
The characteristic poly. is $\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i)$.

(2) (Zaslavsky)

The # of chambers is $\# ch(\mathcal{A}) = \prod_{i=1}^{\ell} (d_i + 1)$

Edelman-Reiner Conj.

$\mathbb{R}^\ell \supset \Phi \supset \Phi^+$: root sys. with exp: $e_1, \cdots, e_\ell$, Coxeter #: h.



$\mathrm{Cat}^1_{B_2}$

$$\widetilde{H}_{\alpha,k} := \{(x, z) \in \mathbb{R}^\ell \times \mathbb{R} \mid \alpha(x) = kz\}$$

$$c\mathrm{Cat}^m_\Phi := \{\widetilde{H}_{\alpha,k} \mid \alpha \in \Phi^+,\ -m \le k \le m\} \cup \{H_0 = \{z=0\}\}$$

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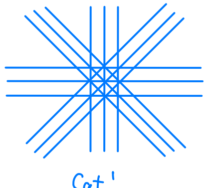
Then, (i) $c\mathrm{Cat}^m_\Phi$ is free with $\exp = (1, e_1 + mh, \cdots, e_\ell + mh)$.

(ii) $c\mathrm{Shi}^m_\Phi$ is free with $\exp = (1, \underbrace{mh, mh, \cdots, mh}_{\ell})$

From now, we look at Cat^m_Φ.



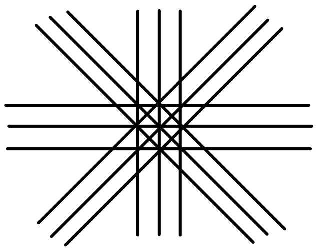
Edelman–Reiner Conj. $\mathbb{R}^\ell > \Phi > \Phi^+$: root sys. with exp: $e_1, \cdots, e_\ell$, Coxeter #: h.

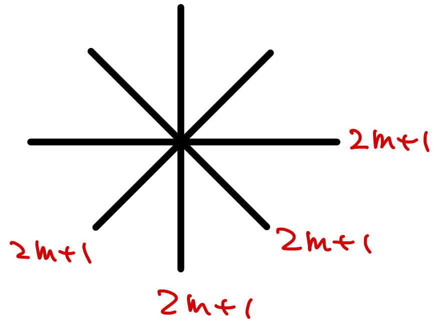
$\tilde{H}_{\alpha,k} := \{(x, z) \in \mathbb{R}^\ell \times \mathbb{R} \mid \alpha(x) = kz\}$. $cCat^m_\Phi := \{\tilde{H}_{\alpha,k} \mid \alpha \in \Phi^+, -m \le k \le m\} \cup \{H_0 = \{z = 0\}\}$

Then, $cCat^m_\Phi$ is free with $\exp = (1, e_1 + mh, \cdots, e_\ell + mh)$.

Crucial Point of the Proof : restricted multiarr. on H_0. (by Ziegler)

$$Cat^m_\Phi \xrightarrow{\text{coning (homogenize)}} cCat^m_\Phi \xrightarrow{\text{restriction } |_{z=0}} (A_\Phi := Cat^0_\Phi, \underbrace{2m+1}_{\text{multiplicity}})$$



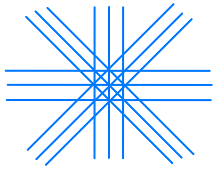


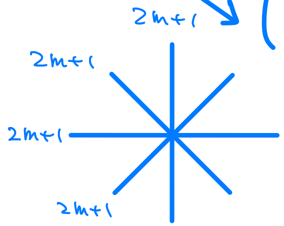
Claim $D(A_\Phi, 2m+1) := \{\delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^{2m+1}), H \in A_\Phi\}$

is free with exp (: deg. of basis) $= (e_1 + mh, \cdots, e_\ell + mh)$

Crucial Point of the Proof : restricted multiarr. on $\mathcal{H}_0$ (by Ziegler)

$$\mathrm{Cat}^m_\Phi \xrightarrow[\text{(homogenize)}]{\text{coning}} c\mathrm{Cat}^m_\Phi \xrightarrow{|_{z=0}} (\mathcal{A}_\Phi = \mathrm{Cat}^0_\Phi, 2m+1)$$





Claim $D(\mathcal{A}_\Phi, 2m+1) := \{ \delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^{2m+1}), H \in \mathcal{A}_\Phi \}$ is free with

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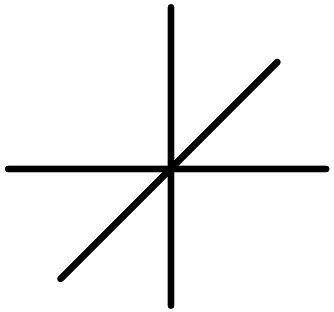
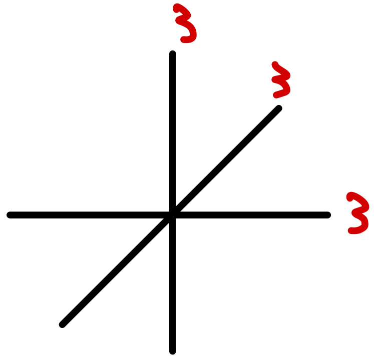
Example ($\Phi = A_2$)



$xy(x-y)$



$x^3y^3(x-y)^3$

$\delta_1 = x\partial_x + y\partial_y$

$\delta_2 = x^2\partial_x + y^2\partial_y$

(1, 2) $\xrightarrow{\text{shift } h(=3)}$ (4, 5)

$\delta_1' = x^3(x-2y)\partial_x + y^3(y-2x)\partial_y$

$\delta_2' = x^3(x-y)^2\partial_x + y^3(x-y)^2\partial_y$

Primitive derivation (K. Saito)

Brieskorn–Grothendieck–Slodowy

$$\begin{array}{ccccc} X_0 : \text{simple (ADE) singularity} & \subset & X : \text{univ. unfolding} & \hookrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \varphi & & \downarrow \text{"adjoint quotient"} \\ 0 & \hookrightarrow & \mathbb{C}^\ell & \xrightarrow{\cong} & \mathfrak{h}/\!/W = \mathfrak{g}/\!/G. \end{array}$$

$\mathfrak{h}/\!/W = \operatorname{Spec} S^W$, W: Weyl gr

$S^W = \mathbb{C}[P_1, \cdots, P_\ell]$.

$\deg P_1 \leq \cdots \leq \deg P_\ell$

The map φ determines the

Gauss–Manin connection

$$\nabla : \mathrm{Der}_{S^W} \times R^2\varphi_*\mathbb{C} \longrightarrow R^2\varphi_*\mathbb{C}.$$

The primitive derivation $\nabla_{\partial/\partial P_\ell}$ plays the crucial role.

These structures are interpreted as follows.

Primitive derivation (K. Saito) Brieskorn-Grothendieck-Slodowy

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Def $\nabla : \mathrm{Der}_S \times \mathrm{Der}_S \longrightarrow \mathrm{Der}_S$ $\qquad \delta_2 = \sum f_i \partial_i$

$$(\delta_1, \delta_2) \longmapsto \nabla_{\delta_1} \delta_2 = \sum_i (\delta_1 f_i) \partial_i.$$

(**Rem.** ∇ is the Levi-Civita conn. of the W-invariant metric.)

Def $$D := \frac{\partial}{\partial P_\ell} := \frac{1}{Q} \cdot \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix},$$

where

$$Q = \frac{\partial(P_1 \cdots P_\ell)}{\partial(x_1 \cdots x_\ell)} = \prod_{H \in \mathcal{A}_\Phi} \alpha_H.$$

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$S^W = \mathbb{C}[P_1, \cdots, P_\ell]$, $\mathfrak{h}/\!/W = \operatorname{Spec} S^W$

$\deg P_1 \le \cdots \le \deg P_\ell$

The map φ determines the Gauss-Manin connection

$$\nabla : \mathrm{Der}_{S^W} \times R^2\varphi_*\mathbb{C} \longrightarrow R^2\varphi_*\mathbb{C}.$$

The primitive derivation $\nabla_{\partial/\partial P_\ell}$ plays the crucial role.

These structures are interpreted as follows.

Def $\nabla : \mathrm{Der}_S \times \mathrm{Der}_S \longrightarrow \mathrm{Der}_S$

$\delta_2 = \sum f_i \partial_i$

$$(\delta_1, \delta_2) \longmapsto \nabla_{\delta_1}\delta_2 = \sum_i (\delta_1 f_i)\partial_i.$$

Def $$D := \frac{\partial}{\partial P_\ell} := \frac{1}{Q}\cdot\det\begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix},$$ where $Q = \frac{\partial(P_1 \cdots P_\ell)}{\partial(x_1 \cdots x_\ell)} = \prod_{H \in \mathcal{A}_\Phi} \alpha_H$.

$(DP_\ell = 1,\quad DP_i = 0 \text{ for } i < \ell.)$

Thm. (K. Saito, Terao, Y.) ∇_D induces the linear isom.

$$\nabla_D : D(\mathcal{A}_\Phi, 2m+1)^W \xrightarrow{\sim} D(\mathcal{A}_\Phi, 2m-1)^W.$$

In particular, $$D(\mathcal{A}_\Phi, 2m+1)^W = \nabla_D^{-m} D(\mathcal{A}_\Phi, 1)^W$$

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In particular, $D(\mathcal{A}_\Phi, 2m+1)^W = \nabla_D^{-m} D(\mathcal{A}_\Phi, 1)^W$

We can also prove that

- $D(\mathcal{A}_\Phi, 2m+1)^W$ is S^W-free module.
- The above free basis provides S-free basis of $D(\mathcal{A}, 2m+1)$.

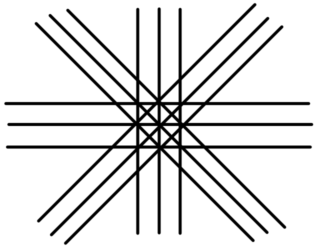
Since $\deg P_\ell = h$, $\deg \nabla_D^{-m} \delta_i = \deg \delta_i + mh$.

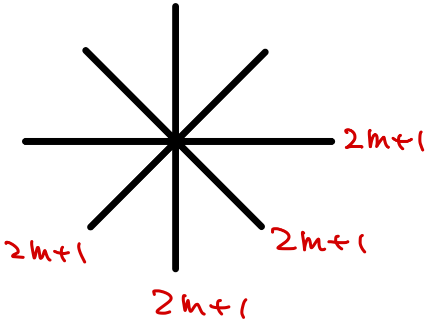
Thm (Terao 2003) $D(\mathcal{A}_\Phi, 2m+1)$ is free with

$\exp : (e_1 + mh, \cdots, e_\ell + mh)$.

Summary

$$\mathrm{Cat}_{\Phi}^{m} \xrightarrow[\text{(homogenize)}]{\text{coning}} c\mathrm{Cat}_{\Phi}^{m} \xrightarrow[|_{z=0}]{\text{restriction}} (\mathcal{A}_{\Phi}, 2m+1)$$





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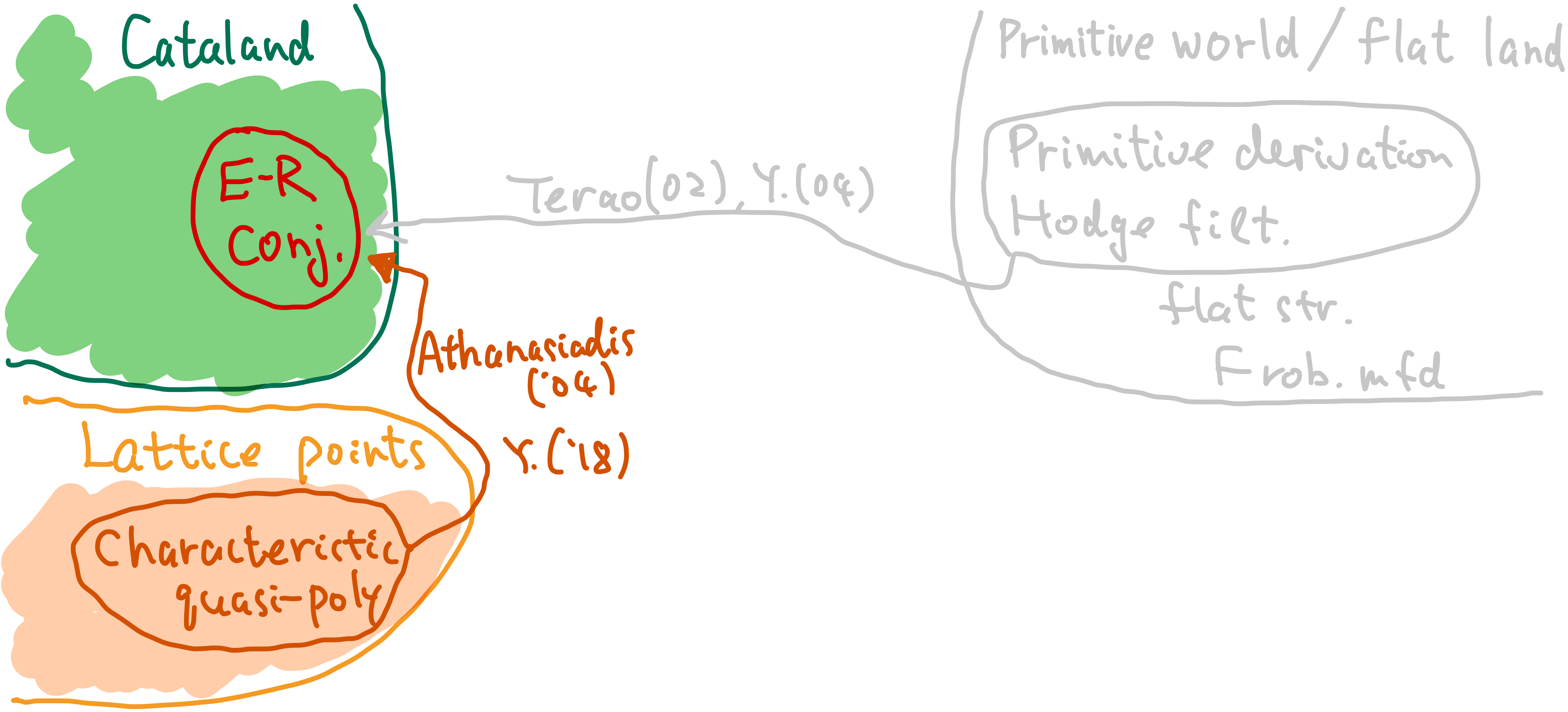
Thm (Terao 2003) $D(\mathcal{A}_{\Phi}, 2m+1)$ is free with

exp: $(e_1+mh, \cdots, e_\ell+mh)$.

⇓

$c\mathrm{Cat}_{\Phi}^{m}$ is free with exp: $(1, e_1+mh, \cdots, e_\ell+mh)$.

→ Towards unitary refl. gr. (Hoge-Mano-Röhrle-Stump, ...)



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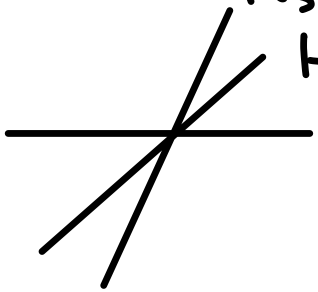
Example



Count mod q complement.

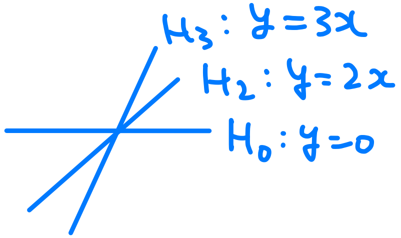
$$\widetilde{H}_i := \{ (x, y) \in (\mathbb{Z}/q)^2 \mid y \equiv ix \bmod q \}$$

$$(\mathbb{Z}/q)^2 \setminus \widetilde{H}_0 \cup \widetilde{H}_2 \cup \widetilde{H}_3 = ?$$

$$= \begin{cases} q^2 - 3q + 2 & \text{if } q \equiv 1 \text{ or } 5 \bmod 6 \\ q^2 - 3q + 3 & \text{if } q \equiv 2 \text{ or } 4 \bmod 6 \\ q^2 - 3q + 4 & \text{if } q \equiv 3 \bmod 6 \\ q^2 - 3q + 5 & \text{if } q \equiv 6 \equiv 0 \bmod 6 \end{cases}$$

"quasi-polynomial".

Example



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Thm (Kamiya-Takemura-Terao 2007)

$\mathcal{A}$: arr. $/\mathbb{Z}$. Then $\exists \rho > 0$ and $\exists f_1, f_2, \cdots, f_\rho \in \mathbb{Z}[t]$

s.t.

(1) $$\left| (\mathbb{Z}/q)^\ell \setminus \bigcup_{H \in \mathcal{A}} \widetilde{H} \right| = \begin{cases} f_1(q) & \text{if } q \equiv 1 \bmod \rho \\ f_2(q) & \text{if } q \equiv 2 \bmod \rho \\ \vdots & \vdots \quad \vdots \quad \vdots \\ f_\rho(q) & \text{if } q \equiv \rho \bmod \rho \end{cases}$$

(for $q \gg 0$)

"$\chi_{\mathrm{quasi}}(\mathcal{A}, q)$"

(2) $\gcd(\rho, i) = \gcd(\rho, j) \Longrightarrow f_i = f_j$.

(3) $f_1(t) = \chi(\mathcal{A}, t)$

(4) (Liu-Tran-Y.) $f_\rho(t)$ is the char. poly. of toric arr.

Thm (Kamiya–Takemura–Terao 2007) $\mathcal{A}$: arr. $/\mathbb{Z}$.

Then $\exists \rho > 0$ and $\exists f_1, f_2, \cdots, f_\rho \in \mathbb{Z}[t]$ s.t.

(1) $$\left| (\mathbb{Z}/q)^\ell \setminus \bigcup_{H \in \mathcal{A}} \bar{H} \right| = \begin{cases} f_1(q) & \text{if } q \equiv 1 \bmod \rho \\ f_2(q) & \text{if } q \equiv 2 \bmod \rho \\ \vdots & \vdots \quad \vdots \quad \vdots \\ f_\rho(q) & \text{if } q \equiv \rho \bmod \rho \end{cases}$$ "$\chi_{quasi}(\mathcal{A}, q)$"

(for $q \gg 0$)

(2) $\gcd(\rho, i) = \gcd(\rho, j) \Longrightarrow f_i = f_j$.

(3) $f_1(t) = \chi(\mathcal{A}, t)$

(4) (Liu–Tran–Y.) $f_\rho(t)$ is the char. poly of toric arr.

Thm. (Athanasiadis 2004, Y. 2018) w.r.t. coweight lattice,

$$\chi_{quasi}(\mathrm{Cat}^m_{\mathcal{A}}, q) = \chi_{quasi}(\mathcal{A}_\Phi, q - mh).$$

Rem. There is a Shi arr. version, which involves Lam–Postnikov's "Eulerian poly. for Φ".

Thm. $\chi_{quasi}(\mathrm{Cat}^m_{\mathcal{A}}, q) = \chi_{quasi}(\mathcal{A}_{\Phi}, q - mh)$.

Proof for $\Phi = A_2,\ m = 1$ $h = 3$

$\mathcal{A}_{A_2}$: $xy(x+y) = 0$.

Idea: Pull the mod q hyperplanes back by the map

$$[1,q]^2 \xrightarrow[\text{bij}]{\sim} (\mathbb{Z}/q)^2,$$

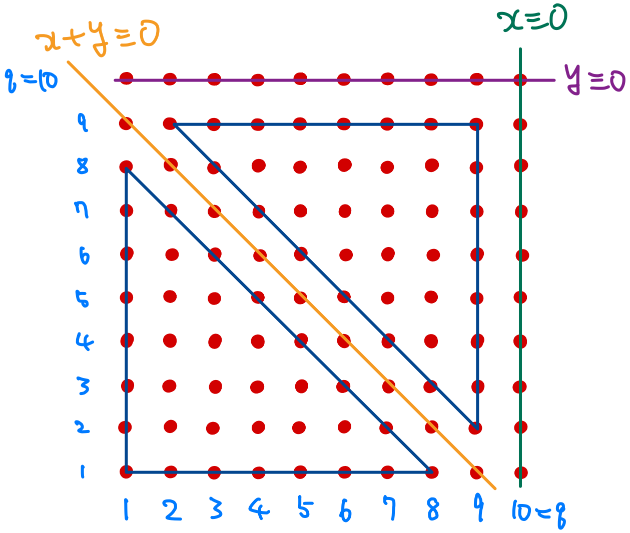
and use Ehrhart (quasi) poly.

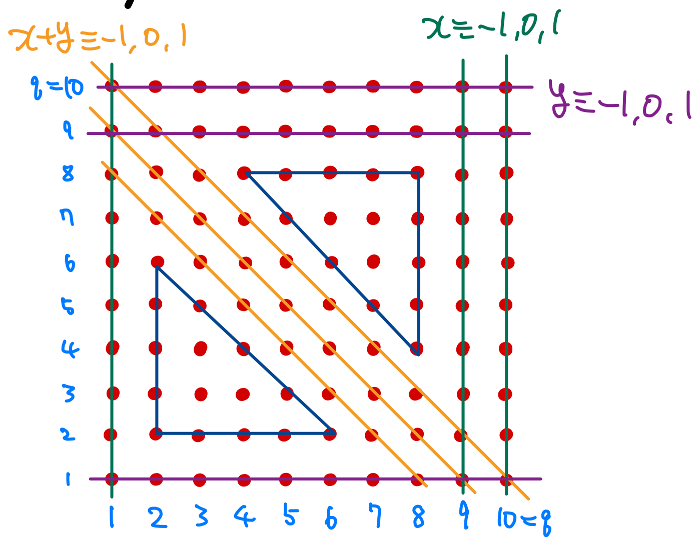
Let  : P

$$P := \left\{ (x,y) \;\middle|\; \begin{matrix} x, y \geq 0 \\ x + y \leq 1 \end{matrix} \right\}$$

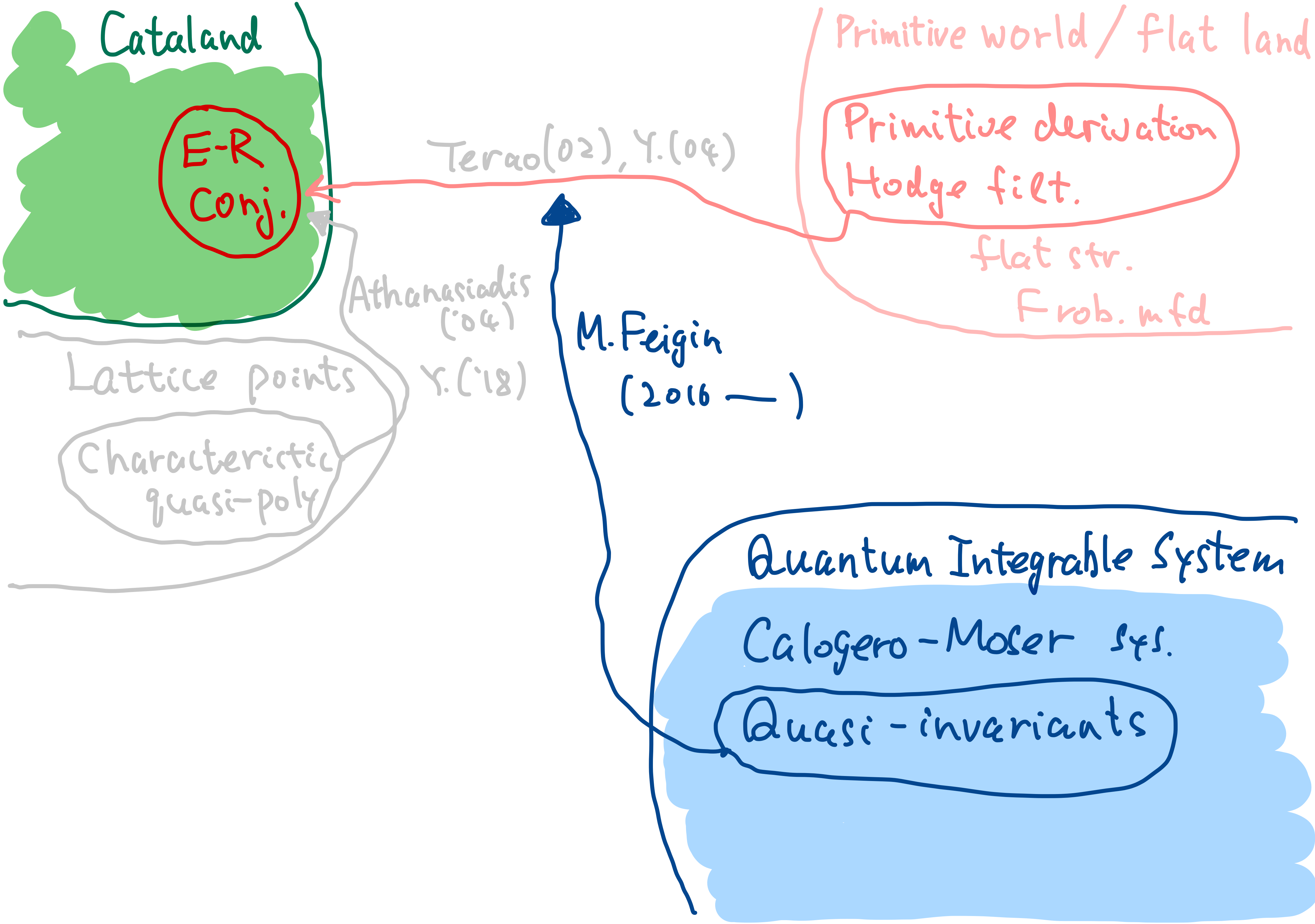
Ehrhart (quasi) poly

$$L_P(q) := |qP \cap \mathbb{Z}^2|,$$





$$\chi_{quasi}(\mathcal{A}_{A_2}, q) = 2 \cdot L_P(q - h) \rightsquigarrow \chi_{quasi}(\mathrm{Cat}^1_{A_2}, q) = 2 \cdot L_P(q - 2h)$$


Cataland
E-R Conj.
Primitive world / flat land
Primitive derivation
Hodge filt.
flat str.
Frob. mfd
Terao(02), Y.(04)
Athanasiadis ('04)
Y.('18)
Lattice points
Characteristic quasi-poly
M.Feigin (2016 —)
Quantum Integrable System
Calogero-Moser sys.
Quasi-invariants

3. Integral expression and "discretization".

3. Integral expression and "discretization".

$S^W = \mathbb{C}[p_1 \cdots p_\ell]$

$D = \partial / \partial p_\ell$.

Recall our description of $D(A_\Phi, 2m+1)$:

$$\nabla_D^{-m} : D(A_\Phi)^W \xrightarrow{\cong} D(A_\Phi, 2m+1)^W.$$

Def (Chalykh, Veselov, Feigin "m-quasi invariants" motivated by quantum integrable sys.)

$$Q_m := \{ f \in S \mid (1 - s_\alpha) f \in (\alpha^{2m+1}),\ \alpha \in \Phi \},$$

where s_α is the reflection w.r.t. $\alpha \in \Phi$.

Prop. (Feigin) Let $\delta = \sum_{i=1}^{\ell} f_i \partial_i \in \mathrm{Der}_S^W$. Then

$$\delta \in D(A_\Phi, 2m+1)^W \iff f_1 \in Q_m.$$

Def $Q_m := \{ f \in S \mid (1-s_\alpha) f \in (\alpha^{2m+1}),\ \alpha \in \Phi \}$, where s_α is the reflection

Prop. (Feigin) Let $\delta = \sum_{i=1}^{\ell} f_i \partial_i \in \mathrm{Der}_S^W$. Then $\delta \in D(A_\Phi, 2m+1)^W \Longleftrightarrow f_i \in Q_m$.

An integral expression of a basis of Q_m is studied by Felder–Veselov, Bandlow–Musiker.

M. Feigin translated as follows:

Prop ($\Phi = A_{\ell-1}$, $Q = \prod_{1 \le i < j \le \ell} (x_i - x_j)$)

Let $g(t) := (t-x_1)(t-x_2)\cdots(t-x_n)$, and

$$\eta_k^m := \sum_{i=1}^{\ell} \left(\int_0^{x_i} t^k \cdot g(t)^m \, dt \right) \partial_i .$$

Then ($\theta_0 = \partial_1 + \cdots + \partial_\ell$, and) $\eta_0^m, \eta_1^m, \ldots, \eta_{\ell-2}^m$ form a basis of $D(A_\Phi, 2m+1)$.

Prop ($\Phi = A_{\ell-1}$, $Q = \prod_{1\le i<j\le \ell}(x_i - x_j)$) Let $g(t) := (t-x_1)(t-x_2)\cdots(t-x_n)$, and

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Then ($\theta_0 = \partial_1 + \cdots + \partial_\ell$, and) $\eta^m_0, \eta^m_1, \ldots, \eta^m_{\ell-2}$ form a basis of $D(A_\Phi, 2m+1)$.

Example $\ell = 2$, $m = 1$. $k = 0$

$$\eta^1_0 = \sum_{i=1}^{\ell}\left(\int_0^{x_i}(t-x_1)(t-x_2)\,dt\right)\partial_i .$$

$$\eta^1_0 \cdot (x_1 - x_2) = -\int_{x_1}^{x_2}(t-x_1)(t-x_2)\,dt = \frac{(x_2-x_1)^3}{6}.$$

Rem $\nabla_D \eta^m_k \doteq \eta^{m-1}_k$. So, $\eta^m_k \doteq \nabla_D^{-m}\eta^0_k$.

The integral expression is consistent with the primitive derivation ∇_D.

Prop ($\Phi = A_{\ell-1}$, $Q = \prod_{1\le i<j\le \ell}(x_i - x_j)$) Let $g(t) := (t-x_1)(t-x_2)\cdots(t-x_n)$, and

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… back to the Cat^m_Φ.

Def (difference operator) $\Delta F(x) = F(x+1) - F(x)$.

Def (discrete integral) Suppose $f(x) = \Delta F(x)$. Then define

$$\sum_a^b f(x)\,\Delta x := F(b) - F(a).$$

Def. Let $\tilde{\eta}_k^m := \sum_{i=1}^{\ell}\left(\sum_0^{x_i} t^k \cdot g(t)^{\underline{m}}\,\Delta t\right)\partial_i$, where

$$g(t)^{\underline{m}} = g(t)\cdot g(t-1)\cdots g(t-m+1). \quad \text{(falling power)}$$

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Thm. (Suyama, Y. 2021) $\boxed{\Phi = A_{\ell-1}}$

θ_0 and $\tilde{\eta}_0^m, \cdots, \tilde{\eta}_{\ell-2}^m$ form a basis of $D(\mathrm{Cat}_\Phi^m)$.

$\cdots$ Is Cat_Φ^m a discretization of $(\mathcal{A}_\Phi, 2m+1)$?

$g(t) := (t-x_1)(t-x_2)\cdots(t-x_n)$ $\qquad$ $g(t)^{\underline{m}} = g(t)\cdot g(t-1)\cdots g(t-m+1)$.

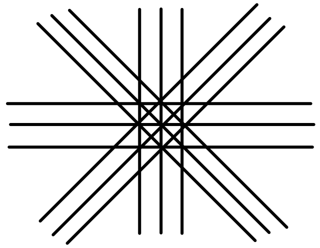
Thm. (Suyama, Y. 2021) θ_0 and $\tilde{\eta}_0^m, \dots, \tilde{\eta}_{\ell-2}^m$ form a basis of $D(\mathrm{Cat}_\Phi^m)$.

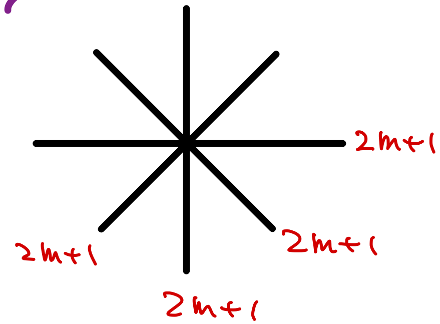
$\Phi = A_{\ell-1}$

Cat_Φ^m —coning (homogenize)→ $c\mathrm{Cat}_\Phi^m$ —restriction $|_{z=0}$→ $(\mathcal{A}_\Phi, 2m+1)$

Cat_Φ^m —L.T. (Leading Term)→ $(\mathcal{A}_\Phi, 2m+1)$

$(\mathcal{A}_\Phi, 2m+1)$ ⇢ discretization ?? ⇢ Cat_Φ^m

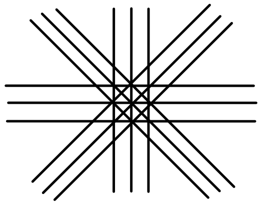


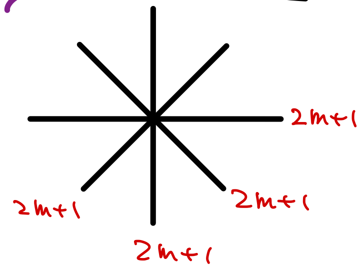


$$\tilde{\eta}_k^m = \sum_{i=1}^{\ell}\left(\sum_0^{x_i} t^k g(t)^{\underline{m}}\,\Delta t\right)\partial_i, \qquad \eta_k^m = \sum_{i=1}^{\ell}\left(\int_0^{x_i} t^k g(t)^m\,dt\right)\partial_i.$$

$\Phi = A_{\ell-1}$

$$\begin{array}{ccccc} & & c\mathrm{Cat}^m_\Phi & & \\ & \nearrow^{\text{coning (homogenize)}} & & \searrow^{\text{restriction } |_{z=0}} & \\ \mathrm{Cat}^m_\Phi & & \xrightarrow{\text{L.T. (Leading Term)}} & & (\mathcal{A}_\Phi, 2m+1) \\ & & \xleftarrow[\text{discretization ??}]{} & & \end{array}$$





$$\tilde{\eta}^m_k = \sum_{i=1}^{\ell}\left(\sum_0^{x_i} t^k g(t)^{\underline{m}} \Delta t\right)\partial_i, \qquad \eta^m_k = \sum_{i=1}^{\ell}\left(\int_0^{x_i} t^k g(t)^m dt\right)\partial_i .$$

<u>Question</u> (Work-in-Progress j.w. Abe, Enomoto)

Can we discretize the primitive derivation?

$$\begin{array}{ccc} D(\mathrm{Cat}^m_\Phi)^W & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_\Phi, 2m+1)^W \\ {\scriptstyle ?\exists ?}\ \vdots & & \downarrow \nabla_D\ ? \\ D(\mathrm{Cat}^{m-1}_\Phi)^W & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_\Phi, 2m-1)^W \end{array}$$

Def. $\Delta_i^{\pm} f = \dfrac{f(x+e_i)-f(x-e_i)}{2}$, where $e_i = (0,\dots,0,\overset{i}{\check{1}},0,\dots,0)$.

Def. (primitive difference op.)

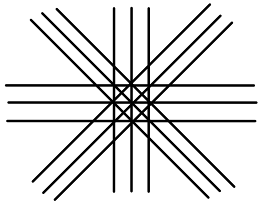
$$\tilde{D} := \frac{1}{\prod_{1\le i<j\le \ell}(x_i-x_j)} \cdot \det\begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \Delta_1^{\pm} \\ \vdots & & & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \Delta_\ell^{\pm} \end{pmatrix}$$

2023+

Thm (j.w. T. Abe, N. Enomoto). For $\Phi = A_{\ell-1}$, $\tilde{D}$ induces the isomorphism.

$$\begin{array}{ccc} D(\mathrm{Cat}_{\Phi}^{m})^{W} & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_{\Phi}, 2m+1)^{W} \\ \downarrow \nabla_{\tilde{D}} & \circlearrowright & \nabla_D \downarrow \;\; \uparrow \nabla_D^{-1} \\ D(\mathrm{Cat}_{\Phi}^{m-1})^{W} & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_{\Phi}, 2m-1)^{W}. \end{array}$$

Summary for $\Phi = A_{\ell-1}$



Cat^m_Φ —coning (homogenize)→ $c\mathrm{Cat}^m_\Phi$ —restriction $|_{z=0}$→ $(\mathcal{A}_\Phi, 2m+1)$

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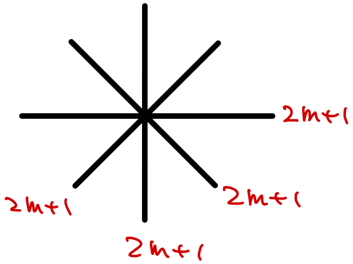
$(\mathcal{A}_\Phi, 2m+1)$ ⇢ discretization ?? ⇢ Cat^m_Φ



$$\tilde{\eta}^m_k = \sum_{i=1}^{\ell} \left(\sum_0^{x_i} t^k g(t)^{\underline{m}} \Delta t \right) \partial_i \qquad \eta^m_k = \sum_{i=1}^{\ell} \left(\int_0^{x_i} t^k g(t)^m dt \right) \partial_i .$$

$$\begin{array}{ccc} D(\mathrm{Cat}^m_\Phi)^W & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_\Phi, 2m+1)^W \\ \nabla_{\tilde{D}} \downarrow & & \downarrow \nabla_D \ \wr \\ D(\mathrm{Cat}^{m-1}_\Phi)^W & \xrightarrow{\text{L.T.}} & D(\mathcal{A}_\Phi, 2m-1)^W \end{array}$$

$$\tilde{D} := \frac{1}{\prod_{1 \le i < j \le \ell} (x_i - x_j)} \cdot \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \Delta_1^{\pm} \\ \vdots & & & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \Delta_\ell^{\pm} \end{pmatrix},$$

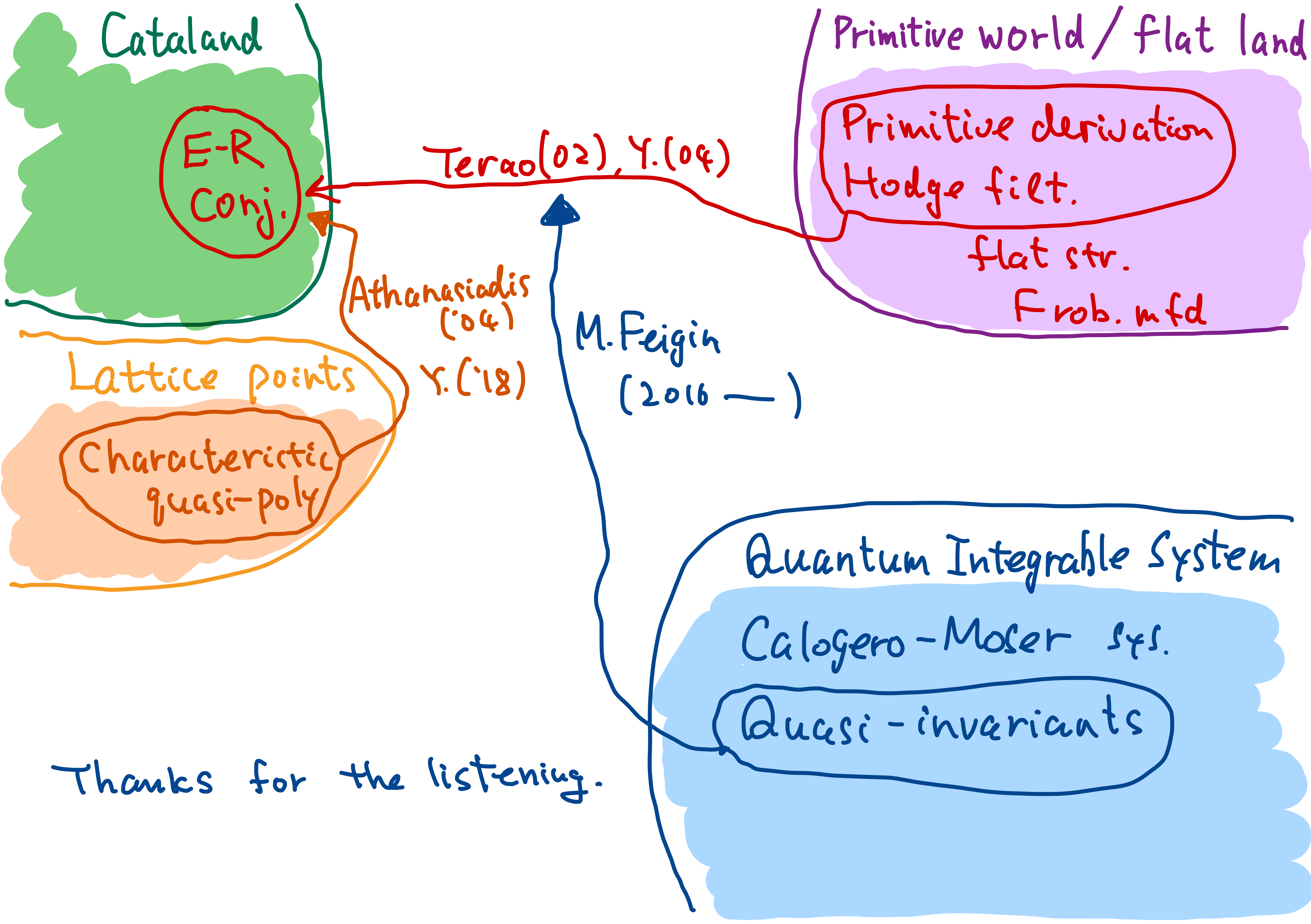
where $\Delta_i^{\pm} f = \dfrac{f(x+e_i) - f(x-e_i)}{2}$, $e_i = (0, \cdots 0, \overset{i}{1}, 0 \cdots 0)$.

$$D := \frac{1}{\prod_{1 \le i < j \le \ell} (x_i - x_j)} \cdot \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \partial_1 \\ \vdots & & & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \partial_\ell \end{pmatrix}$$

Work in progress

- $\nabla_{\tilde{D}}$ for B_ℓ, D_ℓ (Abe-Enomoto-Y.)
- Integral expression for $B_\ell, \cdots$ (Feigin-Wang-Y.)

Q. Discrete analogue of Flat/Frob. str ??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Thanks for the listening.