From Extended Affine Weyl Groups to duVal singularities via hyperplane arrangements

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The Dubrovin Connection and the WDVV equations

Let $\circ$ be an associative/commutative product with
$\eta(a \circ b, c) = \eta(a, b \circ c)$.

Dubrovin connection/Lax pair:

${}^{(\lambda)}\nabla_X Y = \nabla_X Y + \lambda X \circ Y$ flat for all λ.

The WDVV equations are the conditions for the vanishing of the curvature and torsion of this connection.

In flat coordinates for ∇

$$\partial_i \circ \partial_j = F_{ijr} \eta^{rs} \partial_s .$$

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Generalized Legendre Transformations

Assume we have a unity element e, so $e \circ X = X$. Let δ be an invertible vector field (so there exists δ^{-1} so that $\delta^{-1} \circ \delta = e$).

Conjugate the Dubrovin connection:

$$ {}^{(\lambda)}\widetilde{\nabla}_X Y = \delta^{-1} \circ \left({}^{(\lambda)}\nabla_X \right) (\delta \circ Y) $$

Question:

When does ${}^{(\lambda)}\widetilde{\nabla}$ have the same properties as ${}^{(\lambda)}\nabla$?

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Definition

An invertible vector field δ is a generalised Legendre field if:

$$X \circ \nabla_Y \delta = Y \circ \nabla_X \delta \,.$$

(This is equivalent to $\nabla_X \delta = X \circ \nabla_e \delta$, so definition not over-determined) The field δ is a Legendre field if $\nabla_X \delta = 0$.

With this ${}^{(\lambda)}\widetilde{\nabla}$ has the same properties as ${}^{(\lambda)}\nabla$ and hence if one has zero curvature ($\equiv$ WDVV) then the other has zero curvature

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From WDVV to Frobenius Manifolds

Frobenius Manifolds - extra structure:

(a) identity field e such that $e \circ X = X$ and $\nabla e = 0$;

(b) Euler field $\mathcal{L}_E \circ = \circ$.

Interchange roles - almost-duality:

(c) $X \star Y = E^{-1} \circ X \circ Y \quad \text{where } E^{-1} \circ E = e;$

(d) $< X, Y > = (X \circ Y, E^{-1})$.

Dubrovin [2004]

(i) $<,>$ flat (intersection form);

(ii) $\star$ associative, commutative with identity E (but in general new unity field not constant ${}^{<,>}\nabla E \neq 0$);

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Interplay between Legendre transformations and almost-duality

Let δ be a Legendre field. Then the following diagram commutes:

$$\begin{array}{ccc} F & \stackrel{\delta}{\longleftrightarrow} & \mathcal{L}_\delta F \\ \updownarrow & & \updownarrow \\ F^\star & \stackrel{E\circ\delta}{\longleftrightarrow} & (\mathcal{L}_{E\circ\delta})F^\star = (\mathcal{L}_\delta F)^\star \end{array}$$

In particular, $E \circ \delta$ is a generalized Legendre field.

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$$F^\star = \sum_{\alpha \in \mathcal{U}} h_\alpha \alpha(z)^2 \log \alpha(z) \quad \overset{E \circ \delta}{\longleftrightarrow} \quad (\mathcal{L}_\delta F)^\star = \left\{ \begin{array}{c} \textit{cubic terms} \\ \\ + \sum Li_3[e^{\alpha(z)}] \end{array} \right\}$$

Examples in 3 dimensions

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All generalizes to A_n, D_n. Parts exists for $E_{6,7,8}$.

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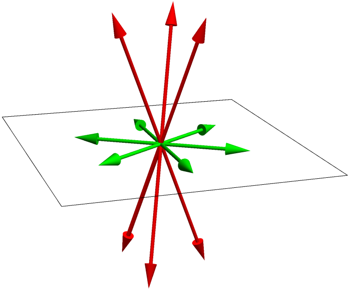
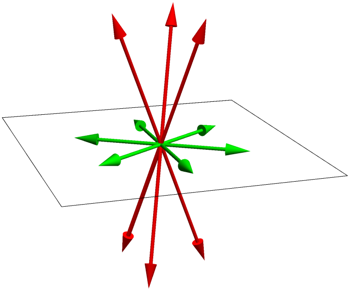


where:

Green vectors: roots of A_n ;

Red vectors: extension into perpendicular of $W(w_1)$.

The Legendre field is the normal to the plane containing in A_n-roots.



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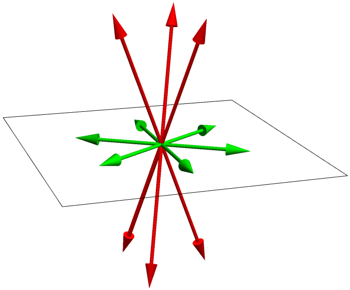
$\bigvee$-systems and Hyperplanes



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Hurwitz spaces: (λ, ω)

Landau Ginzburg superpotential:

$$\lambda(p) = p + \frac{t^2}{p - t^1}\,, \qquad \omega = dp\,.$$

Crucial ingredient: ω primitive form/primary differential
Direct computation using:

$$\begin{aligned} c_{ijk} &= \sum \mathrm{res} \left\{ \frac{\partial_i \lambda \partial_j \lambda \partial_k \lambda}{\lambda'} \omega \right\} \\ \eta_{ij} &= \sum \mathrm{res} \left\{ \frac{\partial_i \lambda \partial_j \lambda}{\lambda'} \omega \right\} \end{aligned}$$

gives

$$F = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{2}(t^2)^2 \log t^2\,.$$

Recent work of Arsie et al. gives the manifold as $\left(\mathbb{C}^2/B_2\right)_{\_}$.

Legendre field: $\delta = \partial_2$

Legendre transformation = change in primary differential ω.

$$d\tilde{p} = \partial_2 \lambda \, dp$$

So new LG potential is

$$\lambda(\tilde{p}) = e^{\tilde{p}} + t^1 + t^2 e^{-\tilde{p}} , \qquad \text{with } \omega = d\tilde{p} .$$

This gives:

$$\mathcal{L}_\delta F = \frac{1}{2}(t^1)^2 t^2 + e^{t^2} .$$

This lives on the extended affine Weyl orbit space $\mathbb{C}^2 / A_1^{(1)}$.

All generalizes to $\mathbb{C}^n / A_{n-1}^{(k)}$.

From EAW to duVal

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- GW invariants lie in a ring $\mathbb{C}[\tau_1, \tau_2]$. This involves an extension of the original work of Bryan and Gholampour (where $\tau_1 = \tau_2$);
- Using ideas of [DSZZ] the result extends to binary dihedral duVal singularities, via a $\mathbb{Z}_2$-invariant rational LG-superpotential;

- For $E_{6,7,8}$ objects on each side known: conjecturally true.

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$$\left\{ \begin{array}{c} \textit{Extended affine} \\ \textit{Weyl orbit space} \\ \\ \mathbb{C}^n / A_{n-1}^{(k)} \end{array} \right\} \overset{\textit{almost}}{\underset{\textit{duality}}{\longleftrightarrow}} \left\{ \begin{array}{c} \textit{GW generating function } F_{(n-k,k)} \\ \textit{for resolution of} \begin{pmatrix} Y \\ \downarrow \\ \mathbb{C}^2/\mathbb{Z}_n \end{pmatrix} \end{array} \right\}$$

- GW invariants lie in a ring $\mathbb{C}[\tau_1, \tau_2]$. This involves an extension of the original work of Bryan and Gholampour (where $\tau_1 = \tau_2$) ;
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