Hyperplane arrangements arising from symplectic singularities

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Plan

Slogan: Symplectic singularities give rise to MANY examples of hyperplane arrangements.

Conic symplectic singularities

Definition

An affine variety $X/\mathbb{C}$ is a conic symplectic singularity if

(i) X is normal.

(ii) X_{reg} has a symplectic form.

(iii) If $\pi\colon Y \to X$ is a resolution of singularities then $\pi^*\omega$ is a regular 2-form on Y.

(iv) $\mathbb{C}[X]$ is $\mathbb{N}$-graded, $\mathbb{C}[X]_0 = \mathbb{C}$ and ω has weight $\ell > 0$.

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We say that $\pi\colon Y \to X$ is a *symplectic resolution* if $\pi^*\omega$ is a symplectic form on Y.

The Cartan space

Example

If $\Gamma \subset SL(2,\mathbb{C})$ then $\mathbb{C}^2/\Gamma$ is a conic symplectic singularity and the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$ is a symplectic resolution.

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Assume $\pi \colon Y \to X$ is a (projective) symplectic resolution (or more generally a $\mathbb{Q}$-factorial terminalization).

Let $\mathfrak{h}^* = H^2(Y,\mathbb{R})$ and $\mathfrak{h}^*_{\mathbb{Q}} = H^2(Y,\mathbb{Q})$.

The resolutions

Theorem (Birkar-Cascini-Hacon-McKernan)

(a) X admits finitely many projective symplectic resolutions $Y = Y_1, \dots, Y_N$.

(b) There exist convex polyhedral cones $\mathrm{Mov}(Y), \mathrm{Amp}(Y_i)$ in $\mathfrak{h}^*$ such that

$$\mathrm{Mov}(Y) = \bigcup_{i=1}^{N} \overline{\mathrm{Amp}(Y_i)}.$$

The Namikawa Weyl group

Theorem (Namikawa)

There exists a finite hyperplane arrangement $\mathcal{A} \subset \mathfrak{h}_{\mathbb{Q}}^*$, with Coxeter subarrangement $\mathcal{B} \subset \mathcal{A}$ such that:

(a) $\mathrm{Mov}(Y)$ is a fundamental domain for the action of $W = \langle s_H \,|\, H \in \mathcal{B} \rangle$.

(b) W acts on $\mathcal{A}$.

(c) $\bigcup_{i=1}^{k} \mathrm{Amp}(Y_i) = \mathrm{Mov}(Y) \setminus \left(\bigcup_{H \in \mathcal{A}} H \right)$.

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We call W the Namikawa Weyl group (it is always a Weyl group).

Counting

Corollary

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- Goal in examples - describe all projective symplectic resolutions.
- First we must compute how many there are.

- If $X = \mathbb{C}^2/\Gamma$ then $(W, \mathfrak{h})$ given via the McKay correspondence and $\mathcal{A} = \mathcal{B}$ is Coxeter arrangement of W_Γ.
- If $X = \mathcal{N}(\mathfrak{g}^*)$ is the nil-cone of a simple Lie algebra then Springer resolution $T^*(G/B) \to \mathcal{N}(\mathfrak{g}^*)$ is symplectic resolution and again $(W, \mathfrak{h})$ usual Weyl group with $\mathcal{A} = \mathcal{B}$ the Coxeter arrangement.

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As a consequence, we see in both these examples there is a unique projective symplectic resolution.

Complex reflection groups

If (Γ, V) is a complex reflection group with reflections S then

(a) $X = (V \times V^*)/\Gamma$ is a conic symplectic singularity.

(b) $\mathfrak{h}^* = \{c \colon S/\Gamma \to \mathbb{R}\}$ (Ito-Reid).

(c) $W = \prod_{[H] \in \mathcal{H}/\Gamma} \mathfrak{S}_{\ell_H}$ (B-Schedler-Thiel).

$\mathcal{A} \neq \mathcal{B}$ in general, but known in many examples.

G_4

Here $W = \mathfrak{S}_3$ acting on $\mathfrak{h}^* = \{(\kappa_0, \kappa_1, \kappa_2) \,|, \kappa_0 + \kappa_1 + \kappa_2 = 0\}$ with arrangement:

$$\kappa_1, \kappa_2, \kappa_1 + \kappa_2, \kappa_1 - 2\kappa_2, \kappa_1 - \kappa_2, 2\kappa_1 - \kappa_2 = 0.$$

Then $\dim H^*(M(\mathcal{A}), \mathbb{C}) = 12$ so $N = 2$.

Other exceptional groups (B-Schedler-Thiel)

Group	$\lvert\mathcal{A}\rvert$	Weyl group	N
G_4	6	$\mathfrak{S}_3$	2
G_5	33	$\mathfrak{S}_3 \times \mathfrak{S}_3$	92
G_6	16	$\mathfrak{S}_2 \times \mathfrak{S}_3$	12
G_7	61	$\mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_3$	3296
G_8	25	$\mathfrak{S}_4$	14
G_9	54	$\mathfrak{S}_2 \times \mathfrak{S}_4$	2
G_{10}	111	$\mathfrak{S}_3 \times \mathfrak{S}_4$	15476
G_{11}	196	$\mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_4$	2851133
G_{13}	6	$\mathfrak{S}_2 \times \mathfrak{S}_2$	3
G_{14}	22	$\mathfrak{S}_2 \times \mathfrak{S}_3$	23
G_{15}	65	$\mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_2$	2596
G_{20}	12	$\mathfrak{S}_3$	4
G_{25}	12	$\mathfrak{S}_3$	4
G_{26}	37	$\mathfrak{S}_2 \times \mathfrak{S}_3$	62
$F_4 = G_{28}$	8	$\mathfrak{S}_2 \times \mathfrak{S}_2$	4

Wreath products (B-Craw)

If $\Gamma \subset SL(2,\mathbb{C})$ and $\Gamma_n = \Gamma^n \rtimes \mathfrak{S}_n$ then $X = \mathbb{C}^{2n}/\Gamma_n$.

(B-Craw)

(a) $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{h}_\Gamma$

(b) $W = \mathfrak{S}_2 \times W_\Gamma$

(c) $\mathcal{A}$ is the $(n-1)$-extended Catalan arrangement

(d) $N = \prod_{i=1}^{r} \frac{(n-1)h+d_i}{d_i}$, $d_1, \dots, d_r$ degrees of W_Γ (Athanasiadis)

All projective symplectic resolutions given explicitly by Nakajima quiver varieties.

Example

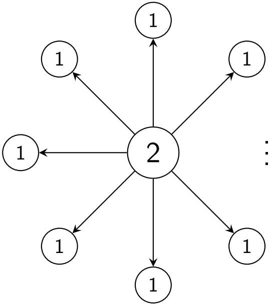
One final quotient singularity:

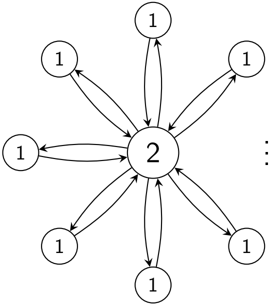
$$X = (\mathbb{C}^2 \otimes \mathbb{C}^2)/(Q_8 \times_{\mathbb{Z}_2} D_8).$$

(B-Schedler)

(a) $\mathfrak{h} = \mathbb{C}^5$
(b) $W = \mathfrak{S}_2^5$
(c) $\mathcal{A} = \{x_i = 0\} \cup \{\sum_{i \in I} x_i = \sum_{j \notin I} x_j\}$.
(d) $N = 81$

Hyperpolygon spaces





Hyperpolygon spaces

- Let V be space of representations of the star shaped quiver with n legs and dimension vector $(2, 1, \dots, 1)$.
- $X_n = \mu^{-1}(0)/\!/G$ is Hamiltonian reduction of $V \times V^*$ by $G = GL(2) \times (\mathbb{C}^\times)^n$.
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(B-Craw-Rayan-Schedler-Weiss)

$$(\mathbb{C}^2 \otimes \mathbb{C}^2)/(Q_8 \times_{\mathbb{Z}_2} D_8) \cong X_5.$$

Hyperpolygon spaces

Theorem (B-Craw-Rayan-Schedler-Weiss)

(a) $\mathfrak{h} = \mathbb{C}^n$

(b) $W = \mathfrak{S}_2^n$ (for $n \geq 5$)

(c) $\mathcal{A} = \{x_i = 0\} \cup \{\sum_{i \in I} x_i = \sum_{j \notin I} x_j\}$.

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(Hubbard, King)

For $n \geq 5$,

$$N_n : 1, 2, 4, 12, 81, 1684, 122921, 33207256, 3444822538 \dots$$

is the sequence counting the "number of self-dual threshold functions of $n + 1$ variables".

Nakajima Quiver Varieties

- Let $Q = (Q_0, Q_1)$ be a finite quiver and $\alpha \in \mathbb{Z}^{Q_0}$ a dimension vector for Q.
- Gives rise to (affine) Nakajima quiver variety $\mathfrak{M}_0(\alpha)$.
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Assume $\alpha \in \Sigma$.

(a) $\mathfrak{h}^* = \{\theta \in \mathbb{R}^{Q_0} \,|\, \theta(\alpha) = 0\}$

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Partial results by Wu on W.

Hypertoric varieties

- For $0 < r < n$, fix a unimodular $r \times n$ matrix A and choose B such that
$$0 \to \mathbb{Z}^{n-1} \stackrel{B}{\to} \mathbb{Z}^n \stackrel{A}{\to} \mathbb{Z}^r \to 0$$
exact.
- Gives action of $T = (\mathbb{C}^\times)^r$ on $V = \mathbb{C}^n$.
- Hypertoric variety $X(A)$ is the Hamiltonian reduction $\mu^{-1}(0)/\!/T$ of $V \times V^*$.

Hypertoric varieties

(Nagaoka)

Order $B^T = [b_1, \dots, b_n]$ such that
$b_1 = \cdots = b_{\ell_1}, b_{\ell_1+1} = \cdots = b_{\ell_1+\ell_2}, \dots$ and assume $b_i \neq -b_j$.

(a) $\mathfrak{h} = \mathbb{R}^r$

(b) $W = \mathfrak{S}_{\ell_1} \times \mathfrak{S}_{\ell_2} \times \cdots$

(c) $\mathcal{A} = \{H \subset \mathbb{R}^r \mid \langle a_{i_1}, \dots, a_{i_{r-1}} \rangle = H, \dim H = r-1\}$.

Questions

(1) If we choose an arbitrary pair $(\mathcal{A} \supset \mathcal{B})$ can we always find a conic symplectic singularity realizing it?

(2) Is there an effective way to compute N for a large class of examples?

(3) W acts on $H^*(M(\mathcal{A}), \mathbb{C})$. What is this graded representation?