Accurate Reflection Arrangements

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Motivation

In 1983, Orlik and Solomon computed $\chi(\mathscr{A}(W)^X, t)$ of every restriction $\mathscr{A}(W)^X$ of each Coxeter arrangement $\mathscr{A}(W)$ to some intersection X of reflecting hyperplanes of $\mathscr{A}(W)$ in long and intricate computations. They showed that each $\chi(\mathscr{A}(W)^X, t)$ factors over $\mathbb{Z}$. They observed that for each $1 \leq d \leq \ell$, where ℓ is the dimension of the reflection representation of W, there exists such a restriction $\mathscr{A}(W)^X$ such that the roots of $\chi(\mathscr{A}(W)^X, t)$ are the first d exponents of $\mathscr{A}(W)$ when ordered by size. In view of Terao's factorization theorem, it is natural to pose:

Conjecture (Orlik-Solomon-Terao 1987)

For every Coxeter arrangement $\mathscr{A}(W)$ the first d exponents of $\mathscr{A}(W)$ when ordered by size are realized as the exponents of some free restriction $\mathscr{A}(W)^X$ of $\mathscr{A}(W)$ for every $1 \leq d \leq \ell$.

Proof: long case-by-case studies by Orlik-Terao (1992, 1993).

Goal: Find a uniform proof at least for Weyl arrangements.

Arrangements

- V an ℓ-dimensional, complex vector space;
- $\mathscr{A}$ a (central) hyperplane arrangement in V;
- $L(\mathscr{A})$ the intersection lattice of $\mathscr{A}$.
- for X in $L(\mathscr{A})$ consider the *localization* $\mathscr{A}_X$ of $\mathscr{A}$ at X: $\mathscr{A}_X := \{H \in \mathscr{A} \mid H \supset X\} \subseteq \mathscr{A}$.
- for X in $L(\mathscr{A})$ consider the *restriction* $\mathscr{A}^X$ of $\mathscr{A}$ to X: $\mathscr{A}^X := \{H \cap X \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}$ is a hyperplane arrangement in X.
- Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V.
- The *defining polynomial* $Q(\mathscr{A})$ of $\mathscr{A}$ is given by

$$Q(\mathscr{A}) := \prod_{H \in \mathscr{A}} \alpha_H \in S,$$

where $\alpha_H \in V^*$ satisfies $H = \ker(\alpha_H)$.

Free Arrangements

Let Der(S) be the S-module of $\mathbb{C}$-derivations of S. It is a free S-module with basis $\partial/\partial x_1, \dots, \partial/\partial x_\ell$.
From the grading of S, we obtain a $\mathbb{Z}$-grading $\mathrm{Der}(S) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Der}(S)_p$.

Definition (K. Saito 1975/1981)

The *module of $\mathscr{A}$-derivations* of $\mathscr{A}$ is defined by

$$D(\mathscr{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(Q(\mathscr{A})) \in Q(\mathscr{A})S\}.$$

We say that $\mathscr{A}$ is *free* if $D(\mathscr{A})$ is a free S-module.

If $\mathscr{A}$ is a free arrangement we may choose a homogeneous basis $\theta_1, \dots, \theta_\ell$ of $D(\mathscr{A})$. The degrees of the θ_i are called the *exponents* of $\mathscr{A}$. They are uniquely determined by $\mathscr{A}$. In that case we write

$$\exp(\mathscr{A}) := (e_1, e_2, \dots, e_\ell)$$

for the exponents of $\mathscr{A}$.

Free Arrangements, II

- If $\mathscr{A}$ is free, so is any localization $\mathscr{A}_X$.
- In general the restriction $\mathscr{A}^X$ of a free arrangement $\mathscr{A}$ need not be free (Edelman-Reiner 1991).
- Every reflection arrangement $\mathscr{A}(G)$ is free, for G a complex reflection group (Arnold/Saito (case of Coxeter groups); Terao (general case) 1980).
- For $\mathscr{A}(G)$ the reflection arrangement of a complex reflection group G, every restriction $\mathscr{A}(G)^X$ of $\mathscr{A}(G)$ is free.
  (Orlik-Terao 1992, 1993: Coxeter, in complex case up to $\dim X = 3$,
  Hoge-R 2013: G_{33}, G_{34} and $\dim X = 4, 5$).

Free Arrangements: Addition-Deletion

Fix $H \in \mathscr{A}$, denote $\mathscr{A}' := \mathscr{A} \setminus \{H\}$ and $\mathscr{A}'' := \mathscr{A}^H$. Call $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ the triple with respect to the hyperplane $H \in \mathscr{A}$.

Theorem (Addition-Deletion Theorem (Terao 1980))

Let $\mathscr{A}$ be a non-empty arrangement and let $H \in \mathscr{A}$. Then two of the following statements imply the third:

(i) $\mathscr{A}$ *is free with* $\exp(\mathscr{A}) = (e_1, \ldots, e_{\ell-1}, e_\ell)$.

(ii) $\mathscr{A}'$ *is free with* $\exp(\mathscr{A}') = (e_1, \ldots, e_{\ell-1}, e_\ell - 1)$.

(iii) $\mathscr{A}''$ *is free with* $\exp(\mathscr{A}'') = (e_1, \ldots, e_{\ell-1})$.

Moreover, all three assertions hold if $\mathscr{A}$ and $\mathscr{A}'$ are both free.

Inductive Freeness

Terao's Theorem above motivates the following concept.

Definition (Orlik-Terao 1992)

The class $\mathcal{IF}$ of *inductively free* arrangements is the smallest class of arrangements which satisfies

(i) the empty arrangement $\varnothing_\ell$ is in $\mathcal{IF}$ for $\ell \geq 0$,

(ii) if there exists $H \in \mathscr{A}$ such that $\mathscr{A}'' \in \mathcal{IF}$, $\mathscr{A}' \in \mathcal{IF}$, and $\exp(\mathscr{A}'') \subseteq \exp(\mathscr{A}')$, then $\mathscr{A} \in \mathcal{IF}$.

Inductive freeness is a combinatorial property (Cuntz-Hoge 2015).
Example: Coxeter arrangements are inductively free; in fact they are hereditarily inductivley free (every restriction is inductivley free) (Barakat-Cuntz 2012).

Accuracy

Definition (Mücksch-R 2021; Mücksch-R-Tran 2023)

Suppose $\mathscr{A}$ is free with exponents $\exp(\mathscr{A}) = (e_1, e_2, \ldots, e_\ell)_\leq$.

1. $\mathscr{A}$ is *accurate* provided for each $1 \leq d \leq \ell$ there exists a flat X_d in $L(\mathscr{A})$ of dimension d such that the restriction $\mathscr{A}^{X_d}$ of $\mathscr{A}$ to X_d is free with $\exp(\mathscr{A}^{X_d}) = (e_1, e_2, \ldots, e_d)_\leq$.
   The tuple $(X_1, X_2, \ldots, X_\ell)$ is a *witness* for the accuracy of $\mathscr{A}$.
2. $\mathscr{A}$ is *flag-accurate* provided there is a witness $(X_1, X_2, \ldots, X_\ell)$ for the accuracy of $\mathscr{A}$ such that $X_1 \subset X_2 \subset \ldots \subset X_\ell$ is a flag in $L(\mathscr{A})$.
3. $\mathscr{A}$ is *ind-flag-accurate* if $\mathscr{A}$ is both inductively free and flag-accurate, and there is a witness $(X_1, X_2, \ldots, X_\ell)$ for the flag-accuracy of $\mathscr{A}$ such that $\mathscr{A}^{X_d}$ is inductively free for every $1 \leq d \leq \ell$. In that case $(X_1, X_2, \ldots, X_\ell)$ is a *witness* for the ind-flag-accuracy of $\mathscr{A}$.

Examples among reflection arrangements

Example

Let $G = G(1, 1, \ell)$ be the symmetric group.
Then $\mathscr{A}(G)$ is free with $\exp \mathscr{A}(G) = \{1, 2, \dots, \ell - 1\}$.
Let $X_d := \{x_1 = \dots = x_{\ell-d+1}\}$ for $1 \leq d \leq \ell - 1$.
Then $\mathscr{A}(G)^{X_d} \cong \mathscr{A}(G(1, 1, d))$.
So the latter is free with $\exp \mathscr{A}(G) = \{1, 2, \dots, d - 1\}$.
Since braid arrangements are inductively free and the flats X_d form a flag in $L(\mathscr{A}(G))$, the braid arrangement $\mathscr{A}(G)$ is ind-flag-accurate.

Rephrase motivating conjecture:

Conjecture (Orlik-Solomon-Terao 1987)

Every Coxeter arrangement is accurate.

Proof: intricate and long case-by-case studies (Orlik-Terao 1992, 1993).
Uniform proof for Weyl arrangements [Mücksch-R 2021] (MAT-freeness).
Are Coxeter arrangements even flag-accurate, or ind-flag-accurate?

Divisional Freeness

Definition/Theorem (Abe 2016)

An ℓ-arrangement $\mathscr{A}$ is *divisionally free* if there is a flag

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{\ell-1} \subseteq X_\ell = V$$

with $\dim(X_i) = i$ for $1 \leq i \leq \ell$ and $\chi(\mathscr{A}^{X_i}, t) \mid \chi(\mathscr{A}^{X_{i+1}}, t)$ for each $1 \leq i \leq \ell - 1$. Such a flag is called a *divisional flag*. Such $\mathscr{A}$ are free.

Remarks

(i). If $\mathscr{A}$ is flag-accurate, then it is both accurate and divisionally free, since any witness for the flag-accuracy is a divisional flag and a witness for accuracy. The converse is false (ex. ideal Shi arrangement in type F_4).
(ii). Flag-accuracy only depends on $L(\mathscr{A})$ and thus is combinatorial, ditto for ind-flag-accuracy. Likewise, divisional freeness is also combinatorial. But it is not known whether this is also the case for accuracy itself.

Accurate Reflection Arrangements

Theorem (Mücksch-R 2021; Mücksch-R-Tran 2023)

Let G be a complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(G)$. Then $\mathscr{A}$ is flag-accurate if and only if it is divisionally free. This is the case if and only if G has no irreducible factor isomorphic to one of the monomial groups $G(r, r, \ell)$, $r > 2$, $\ell > 2$, or $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$.

Proof uses classification of divisionally free reflection arrangements due to Abe (2016). [Mücksch-R 2021]: accuracy and divisional freeness coincide for reflection arrangements. Proof extends to flag-accuracy.

Corollary

Coxeter arrangements are ind-flag-accurate.

Proof uses hereditary inductive freeness of Coxeter arrangements (Barakat-Cuntz 2012).

Accurate Reflection Arrangements

Theorem (Mücksch-R-Tran 2023)

Let G be an irreducible complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(G)$. Suppose $G \neq G_{31}$. TFAE:

(i) *$\mathscr{A}$ is accurate;*

(ii) *$\mathscr{A}$ is flag-accurate;*

(iii) *$\mathscr{A}$ is ind-flag-accurate;*

(iv) *$\mathscr{A}$ is divisionally free;*

(v) *$\mathscr{A}$ is inductively free.*

Proof uses theorem above and classification of inductively free reflection arrangements (Hoge-R 2015).

G_{31} is excluded, as $\mathscr{A}(G_{31})$ itself is not inductively free, but $\mathscr{A}(G_{31})$ does satisfy the properties in parts (i), (ii), and (iv) of the theorem. In particular, $\mathscr{A}(G_{31})$ is flag-accurate, but not ind-flag-accurate.

A useful lemma

The following is very helpful in inductive arguments:

Lemma

Let $\mathscr{A}$ be (inductively) free with exponents $\exp(\mathscr{A}) = (e_1, \ldots, e_\ell)_\leq$. Then $\mathscr{A}$ is (ind-)flag-accurate if and only if there exist k linearly independent hyperplanes $H_1, \ldots, H_k \in \mathscr{A}$ for some $1 \leq k \leq \ell$ such that $\mathscr{A}^{X_i}$ is (inductively) free with $\exp(\mathscr{A}^{X_i}) = (e_1, \ldots, e_{\ell-i})_\leq$ for each $1 \leq i \leq k$ where $X_i := \bigcap_{j=1}^{i} H_j$ and that $\mathscr{A}^{X_k}$ is (ind-)flag-accurate.
In particular, $\mathscr{A}$ is (ind-)flag-accurate if and only if there exists an H in $\mathscr{A}$ such that $\mathscr{A}^H$ is (ind-)flag-accurate with $\exp(\mathscr{A}^H) = (e_1, \ldots, e_{\ell-1})_\leq$.

Reverse implication of last statement also applies for accuracy, but not forward implication: there are examples where $\mathscr{A}$ is accurate, but $\mathscr{A}^H$ is not accurate for any H.

MAT-freeness

In 2020, Cuntz-Mücksch introduced the notion of *MAT-freeness* to investigate arrangements whose freeness can be derived using an iterative application of the Multiple Addition Theorem (due to Abe et al 2016).

Theorem (Mücksch-R 2021)

MAT-free arrangements are accurate.

MAT-freeness is a combinatorial property only relying on $L(\mathscr{A})$.
As *ideal subarrangements* of Weyl arrangements are MAT-free, due to Abe-Barakat-Cuntz-Hoge-Terao 2016, we get:

Theorem (Mücksch-R 2021)

Ideal arrangements are accurate.

Theorem (Mücksch-R-Tran 2023)

Ideal arrangements of rank at most 8 are flag-accurate.

Extended Shi and extended Catalan arrangements

Theorem (Mücksch-R 2021)

Extended Shi arrangements Shi^k, *ideal-Shi arrangements* $\mathrm{Shi}^k_{\mathcal{I}}$ *and extended Catalan arrangements* Cat^k *are accurate.*

Theorem (Mücksch-R-Tran 2023)

Extended Shi arrangements Shi^m *are flag-accurate. Extended Catalan arrangements* Cat^m *of Dynkin type* A, B, *or* C *are flag-accurate.*

Conjecture

Extended Catalan arrangements are flag-accurate.

Multi-Addition-Theorem (MAT-Theorem)

Theorem (Abe-Barakat-Cuntz-Hoge-Terao 2016)

Let $\mathscr{A}' = (\mathscr{A}', V)$ be a free arrangement with $\exp(\mathscr{A}') = (e_1, \dots, e_\ell)_\leq$ and let $1 \leq p \leq \ell$ be the multiplicity of the highest exponent, i.e.

$$e_{\ell-p} < e_{\ell-p+1} = \cdots = e_\ell =: e.$$

Let $H_1, \dots, H_q$ be hyperplanes in V with $H_i \notin \mathscr{A}'$ for $i = 1, \dots, q$. Let

$$\mathscr{A}_j'' := (\mathscr{A}' \cup \{H_j\})^{H_j} = \{H \cap H_j \mid H \in \mathscr{A}'\}, \quad \text{for } j = 1, \dots, q.$$

Assume that the following conditions are satisfied:

(1) $X := H_1 \cap \cdots \cap H_q$ *is* q*-codimensional.*

(2) $X \nsubseteq \bigcup_{H \in \mathscr{A}'} H$.

(3) $|\mathscr{A}'| - |\mathscr{A}_j''| = e$ *for* $1 \leq j \leq q$.

Then $q \leq p$ and $\mathscr{A} := \mathscr{A}' \cup \{H_1, \dots, H_q\}$ is free with

$$\exp(\mathscr{A}) = (e_1, \dots, e_{\ell-q}, e+1, \dots, e+1)_\leq.$$

MAT-freeness

Definition

Let $\mathscr{A}'$ and $\{H_1, \ldots, H_q\}$ be as in MAT-Theorem such that conditions (1)–(3) are satisfied. Then the addition of $\{H_1, \ldots, H_q\}$ to $\mathscr{A}'$ resulting in $\mathscr{A} = \mathscr{A}' \cup \{H_1, \ldots, H_q\}$ is called an *MAT-step*.

An iterative application of the MAT-Theorem motivates the concept of MAT-freeness.

MAT-freeness

Definition (Cuntz-Mücksch 2019)

An arrangement $\mathscr{A}$ is called *MAT-free* if there exists an ordered partition

$$\pi = (\pi_1 | \cdots | \pi_n)$$

of $\mathscr{A}$ such that the following hold. Set $\mathscr{A}_0 := \varnothing_\ell$ and

$$\mathscr{A}_k := \bigcup_{i=1}^{k} \pi_i \quad \text{for } 1 \leq k \leq n.$$

Then for every $0 \leq k \leq n-1$ suppose that

(1) $\operatorname{rk}(\pi_{k+1}) = |\pi_{k+1}|$,

(2) $\cap_{H \in \pi_{k+1}} H \nsubseteq \bigcup_{H' \in \mathscr{A}_k} H'$,

(3) $|\mathscr{A}_k| - |(\mathscr{A}_k \cup \{H\})^H| = k$ for each $H \in \pi_{k+1}$,

i.e. $\mathscr{A}_{k+1} = \mathscr{A}_k \cup \pi_{k+1}$ is an MAT-step.

MAT-free Reflection arrangements

Theorem (Cuntz-Mücksch 2020)

Let G be an irreducible complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(G)$. Suppose $G \neq G_{32}$. Then $\mathscr{A}$ is MAT-free if and only if it is inductively free.
This is the case if and only if G has no irreducible factor isomorphic to one of the monomial groups $G(r, r, \ell)$, $r > 2$, $\ell > 2$, or $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$.

Theorem (Abe-Barakat-Cuntz-Hoge-Terao 2016)

Ideal subarrangements of Weyl arrangements are MAT-free.