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E.M. Feichtner: A Leray model for the OS-algebra of a matroid

joint work w/ Chr. Bibby, G. Denham
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Goals:

- discuss Leray model for $M(\mathcal{A})$, thereby reconciling OS- and nested set combinatorics of arr'ts
- present a combinatorial lift to matroids, a CDGA interpolating between $OS(M)$ and $Ch^*(M)$.

① Introducing the players

$\mathcal{A}$ arr't of hyperplanes in $\mathbb{C}P^n$

$\mathcal{L}$ intersection lattice

$M(\mathcal{A}) := \mathbb{C}P^n \setminus \bigcup_{H \in \mathcal{A}} H$ arr't complement

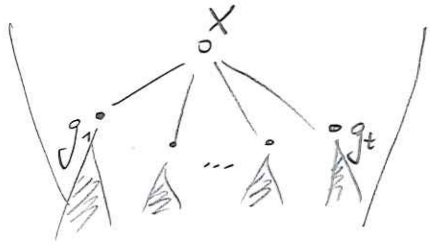
$OS(\mathcal{A}) = H^*(M(\mathcal{A}), \mathbb{Q})$ Orlik-Solomon algebra

$\mathcal{G} \subseteq \mathcal{L}$ Building set, a „good" subset of $\mathcal{L}$

Ex: $\mathcal{G}_{max} = \mathcal{L}$

$\mathcal{G}_{min}$ = irred. in $\mathcal{L}$

e.g. the 1-block partitions in Π_n



$$[\hat{0}, x] = \prod_{g_i \in \max \mathcal{G}_{\leq x}} [\hat{0}, g_i]$$

$\mathcal{S} \subseteq \mathcal{G}$ nested if for any pairwise incomparable $g_1, \dots, g_k \in \mathcal{S}$, $k \geq 2$,

$$\bigvee_{i=1}^{k} g_i \notin \mathcal{G}.$$

$N(\mathcal{G})$: complex of nested sets

Ex: $N(\mathcal{G}_{max}) = \Delta(\mathcal{L})$, the order complex of $\mathcal{L}$

in general: $N(\mathcal{G}) \cong \Delta(\mathcal{L})$.

Consider

$$\phi : M(\mathcal{A}) \longrightarrow \mathbb{CP}^r \times \prod_{g \in \mathcal{G}} \mathbb{P}(\mathbb{C}^{r+1}/g)$$

$$x \longmapsto (x, (\overline{x+g})_{g \in \mathcal{G}})$$

$Y(\mathcal{A}, \mathcal{G}) := \overline{\operatorname{Im} \phi}$, the WONDERFUL COMPACTIFICATION of $M(\mathcal{A})$

[DeConcini–Procesi '95]

Facts:

- $Y(\mathcal{A},\mathcal{G})$ smooth proj. variety
- $Y(\mathcal{A},\mathcal{G}) \setminus M(\mathcal{A}) = \bigcup_{G\in\mathcal{G}} D_G$ divisor w/ normal crossings
- $\bigcap_{G\in\mathcal{S}} D_G \neq \emptyset \iff \mathcal{S}\in\mathcal{N}(\mathcal{G})$.

Defn

$$DP(\mathcal{A},\mathcal{G}) := \mathbb{Q}[x_G \mid G\in\mathcal{G}] \Big/ \left\langle \begin{array}{ll} x_S := \prod_{G\in S} x_G, & S\in\mathcal{N}(\mathcal{G}) \\ \sum_{G\geq H} x_G, & H \text{ h'plane in } \mathcal{A} \end{array} \right\rangle$$

DeConcini-Procesi algebra

$$DP(\mathcal{A},\mathcal{G}) \cong H^*(Y(\mathcal{A},\mathcal{G}),\mathbb{Q})$$

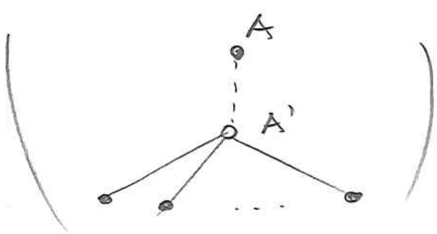
[DeConcini-Procesi '95]

monomial Basis:

$$\prod_{A\in S} x_A^{m_A}, \qquad S\in\mathcal{N}(\mathcal{G})$$

$m_A < \operatorname{rk} A - \operatorname{rk} A'$

where $A' = \bigvee S_{<A}$ $(<A!)$



[Yuzvinsky '97]

[F.-Yuzvinsky '04]

② A rational model for $M(\mathcal{A})$

$$\boxed{\mathcal{B}(\mathcal{A},\mathcal{G})} := \mathbb{Q}[e_g, x_g \mid g \in \mathcal{G}] \big/ (I + J)$$

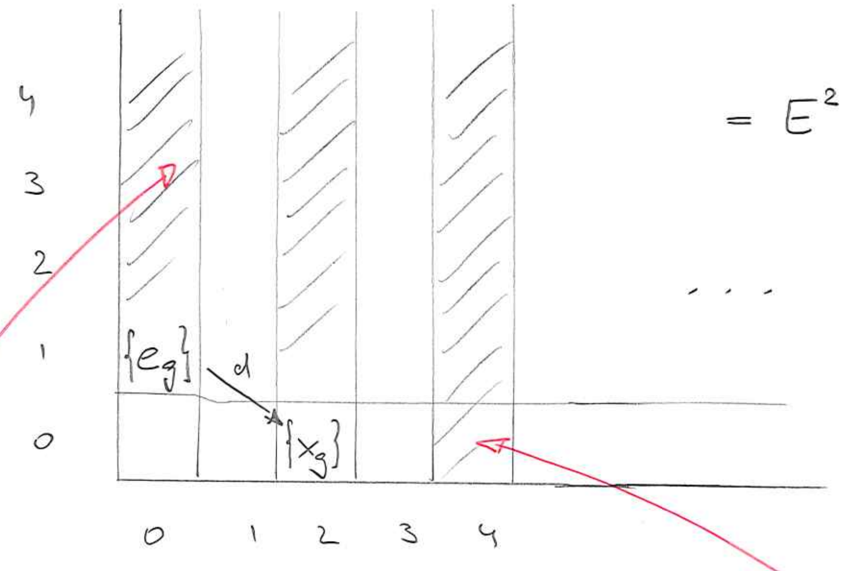
Bigraded: $\deg e_g = (0,1)$, $\deg x_g = (2,0)$ no $e_{\hat{1}}$

$$I := \left(e_S \cdot x_T \ : \ S \cup T \notin N(\mathcal{G}),\ S \cap T = \emptyset \right)$$

$$J := \left(c_i := \sum_{g \geq H_i} x_g \ ,\ H_i \text{ hyple in } \mathcal{A} \right)$$

$$d(e_g) = x_g, \quad d(x_g) = 0 \quad \text{f.a. } g \in \mathcal{G}.$$

Remarks



$= E^2(M(\mathcal{A}) \hookrightarrow Y(\mathcal{A},\mathcal{G}))$ ... Leray spectral sequence

$\mathcal{B}^{0,*}$ = ext. face ring of $N(\mathcal{G})$

$\mathcal{B}^{*,0} = DP(\mathcal{A},\mathcal{G}) = H^*(Y(\mathcal{A},\mathcal{G}))$

$E^3 = E^\infty$

$E^3_{0,*} = OS(\mathcal{A})$ $\quad E^3_{p,q} = 0$ for $p \geq 1$.

$$(OS(\mathcal{A}), 0) \longrightarrow (\mathcal{B}(\mathcal{A},\mathcal{G}), d)$$

$$\omega_{H_i} \longmapsto \sum_{g \geq H_i} e_g$$

is a quasi-isomorphism.

[Looijenga '93, Totaro '96, Bibby '14, Dupont '14]

③ A Leray model for matroids

$Y(\mathcal{A},\mathcal{G})$ is the result of successive blowups of subspaces indexed by $\mathcal{G}$ in some non-increasing order (w.r.t. $\mathcal{L}$)

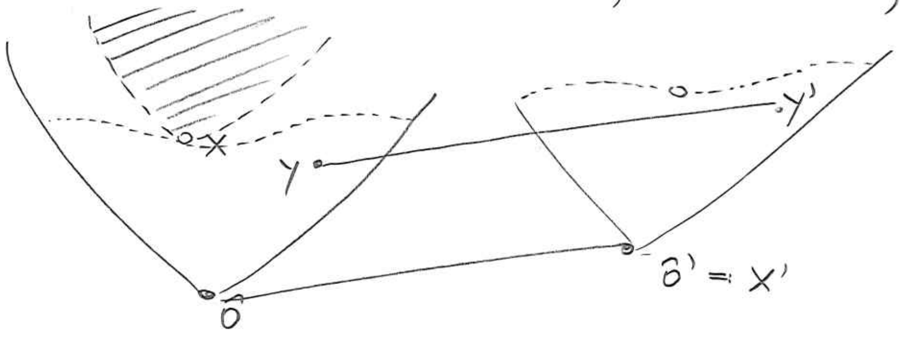
$$\mathbb{CP}^r \text{ stratif. by } \mathcal{A} \xrightarrow{\text{geometric blowup}} Y(\mathcal{A},\mathcal{G}) \text{ stratif. by } \{D_g\}_{g\in\mathcal{G}}$$

$$\mathcal{L}(\mathcal{A}) \xrightarrow{?} \text{face poset of } N(\mathcal{G})$$

Combinatorial blowups of $\mathcal{L}$ in $X \in \mathcal{L}$

[F.-Kozlov, '04]

$$Bl_X(\mathcal{L}) := \{Y \mid Y \not\geq X\} \cup \{Y' \mid Y \not\geq X,\ Y \vee X \text{ ex. in } \mathcal{L}\}$$



Prop [F. Kozlov '04]: Comb. blowups along a building set $\mathcal{G}$ in non-increasing order result in $\mathcal{F}(N(\mathcal{G}))$.

M simple matroid on $\{1, \dots, n\}$

L lattice of flats

$\mathcal{G} = \{g_1, \dots, g_t\}$ building set with a suitable linear order

$L^{(0)} := L, \dots, L^{(j)} := Bl_{g_{j-1}} L^{(j-1)}, \dots \quad L^{(t)} = \mathcal{F}(N(\mathcal{G}))$

$N^{(0)} = at(L) \qquad N^{(j)} := \{ S \subset \text{atoms}(L^{(j)}) \mid \bigvee S \text{ ex. in } L^{(j)} \}, \dots N^{(t)} = N(\mathcal{G})$

$$\boxed{B^{(j)}(M, \mathcal{G})} := \mathbb{Q}[e_g, x_g \mid g \text{ atom in } L^{(j)}] \Big/ I^{(j)}$$

$\deg e_g = (0,1)$, $\deg x_g = (2,0)$

$I^{(j)}$ generated by

(1) $e_S x_T \qquad S \cap T = \emptyset, \quad S \cup T \notin N(\mathcal{G})$

(2) $\partial e_S \qquad S \in N^{(j)}, \ \operatorname{rk} \bigvee_{(j)} S < |S|$

(3) $\displaystyle\sum_{\substack{g \text{ atom in } L^{(j)} \\ g \geq_L i}} x_g \quad , \quad i = 1, \dots, n$

$d^{(j)}(e_g) = x_g,$

$d^{(j)}(x_g) = 0.$

OBS: $\mathcal{B}^{(0)}(M, g) = OS(M)$

$\mathcal{B}^{(t)}(M, g) = \mathcal{B}(M, \rho)$ w/ bottom row $Ch^*(M)$.

$$\mathcal{B}^{(0)} \xrightarrow{\phi^{(0)}} \mathcal{B}^{(1)} \xrightarrow{\phi^{(1)}} \cdots \xrightarrow{\phi^{(t-1)}} \mathcal{B}^{(t)}$$

is a series of quasi-isomorphisms.

[Bibby, Denham, F. '22]

Key ingredients

(1) Gröbner bases for $I^{(j)}$:

(1), (2) & (3'): $e_{A_1} x_{A_2} \Big(\sum_{\substack{g \text{ atom in } L^{(j)} \\ g \geq_L B}} x_g \Big)^d$

$A_1 \cup A_2 \in \mathcal{N}^{(j)}$, $\bigvee A_1 \cup A_2 < B$

$d = $ rk-difference

(2) monomial basis for $\mathcal{B}^{(j)}$:

$$e_S x_T^b \;, \qquad \begin{array}{l} S \cup T \in \mathcal{N}^{(j)} \\ S \in \mathrm{nbc}\left(L^{(j)}\right) \\ 0 < b(g) < d\left(V(S\cup T)_{<g}, g\right) \end{array}$$

This basis interpolates between $\mathrm{nbc}(\Pi)$ and Yuzvinsky's basis for $Ch^*(M)$.

(3) $\mathcal{B}^{(j)}$ has a fine grading inspired by the Brieskorn decomposition of the OS-algebra:

$$\left(\mathcal{B}^{(j)}\right)^{p,q} = \bigoplus_{y \in L^{(j)}} OS^q\left(L^{(j)}_{\leq y}\right) \otimes_{\mathbb{Q}} \mathcal{D}\mathcal{P}_y^{\,p}\left(L^{(j)}\right)$$

$\square$