Upper Bounds on the Orders of Cages

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$$n(k,g) \geq M(k,g) = \begin{cases} 1 + k\frac{(k-1)^{(g-1)/2}-1}{k-2}, & g \text{ odd} \\ 2\frac{(k-1)^{g/2}-1}{k-2}, & g \text{ even} \end{cases}$$

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- a (k, g)-Moore-graph is necessarily a (k, g)-cage

Known Cages

girth	5	6	7	8	9	10	11	12
order	10	14	24	30	58	70	112	126
# of cages	1	1	1	1	18	3	1	1

Table: Known trivalent cages.

k	3	4	5	6	7
$n(k,5)$	10	19	30	40	50
number of cages	1	1	4	1	1

Table: Known cages of girth 5.

- $(7,6)$- and $(4,7)$-cage
- $(q+1,6)$-, $(q+1,8)$-, and $(q+1,12)$-cages, q a prime power

The $(3, 13)$-case

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- The record is a Cayley graph, and no smaller Cayley $(3, 13)$-graph exists, Royle

Record $(3, g)$-graphs

Girth g	Lower Bound	Smallest Known (k, g)-Graph	Author(s)
13	202	272	McKay-Myrvold; Hoare
14	258	384	McKay; Exoo
15	384	620	Biggs
16	512	960	Exoo
17	768	2 176	Exoo
18	1 024	2 560	Exoo
19	1 536	4 324	Hoare, H(47)
20	2 048	5 376	Exoo
21	3 072	16 028	Exoo
22	4 096	16 206	Biggs-Hoare, S(73)
23	6 144	35 446	Erskine-Tuite
24	8 192	35 640	Erskine-Tuite
25	12 288	108 906	Exoo
26	16 384	109 200	Bray-Parker-Rowley

Record $(3, g)$-graphs

Girth g	Lower Bound	Smallest Known (k, g)-Graph	Author(s)
27	24 576	285 852	Bray-Parker-Rowley
28	32 768	368 640	Erskine-Tuite
29	49 152	805 746	Erskine-Tuite
30	65 536	806 736	Erskine-Tuite
31	98 304	1 440 338	Erskine-Tuite
32	131 072	1 441 440	Erskine-Tuite

Bermond and Bollobás Question for Cages

Does there exist a constant C such that for every pair of parameters k, g there exists a (k, g)-graph of order not exceeding $M(k, g) + C$?

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- Computer evidence suggests negative answer
- The **excess** of a (k, g)-graph of order n is the difference $n - M(k, g)$; which seems to grow with the growth of both degree and girth

Two (Three?) Closely Related Question:

For odd $g \geq 3$, $g = 2r + 1$, and $k \geq 3$, let $\mathcal{T}_{k,g}$ be the **Moore tree** consisting of $M(k, g)$ vertices, a root u of degree k, all the non-leaf vertices of degree k, and all the leaves being of distance r from u; a k-tree of depth r.

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- Are these two numbers equal?

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- If one picks a vertex u in a (k, g)-graph Γ, $g = 2r + 1$, and removes all the vertices of Γ of distance larger than r from u, keeping semi-edges for those vertices that lost their neighbors, one obtains a (k, g)-multipole of order $M(k, g)$

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- The minimum number $s_{k,g}$ of semi-edges in a (k,g)-multipole of order $M(k,g)$ gives a lower bound on the order of a (k,g)-cage

$$M(k,g) + \lceil \frac{s_{k,g}}{k} \rceil \leq n(k,g)$$

A 'Much Easier' Bermond and Bollobás Question for Cages

Does there exist a $k \geq 3$ and a constant C_k such that **there exist infinitely many** $g \geq 3$ with the property that there exists a (k, g)-graph of order not exceeding the **product** $C_k M(k, g)$?

The answer to the second question should be ...

- For odd $g \geq 3$, $g = 2r + 1$, and $k \geq 3$, let $\mathcal{T}_{k,g}$ be the corresponding Moore tree consisting of $M(k, g)$ vertices, a root u of degree k, all the non-leaf vertices of degree k, and all the leaves being of distance r from u (a k-tree of depth r)

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- Take k disjoint copies of $\mathcal{T}_{k,g}$ and complete the graph into a (k, g)-graph by connecting each leave in each tree to one leaf in each other tree
- This surely should be possible, and it would give the constant $C_k = k$; and maybe one would not even need to use k trees

Or ...

- In order to get a (k, g)-graph, one could, instead of starting from $\mathcal{T}_{k,g}$, start from $\mathcal{T}_{k,g+2}$ and 'carelessly' complete the edges among the trees without forming cycles smaller than g

Or ...

- In order to get a (k, g)-graph, one could, instead of starting from $\mathcal{T}_{k,g}$, start from $\mathcal{T}_{k,g+2}$ and 'carelessly' complete the edges among the trees without forming cycles smaller than g
- That would lead to a construction of a (k, g)-graph of order $M(k, g+2)$, and since

$$\frac{M(k, g+2)}{M(k, g)} = \frac{1 + k\frac{(k-1)^{r+1}-1}{k-2}}{1 + k\frac{(k-1)^{r}-1}{k-2}} \approx (k-1),$$

it would give $C_K \approx (k-1)$

Historic Upper Bounds

Theorem (Erdős, Sachs, 1963)

For every $k \geq 2$, $g \geq 3$,

$$n(k,g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t$$

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$$n(k,g) \leq 4\sum_{t=1}^{g-2}(k-1)^t$$

$$\frac{4\sum_{t=1}^{g-2}(k-1)^t}{M(k,g)} = \frac{4\sum_{t=1}^{g-2}(k-1)^t}{1+k\frac{(k-1)^{(g-1)/2}-1}{k-2}} \approx M(k,g)^2$$

Historic Upper Bounds

Theorem (Sauer 1967)

For every $k \geq 2$, $g \geq 3$,

$$n(k,g) \leq \begin{cases} 2(k-2)^{g-2}, & g \text{ odd}, \\ 4(k-1)^{g-3}, & g \text{ even}. \end{cases}$$

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Still,

$$\frac{2(k-2)^{g-2}}{M(k,g)} = \frac{2(k-2)^{g-2}}{1 + k\frac{(k-1)^{(g-1)/2}-1}{k-2}} \approx M(k,g)^2$$

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- Biggs, 1998, suggested to call an infinite family of k-regular graphs of increasing girths g_i and orders v_i a **family of large girth** if there exists a positive $\gamma > 0$ such that

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- The graphs whose existence is guaranteed by the results of Erdős, Sachs, and Sauer, have $\gamma = 1$

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- 1995 Lazebnik, Ustimenko, and Woldar, $\gamma \geq \frac{4}{3}$, for k a prime power
- the hopeless obvious constructions would yield $\gamma \approx 2$, but we have not had any improvements since 1995

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  For every even $k \geq 2$, $g \geq 3$,

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g	4	5	6	7	8	9	10	11
$n(3, g)$	6	10	14	24	30	58	70	116
$n(3, 4; g)$	7	13	18	29	39	61	82	125
$n(4, g)$	8	19	26	67	80	275	384	

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- then one would prove

$$n(k, g) \leq n(k, k+1; g) \leq n(k+1, g).$$

Degree Monotonicity via Recursive Constructions (?)

- Lemma (Eze and RJ, 2022)

  If Γ *is a bipartite* k*-regular graph of girth* 6*, then there exists a* $(k+1)$*-regular graph of girth* 6 *and order the* 3*-multiple of the order of* Γ*.*

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  - $n(k+1,6) \leq 3n(k,6)$

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A **voltage assignment** on G is any mapping α from $D(G)$ into a group Γ that satisfies the condition $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for all $e \in D(G)$.

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The **derived regular cover (lift)** of G with respect to the voltage assignment α is the graph denoted by G^α.

- $V(G^\alpha) = V(G) \times \Gamma$,
- u_g and v_f are adjacent iff $e = (u, v) \in D(G)$ and $f = g \cdot \alpha(e)$.

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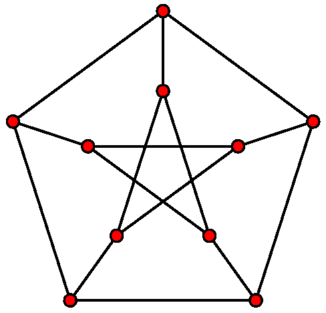
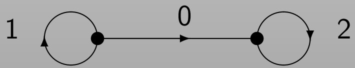
The degree of the derived regular cover of a k-regular G is k-regular.

Recursive Constructions – Voltage Graph Construction – Example



$\Uparrow$



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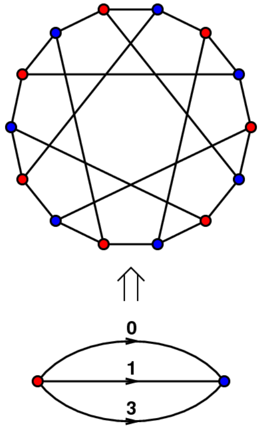

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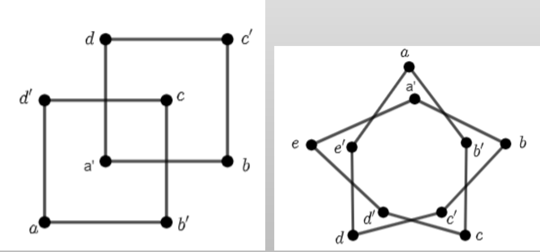

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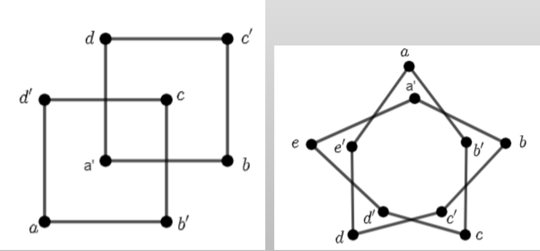

Figure: Canonical Double Covers of C_4 and C_5

- The canonical double cover of the Petersen graph of girth 5 is the Desargues graph, which has 20 vertices and girth 6.

Recursive Constructions – Canonical Double Cover

- Theorem (Erdős and Sachs, 1963)

  For every $k \geq 3$*, and* odd $g \geq 3$*,*

$$n(k, g+1) \leq 2n(k, g)$$

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- Theorem (Balbuena, González-Moreno and Montellano-Ballesteros, 2013)

Let $k \geq 2$ *and* $g \geq 5$, *with* g *odd. Then*

$$n(k, g+1) \leq \begin{cases} 2n(k,g) - 2\left(\frac{k(k-1)^{(g-3)/4}-2}{k-2}\right), & g \equiv 3 \pmod 4 \\ 2n(k,g) - 4\left(\frac{(k-1)^{(g-1)/4}-1}{k-2}\right), & \textit{otherwise}. \end{cases}$$

Recursive Constructions – Voltage Graph Construction

- Theorem (Exoo and RJ, 2011)

  Let Γ *be a base graph of girth* g*. Then there exists a voltage graph lift of* Γ *of girth at least* $2g$*.*

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$$\frac{2^{\beta(\Gamma)}|V(\Gamma)|}{M(k,2g)} \geq \frac{2^{(\frac{k}{2}-1)M(k,g)}M(k,g)}{M(k,2g)}$$

$$\approx \frac{2^{(\frac{k}{2}-1)(k-1)^{(g-1)/2}}(k-1)^{(g-1)/2}}{(k-1)^{g-1}} = \frac{2^{(\frac{k}{2}-1)(k-1)^{(g-1)/2}}}{(k-1)^{(g-1)/2}}$$

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How about the step from even to odd girth?

- **Theorem (Eze and RJ, 2022)**

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  - Proof by analysis of the asymptotics of the Moore bound

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- If a (k, g)-graph contains a perfect matching, removing it will result in a $(k-1, g')$-graph, $g' \geq g$, of the same order

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- A t-good structure in a generalized n-gon is a pair (P, L) consisting of a set of points P, and a set of lines L, subject to the condition that there are t lines in L through any point not in P, and t points in P on any line not in L.

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- Removing the points and lines of a t-good structure from the incidence graph of a generalized n-gon results in a $(q+1-t)$-regular graph of girth at least $2n$.

Recursive Degree Decreasing Constructions

Theorem (Gács and Héger, 2008)

- *For any prime power* q *and* $1 \le t \le q$*, there is a* $(q+1-t,6)$*-graph of order* $2(q^2+q+1-(tq+1))$
- $$\begin{aligned} n(q,8) &\le 2(q^3-2q), & q \text{ odd} \\ n(q,8) &\le 2(q^3-3q-2), & q \text{ even} \\ n(q,12) &\le 2(q^5-q^3) \end{aligned}$$

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- **Theorem**

  Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then

$$n(k, g) \leq 2kq^{\frac{3}{4}g-a}, \quad (2)$$

  where $a = 4,\ 11/4,\ 7/2,\ 13/4$ for $g \equiv 0, 1, 2, 3 \bmod 4$, respectively.

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- **Suggestion for further research:** Can an idea similar to the idea of a t-good structure be used with the $CD(k, q)$-graphs?

Announcement

- Together with Geoff Exoo, we are preparing the next version of the Dynamic Cage Survey
- Beside updating the tables, we also intend to add sections on
  - spectral methods
  - connections to designs and geometries
  - connectedness and cyclic connectedness
  - biregular cages and bipartite biregular cages
  - mixed cages

Please, if you have results that should be included in the Survey and have been published after 2013, send them to me or Geoff.

Call for papers for special issue of ADAM

devoted to results obtained at or related to BIRS workshop **Extremal Graphs Arising from Designs and Configurations**. Papers submitted for this special issue should be on topics presented or discussed at the workshop **Extremal Graphs Arising from Designs and Configurations**, organized in Banff, Canada, May 14 - 19, 2023.

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- The ideal length of papers is 10 to 15 pages, but longer or shorter papers will be considered. Papers that are not processed in time for the special issue may still be accepted and published in subsequent regular issues of ADAM.

Thank you.