On the spectra and spectral radii of token graphs

Cristina Dalfó
Universitat de Lleida, Igualada (Barcelona), Catalonia

joint work with
Miquel Àngel Fiol
Universitat Politècnica de Catalunya, Barcelona, Catalonia
Mónica A. Reyes
Universitat de Lleida, Igualada (Barcelona), Catalonia

International Workshop on Optimal Network Topologies
(IWONT 2023)
ICMS, Bayes Centre, Edinburgh, Scotland, 17 – 21 July 2023

Outline

1. Introduction
2. k-algebraic connectivity and k-spectral radius
3. Spectral radius of token graphs
4. Distance-regular graphs

Outline

1. Introduction
2. k-algebraic connectivity and k-spectral radius
3. Spectral radius of token graphs
4. Distance-regular graphs

Introduction: Some results on token graphs

Theorem (Audenaert, Godsil, Royle, and Rudolph, 2007)

All the strongly regular graphs with the same parameters have cospectral 2*-token graphs.*

Introduction: Some results on token graphs

Theorem (Audenaert, Godsil, Royle, and Rudolph, 2007)

All the strongly regular graphs with the same parameters have cospectral 2*-token graphs.*

Theorem (Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, Zaragoza Martínez, 2021)

For any graph G *on* n *vertices, the Laplacian spectrum of its* h*-token is contained in the Laplacian spectrum of its* k*-token for every* $1 \leq h < k \leq n/2$*:*

$$\mathrm{sp} F_h(G) \subset \mathrm{sp} F_k(G).$$

Introduction: Some results on token graphs

Theorem (Audenaert, Godsil, Royle, and Rudolph, 2007)

All the strongly regular graphs with the same parameters have cospectral 2-token graphs.

Theorem (Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, Zaragoza Martínez, 2021)

For any graph G on n vertices, the Laplacian spectrum of its h-token is contained in the Laplacian spectrum of its k-token for every $1 \leq h < k \leq n/2$:

$$\mathrm{sp} F_h(G) \subset \mathrm{sp} F_k(G).$$

Theorem (Lew, 2023)

Let G have Laplacian eigenvalues $\lambda_1 (= 0) < \lambda_2 \leq \cdots \leq \lambda_n$. Let λ be an eigenvalues of $F_k(G)$ not in $F_{k-1}(G)$. Then,

$$k(\lambda_2 - k + 1) \leq \lambda \leq k\lambda_n.$$

Introduction: Spectral radius

- The **index** or **spectral radius** of a graph is the **largest eigenvalue** of its adjacency matrix. In the case of the Laplacian matrix, it is called **Laplacian spectral radius**.

Introduction: Spectral radius

- The **index** or **spectral radius** of a graph is the **largest eigenvalue** of its adjacency matrix. In the case of the Laplacian matrix, it is called **Laplacian spectral radius**.
- It has special relevance in the study of the: diameter, radius, domination number, matching number, clique number, independence number, chromatic number, or the sequence of vertex degrees.

Introduction: Spectral radius

- The **index** or **spectral radius** of a graph is the **largest eigenvalue** of its adjacency matrix. In the case of the Laplacian matrix, it is called **Laplacian spectral radius**.
- It has special relevance in the study of the: diameter, radius, domination number, matching number, clique number, independence number, chromatic number, or the sequence of vertex degrees.
- This leads to studying the structure of graphs having an extremal spectral radius and fixed values of some of such parameters.

Introduction: Notation

- G: a (simple and connected) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$.

Introduction: Notation

- G: a (simple and connected) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$.
- The **spectrum of the adjacency matrix $\boldsymbol{A}$** is $\operatorname{sp} G \equiv \operatorname{sp} \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$, where $\theta_0 > \theta_1 > \cdots > \theta_d$. By the Perron-Frobenius theorem, G has spectral radius $\rho(G) = \theta_0$.

Introduction: Notation

- G: a (simple and connected) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$.
- The **spectrum of the adjacency matrix $\boldsymbol{A}$** is $\operatorname{sp} G \equiv \operatorname{sp} \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$, where $\theta_0 > \theta_1 > \cdots > \theta_d$.
  By the Perron-Frobenius theorem, G has spectral radius $\rho(G) = \theta_0$.
- The **spectrum of the Laplacian matrix $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$** is $\lambda_1 (= 0) < \lambda_2 \leq \cdots \leq \lambda_n$.
  λ_2 is the algebraic connectivity.

Introduction: Local spectrum of a vertex

- Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.

Introduction: Local spectrum of a vertex

- Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.
- If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, for $i = 0, 1, \ldots, d$, is the (principal) **idempotent** of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$.

Introduction: Local spectrum of a vertex

- Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.
- If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, for $i = 0, 1, \ldots, d$, is the (principal) **idempotent** of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$.
- The $(u$-$)$**local multiplicities** of the eigenvalue θ_i are defined as $m_u(\theta_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu}$ for $u \in V$ and $i = 0, 1, \ldots, d$.

Introduction: Local spectrum of a vertex

- Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.
- If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, for $i = 0, 1, \ldots, d$, is the (principal) **idempotent** of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$.
- The (u-)**local multiplicities** of the eigenvalue θ_i are defined as $m_u(\theta_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu}$ for $u \in V$ and $i = 0, 1, \ldots, d$.
- $\displaystyle\sum_{i=0}^{d} m_u(\theta_i) = 1$ and $\displaystyle\sum_{u \in V} m_u(\theta_i) = m_i$, for $i = 0, 1, \ldots, d$. The number $a_{uu}^{(\ell)}$ of **closed walks of length ℓ rooted at vertex u** can be computed as $\displaystyle a_{uu}^{(\ell)} = \sum_{i=0}^{d} m_u(\theta_i)\theta_i^\ell$ (Fiol and Garriga, 1997).

Introduction: Local spectrum of a vertex

- Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$.
- If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, for $i = 0, 1, \ldots, d$, is the (principal) **idempotent** of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$.
- The $(u$-)**local multiplicities** of the eigenvalue θ_i are defined as $m_u(\theta_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu}$ for $u \in V$ and $i = 0, 1, \ldots, d$.
- $\displaystyle\sum_{i=0}^{d} m_u(\theta_i) = 1$ and $\displaystyle\sum_{u \in V} m_u(\theta_i) = m_i$, for $i = 0, 1, \ldots, d$. The number $a_{uu}^{(\ell)}$ of **closed walks of length ℓ rooted at vertex u** can be computed as $a_{uu}^{(\ell)} = \displaystyle\sum_{i=0}^{d} m_u(\theta_i)\theta_i^\ell$ (Fiol and Garriga, 1997).
- By picking up the eigenvalues with non-null local multiplicities, $\mu_0 (= \theta_0) > \mu_1 > \cdots > \mu_{d_u}$, the (u)-**local spectrum** of G is $\mathrm{sp}_u\, G := \{\mu_0^{m_u(\mu_0)}, \mu_1^{m_u(\mu_1)}, \ldots, \mu_{d_u}^{m_u(\mu_{d_u})}\}$.

First result

Lemma

Let G be a finite graph with different eigenvalues $\theta_0 > \cdots > \theta_d$. Let $w_u^{(\ell)}$ be the number of ℓ-walks starting from (any fixed) vertex u, and let $w_{uu}^{(\ell)}$ be the number of closed ℓ-walks rooted at u. Then,

$$\rho(G) = \lim_{\ell\to\infty} \sqrt[\ell]{w_u^{(\ell)}} = \lim_{\ell\to\infty} \sup \sqrt[\ell]{w_{uu}^{(\ell)}},$$

where '$\sup$' denotes the supremum.

Introduction: Regular o equitable partitions

- A **partition** π of the vertex set V into r cells $C_1, C_2, \ldots, C_r$ is called **regular** or **equitable** whenever, for any $i, j = 1, \ldots, r$, the **intersection numbers** $b_{ij}(u) = |G(u) \cap C_j|$, where $u \in V_i$, do not depend on the vertex u but only on the cells C_i and C_j. In this case, such numbers are simply written as b_{ij}, and the $r \times r$ matrix $\boldsymbol{Q}_A = \boldsymbol{A}(G/\pi)$ is called **quotient matrix** with entries $(\boldsymbol{Q}_A)_{ij} = b_{ij}$.

Introduction: Regular o equitable partitions

- A **partition** π of the vertex set V into r cells $C_1, C_2, \ldots, C_r$ is called **regular** or **equitable** whenever, for any $i, j = 1, \ldots, r$, the **intersection numbers** $b_{ij}(u) = |G(u) \cap C_j|$, where $u \in V_i$, do not depend on the vertex u but only on the cells C_i and C_j. In this case, such numbers are simply written as b_{ij}, and the $r \times r$ matrix $\boldsymbol{Q}_A = \boldsymbol{A}(G/\pi)$ is called **quotient matrix** with entries $(\boldsymbol{Q}_A)_{ij} = b_{ij}$.
- With the Laplacian matrix, we get the **quotient Laplacian matrix** $\boldsymbol{Q}_L = \boldsymbol{L}(G/\pi)$ with entries

$$(\boldsymbol{Q}_L)_{ij} = \begin{cases} -b_{ij} & \text{if } i \neq j, \\ b_{ii} - \sum_{j=1}^{r} b_{ij} & \text{if } i = j, \end{cases}$$

Introduction: Walk-regular graphs

- $a_u^{(\ell)}$ is the **number of closed walks of length ℓ** rooted at vertex u.

Introduction: Walk-regular graphs

- $a_u^{(\ell)}$ is the **number of closed walks of length** ℓ rooted at vertex u.
- If $a_u^{(\ell)}$ only depend on ℓ, for each $\ell \geq 0$, then G is called **walk-regular** (Godsil and McKay, 1980).

Introduction: Walk-regular graphs

- $a_u^{(\ell)}$ is the **number of closed walks of length** ℓ rooted at vertex u.
- If $a_u^{(\ell)}$ only depend on ℓ, for each $\ell \geq 0$, then G is called **walk-regular** (Godsil and McKay, 1980).
- Since $a_u^{(2)} = \delta_u$, the degree of vertex u, a **walk-regular graph** is necessarily **regular**.

Introduction: Walk-regular graphs

- $a_u^{(\ell)}$ is the **number of closed walks of length ℓ** rooted at vertex u.
- If $a_u^{(\ell)}$ only depend on ℓ, for each $\ell \geq 0$, then G is called **walk-regular** (Godsil and McKay, 1980).
- Since $a_u^{(2)} = \delta_u$, the degree of vertex u, a **walk-regular graph** is necessarily **regular**.
- A graph G is called **spectrally regular** when all vertices have the same local spectrum: $\mathrm{sp}_u\, G = \mathrm{sp}_v\, G$ for any $u, v \in V$.

Introduction: Walk-regular graphs

- $a_u^{(\ell)}$ is the **number of closed walks of length ℓ** rooted at vertex u.
- If $a_u^{(\ell)}$ only depend on ℓ, for each $\ell \geq 0$, then G is called **walk-regular** (Godsil and McKay, 1980).
- Since $a_u^{(2)} = \delta_u$, the degree of vertex u, a **walk-regular graph** is necessarily **regular**.
- A graph G is called **spectrally regular** when all vertices have the same local spectrum: $\mathrm{sp}_u\, G = \mathrm{sp}_v\, G$ for any $u, v \in V$.

Lemma (Delorme and Tillich (1997), Fiol and Garriga (1998), Godsil and McKay (1980))

Let $G = (V, E)$ be a graph. The following statements are equivalent.

(i) *G is **walk-regular**.*

(ii) *G is **spectrally regular**.*

(iii) *The spectra of the vertex-deleted subgraphs are all equal: $\mathrm{sp}\,(G \setminus u) = \mathrm{sp}\,(G \setminus v)$ for any $u, v \in V$.*

Outline

k-algebraic connectivity and k-spectral radius

- We consider the Laplacian spectrum. Let G be a graph on n vertices, and $F_k(G)$ its k-token graph for $k \in \{0, 1, \ldots, n\}$. Recall that $F_k(G) \cong F_{n-k}(G)$ where, by convenience, $F_0(G) \cong F_n(G) = K_1$ (a singleton). Moreover, $F_1(G) \cong G$.

k-algebraic connectivity and k-spectral radius

- We consider the Laplacian spectrum. Let G be a graph on n vertices, and $F_k(G)$ its k-token graph for $k \in \{0, 1, \ldots, n\}$. Recall that $F_k(G) \cong F_{n-k}(G)$ where, by convenience, $F_0(G) \cong F_n(G) = K_1$ (a singleton). Moreover, $F_1(G) \cong G$.
- From DDFFHTZ (2021), it is known that the Laplacian spectra of the token graphs of G satisfy $\{0\} = \operatorname{sp} F_0(G) \subset \operatorname{sp} F_1(G) \subset \operatorname{sp} F_2(G) \subset \cdots \subset F_{\lfloor n/2 \rfloor}(G)$.

k-algebraic connectivity and k-spectral radius

- We consider the Laplacian spectrum. Let G be a graph on n vertices, and $F_k(G)$ its k-token graph for $k \in \{0, 1, \ldots, n\}$. Recall that $F_k(G) \cong F_{n-k}(G)$ where, by convenience, $F_0(G) \cong F_n(G) = K_1$ (a singleton). Moreover, $F_1(G) \cong G$.
- From DDFFHTZ (2021), it is known that the Laplacian spectra of the token graphs of G satisfy $\{0\} = \operatorname{sp} F_0(G) \subset \operatorname{sp} F_1(G) \subset \operatorname{sp} F_2(G) \subset \cdots \subset F_{\lfloor n/2 \rfloor}(G)$.
- Let denote $\alpha(G)$ and $\rho(G)$ the **algebraic connectivity** and the **spectral radius** of a graph G, respectively. Then, we have

$$\alpha(G) \geq \alpha(F_2(G)) \geq \cdots \geq \alpha(F_{\lfloor n/2 \rfloor}(G)),$$
$$\rho(G) \leq \rho(F_2(G)) \leq \cdots \leq \rho(F_{\lfloor n/2 \rfloor}(G)).$$

k-algebraic connectivity and k-spectral radius

- We consider the Laplacian spectrum. Let G be a graph on n vertices, and $F_k(G)$ its k-token graph for $k \in \{0, 1, \ldots, n\}$. Recall that $F_k(G) \cong F_{n-k}(G)$ where, by convenience, $F_0(G) \cong F_n(G) = K_1$ (a singleton). Moreover, $F_1(G) \cong G$.
- From DDFFHTZ (2021), it is known that the Laplacian spectra of the token graphs of G satisfy $\{0\} = \operatorname{sp} F_0(G) \subset \operatorname{sp} F_1(G) \subset \operatorname{sp} F_2(G) \subset \cdots \subset F_{\lfloor n/2 \rfloor}(G)$.
- Let denote $\alpha(G)$ and $\rho(G)$ the **algebraic connectivity** and the **spectral radius** of a graph G, respectively. Then, we have

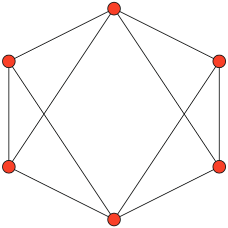
$$\alpha(G) \geq \alpha(F_2(G)) \geq \cdots \geq \alpha(F_{\lfloor n/2 \rfloor}(G)),$$
$$\rho(G) \leq \rho(F_2(G)) \leq \cdots \leq \rho(F_{\lfloor n/2 \rfloor}(G)).$$

Definition

Given a graph G on n vertices and an integer k such that $1 \leq k \leq \lfloor n/2 \rfloor$, the **$k$-algebraic connectivity** $\alpha_k = \alpha_k(G)$ and the **k-spectral radius** $\rho_k = \rho_k(G)$ of G are, respectively, the minimum and maximum eigenvalues of the multiset $\operatorname{sp} F_k(G) \setminus \operatorname{sp} F_{k-1}(G)$.

k-algebraic connectivity and k-spectral radius

Example



G

Spectrum	ev G
$\mathrm{sp}(F_0) = \mathrm{sp}(K_1)$	0
$\mathrm{sp}(F_1) = \mathrm{sp}(G)$	2$= \alpha_1 = \alpha$ 4 4 4 6$= \rho_1$
$\mathrm{sp}(F_2) - \mathrm{sp}(F_1)$	4$= \alpha_2$ 4 6 6 6 8 8 8 10$= \rho_2$
$\mathrm{sp}(F_3) - \mathrm{sp}(F_2)$	4$= \alpha_3$ 8 8 10 10$= \rho_3 = \rho$

k-algebraic connectivity and k-spectral radius: Some facts

(i) $\rho_k(G) \geq \alpha_k(G) \geq 0$.

(ii) $\alpha_1(G) = \alpha(G)$ (the standard algebraic connectivity of G) and $\rho_{\lfloor n/2 \rfloor}(G) = \rho(G)$ (the standard spectral radius of G).

(iii) Since $F_k(K_n) \cong J(n,k)$ (the Johnson graph), we have

$$\alpha_k(K_n) = \rho_k(K_n) = k(n+1-k), \qquad k = 1, \ldots, \lfloor n/2 \rfloor.$$

In particular, $\alpha_1(K_n) = \rho_1(K_n) = n$, $\alpha_2(K_n) = \rho_2(K_n) = 2(n-1)$, and so on.

k-algebraic connectivity and k-spectral radius: Conjectures

Conjecture

For any graph G, $\alpha_1(G) \le \alpha_2(G) \le \cdots \le \alpha_{\lfloor n/2 \rfloor}(G)$.

k-algebraic connectivity and k-spectral radius: Conjectures

Conjecture

For any graph G, $\alpha_1(G) \leq \alpha_2(G) \leq \cdots \leq \alpha_{\lfloor n/2 \rfloor}(G)$.

Conjecture

For any graph G, $\rho_1(G) \leq \rho_2(G) \leq \cdots \leq \rho_{\lfloor n/2 \rfloor}(G)$.

k-algebraic connectivity and k-spectral radius: Conjectures

Conjecture

For any graph G, $\alpha_1(G) \leq \alpha_2(G) \leq \cdots \leq \alpha_{\lfloor n/2 \rfloor}(G)$.

Conjecture

For any graph G, $\rho_1(G) \leq \rho_2(G) \leq \cdots \leq \rho_{\lfloor n/2 \rfloor}(G)$.

Conjecture

If $\rho_1(G) \leq \rho_2(G) \leq \cdots \leq \rho_{\lfloor n/2 \rfloor}(G)$, *then* $\rho_k(G) = \rho(F_k(G))$ *for any* $k \leq \lfloor n/2 \rfloor$.

k-algebraic connectivity and k-spectral radius: Results

Lemma

For any graph G and its complementary graph $\overline{G}$, the k-algebraic connectivity and k-spectral radius of $\overline{G}$ satisfy

$$\alpha_k(G) + \rho_k(\overline{G}) = k(n - k + 1).$$

For $k = 1$: $\alpha(G) + \rho(\overline{G}) = n$.

k-algebraic connectivity and k-spectral radius: Results

Lemma

For any graph G and its complementary graph $\overline{G}$, the k-algebraic connectivity and k-spectral radius of $\overline{G}$ satisfy

$$\alpha_k(G) + \rho_k(\overline{G}) = k(n-k+1).$$

For $k=1$: $\alpha(G) + \rho(\overline{G}) = n$.

Corollary

For any graph G on n vertices, for $k = 1, \ldots, \lfloor n/2 \rfloor$,

$$\alpha_k(G) \leq k(n-k+1), \qquad \rho_k(\overline{G}) \leq k(n-k+1).$$

k-algebraic connectivity and k-spectral radius: Results

Lemma

For any graph G and its complementary graph $\overline{G}$, the k-algebraic connectivity and k-spectral radius of $\overline{G}$ satisfy

$$\alpha_k(G) + \rho_k(\overline{G}) = k(n-k+1).$$

For $k=1$: $\alpha(G) + \rho(\overline{G}) = n$.

Corollary

For any graph G on n vertices, for $k = 1, \ldots, \lfloor n/2 \rfloor$,

$$\alpha_k(G) \leq k(n-k+1), \qquad \rho_k(\overline{G}) \leq k(n-k+1).$$

Corollary

Let G be a bipartite distance-regular graph. Let $\boldsymbol{L}(F_2/\pi)$ be the quotient matrix with spectral radius $\rho_L(F_2/\pi)$. Then,

$$\alpha_2(\overline{G}) = \binom{n}{2} - \rho_L(F_2/\pi).$$

Outline

1. Introduction
2. k-algebraic connectivity and k-spectral radius
3. Spectral radius of token graphs
4. Distance-regular graphs

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.
- By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the two following parameters:

$$\rho^k_M(G) = \max\{\rho(G \setminus U) : U \subset V,\ |U| = k\},$$
$$\rho^k_m(G) = \min\{\rho(G \setminus U) : U \subset V,\ |U| = k\}.$$

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.
- By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the two following parameters:

$$\rho_M^k(G) = \max\{\rho(G \setminus U) : U \subset V,\ |U| = k\},$$
$$\rho_m^k(G) = \min\{\rho(G \setminus U) : U \subset V,\ |U| = k\}.$$

- If G is **walk-regular**, then $\rho_M^1(G) = \rho_m^1(G) = \rho(G \setminus u)$ for every vertex u.

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.
- By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the two following parameters:

$$\rho_M^k(G) = \max\{\rho(G \setminus U) : U \subset V,\ |U| = k\},$$
$$\rho_m^k(G) = \min\{\rho(G \setminus U) : U \subset V,\ |U| = k\}.$$

- If G is **walk-regular**, then $\rho_M^1(G) = \rho_m^1(G) = \rho(G \setminus u)$ for every vertex u.
- If G is **distance-regular** with degree δ, then $\kappa(G) = \delta$ (Brouwer and Koolen, 2009).

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.
- By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the two following parameters:

$$\rho^k_M(G) = \max\{\rho(G \setminus U) : U \subset V,\ |U| = k\},$$
$$\rho^k_m(G) = \min\{\rho(G \setminus U) : U \subset V,\ |U| = k\}.$$

- If G is **walk-regular**, then $\rho^1_M(G) = \rho^1_m(G) = \rho(G \setminus u)$ for every vertex u.
- If G is **distance-regular** with degree δ, then $\kappa(G) = \delta$ (Brouwer and Koolen, 2009).
- D., Van Dam, and Fiol (2011) showed that $\mathrm{sp}(G \setminus U)$ only depends on the **distances** in G between the vertices of U.

Spectral radius of token graphs

- We consider the spectral radius of the adjacency matrix of a (connected) graph G with spectral radius $\rho(G)$ and vertex-connectivity κ.
- By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the two following parameters:

$$\rho_M^k(G) = \max\{\rho(G \setminus U) : U \subset V,\ |U| = k\},$$
$$\rho_m^k(G) = \min\{\rho(G \setminus U) : U \subset V,\ |U| = k\}.$$

- If G is **walk-regular**, then $\rho_M^1(G) = \rho_m^1(G) = \rho(G \setminus u)$ for every vertex u.
- If G is **distance-regular** with degree δ, then $\kappa(G) = \delta$ (Brouwer and Koolen, 2009).
- D., Van Dam, and Fiol (2011) showed that $\mathrm{sp}(G \setminus U)$ only depends on the **distances** in G between the vertices of U.
- For every $k \leq \delta - 1$, the computation of $\rho_M^k(G)$ and $\rho_m^k(G)$ can be drastically reduced by considering only the subsets U with different **distance-pattern** between vertices.

Spectral radius of token graphs

- For instance, if G has diameter D,

$$\rho_M^2(G) = \max_{1\le \ell \le D} \{\rho(G \setminus \{u,v\}) : \mathrm{dist}_G(u,v) = \ell\},$$
$$\rho_m^2(G) = \min_{1\le \ell \le D} \{\rho(G \setminus \{u,v\}) : \mathrm{dist}_G(u,v) = \ell\}.$$

Spectral radius of token graphs

- For instance, if G has diameter D,

$$\rho_M^2(G) = \max_{1\le \ell\le D}\{\rho(G\setminus\{u,v\}) : \mathrm{dist}_G(u,v)=\ell\},$$
$$\rho_m^2(G) = \min_{1\le \ell\le D}\{\rho(G\setminus\{u,v\}) : \mathrm{dist}_G(u,v)=\ell\}.$$

Lemma

Let G be a graph with n vertices, vertex-connectivity κ, and eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then, for every $k = 1, \ldots, \kappa - 1$,

$$\lambda_{k+1} \le \rho_M^k(G) \le \lambda_1,$$
$$\lambda_n \le \rho_m^k(G) \le \lambda_{n-k}.$$

Proof. By using interlacing (Haemers (1995) or Fiol (1999)).

Spectral radius of token graphs: Main result

Theorem

Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \leq k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ be the maximum and minimum of the spectral radii of the U-deleted subgraphs of G, where $|U| = k$.

Spectral radius of token graphs: Main result

Theorem

Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \le k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ be the maximum and minimum of the spectral radii of the U-deleted subgraphs of G, where $|U| = k$.

(i) *The spectral radius of the k-token graph $F_k(G)$ satisfies*

$$k\rho_m^{k-1}(G) \le \rho(F_k(G)) \le k\rho_M^{k-1}(G).$$

Spectral radius of token graphs: Main result

Theorem

Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \leq k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ be the maximum and minimum of the spectral radii of the U-deleted subgraphs of G, where $|U| = k$.

(i) *The spectral radius of the k-token graph $F_k(G)$ satisfies*

$$k\rho_m^{k-1}(G) \leq \rho(F_k(G)) \leq k\rho_M^{k-1}(G).$$

(ii) *If G is a graph of order n and diameter D, the spectral radius of the k-token graph $F_k(G)$ satisfies*

$$\rho(F_k(G)) < k\left(\rho(G) - \frac{1}{n\rho(G)^{2D}}\right).$$

Spectral radius of token graphs: Main result

Theorem

Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \le k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ be the maximum and minimum of the spectral radii of the U-deleted subgraphs of G, where $|U| = k$.

(i) *The spectral radius of the k-token graph $F_k(G)$ satisfies*

$$k\rho_m^{k-1}(G) \le \rho(F_k(G)) \le k\rho_M^{k-1}(G).$$

(ii) *If G is a graph of order n and diameter D, the spectral radius of the k-token graph $F_k(G)$ satisfies*

$$\rho(F_k(G)) < k\left(\rho(G) - \frac{1}{n\rho(G)^{2D}}\right).$$

(iii) *If G is walk-regular and $k = 2$ ($F_2(G)$ is the 2-token graph of G), then*

$$\rho(F_2(G)) = 2\rho_m^1(G) = 2\rho_M^1(G).$$

Spectral radius of token graphs: Consequences

- Eigenvalues of P_n: $\theta_i = 2\cos\left(\frac{i\pi}{n+1}\right)$ for $i = 1, \ldots, n$.

Spectral radius of token graphs: Consequences

- Eigenvalues of P_n: $\theta_i = 2\cos\left(\frac{i\pi}{n+1}\right)$ for $i = 1, \ldots, n$.
- Spectral radius of $K_{m,n}$: $\rho(K_{m,n}) = \sqrt{mn}$.

Spectral radius of token graphs: Consequences

- Eigenvalues of P_n: $\theta_i = 2\cos\left(\frac{i\pi}{n+1}\right)$ for $i = 1, \ldots, n$.
- Spectral radius of $K_{m,n}$: $\rho(K_{m,n}) = \sqrt{mn}$.

Corollary

Let P_n and C_n be the **path** *and* **cycle** *graphs on n vertices. Let P_∞ and C_∞ be the infinite path and cycle graphs.*

(i) $\rho(F_2(P_n)) \leq 4\cos(\pi/n)$ *and* $\rho(F_2(P_\infty)) = 4$,

(ii) $\rho(F_2(C_n)) = 4\cos(\pi/n)$ *and* $\rho(F_2(C_\infty)) = 4$,

(iii) $\rho(F_2(K_{n,n})) = 2\sqrt{n(n-1)}$.

n	3	4	$\cdots$	8	9	10	11
$\rho(P_{n-1})$	1	1.41421	$\cdots$	1.84776	1.87938	1.92113	1.91898
$\rho(F_2(C_n))$	2	2.82842	$\cdots$	3.69552	3.75877	3.84226	3.83796

Table: Spectral radii of the 2-token graphs of the cycles C_n with respect to spectral radii of the paths graphs P_{n-1}.

Outline

Distance-regular graphs

- We consider both the adjacency and Laplacian spectra, with their respective spectral radii ρ_A and ρ_L.

Distance-regular graphs

- We consider both the adjacency and Laplacian spectra, with their respective spectral radii ρ_A and ρ_L.
- G is a **distance-regular** graph with degree $\delta = b_0$, diameter d, intersection array

$$\iota(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$

or intersection matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix},$$

where $a_i = \delta - b_i - c_i$, for $i = 1, \ldots, d$.

Distance-regular graphs

Lemma

Let $F_2(G)$ be the 2-token graph of a distance-regular graph G with degree $\delta = b_0$, diameter d, and intersection array $\iota(G)$. Then, $F_2 = F_2(G)$ has a regular partition π with **quotient matrix** *and* **quotient Laplacian matrix**

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} a_1 & c_2 & & & \\ b_1 & a_2 & c_3 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix},$$

$$\boldsymbol{L}(F_2/\pi) = 2\begin{pmatrix} c_2 & -c_2 & & & \\ -b_1 & b_1 + c_3 & -c_3 & & \\ & -b_2 & b_2 + c_4 & \ddots & \\ & & \ddots & \ddots & -c_d \\ & & & -b_{d-1} & b_{d-1} \end{pmatrix},$$

where $c_i + a_i + b_i = \delta$, for $i = 0, 1, \ldots, d$.

Distance-regular graphs

Proposition

Let G be a distance-regular graph with adjacency and Laplacian matrices $\boldsymbol{A}$ and $\boldsymbol{L}$. Let $F_2(G)$ be its 2-token graph with adjacency and Laplacian matrices $\boldsymbol{A}(F_2)$ and $\boldsymbol{L}(F_2)$ with respective spectral radii $\rho_A(F_2)$ and $\rho_L(F_2)$. Let $\boldsymbol{A}(F_2/\pi)$ and $\boldsymbol{L}(F_2/\pi)$ be the quotient matrices with respective spectral radii $\rho_A(F_2/\pi)$ and $\rho_L(F_2/\pi)$. Then, the following holds:

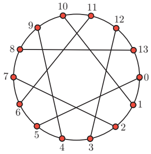
(a) $\rho_A(F_2) = \rho_A(F_2/\pi)$.

(b) $\rho_L(F_2) \geq \rho_L(F_2/\pi)$, *with equality if G is bipartite.*

Distance-regular graphs

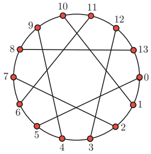
Example (Heawood graph)

- H is a **bipartite distance-regular** graph with $n = 14$ vertices, diameter 3, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$.



Distance-regular graphs

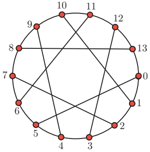
Example (Heawood graph)



- H is a **bipartite distance-regular** graph with $n = 14$ vertices, diameter 3, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$.
- The Laplacian spectral radius of H is $\rho_L(H) = 6$, and the algebraic connectivity of $\overline{H}$ is $\alpha_1(\overline{H}) = n - \rho(H) = 8$.

Distance-regular graphs

Example (Heawood graph)

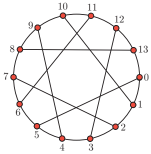


- H is a **bipartite distance-regular** graph with $n = 14$ vertices, diameter 3, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$.
- The Laplacian spectral radius of H is $\rho_L(H) = 6$, and the algebraic connectivity of $\overline{H}$ is $\alpha_1(\overline{H}) = n - \rho(H) = 8$.
- By the last proposition, the 2-token graph $F_2 = F_2(H)$ has a regular partition π with quotient and quotient Laplacian matrices

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \boldsymbol{L}(F_2/\pi) = 2\begin{pmatrix} 1 & -1 & 0 \\ -2 & 5 & -3 \\ 0 & -2 & 2 \end{pmatrix}.$$

Distance-regular graphs

Example (Heawood graph)



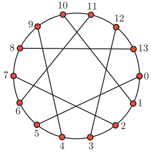
- H is a **bipartite distance-regular** graph with $n = 14$ vertices, diameter 3, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$.
- The Laplacian spectral radius of H is $\rho_L(H) = 6$, and the algebraic connectivity of $\overline{H}$ is $\alpha_1(\overline{H}) = n - \rho(H) = 8$.
- By the last proposition, the 2-token graph $F_2 = F_2(H)$ has a regular partition π with quotient and quotient Laplacian matrices

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \boldsymbol{L}(F_2/\pi) = 2\begin{pmatrix} 1 & -1 & 0 \\ -2 & 5 & -3 \\ 0 & -2 & 2 \end{pmatrix}.$$

- ev$\boldsymbol{A}(F_2/\pi) = 0, \pm 4\sqrt{2}$, ev$\boldsymbol{L}(F_2/\pi) = 0, 8 \pm 2\sqrt{7}$. Thus, $\rho_A(F_2(H)) = 4\sqrt{2}$ and $\rho_2(H) = \rho_L(F_2(H)) = 8 + 2\sqrt{7}$.

Distance-regular graphs

Example (Heawood graph)



- H is a **bipartite distance-regular** graph with $n = 14$ vertices, diameter 3, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$.
- The Laplacian spectral radius of H is $\rho_L(H) = 6$, and the algebraic connectivity of $\overline{H}$ is $\alpha_1(\overline{H}) = n - \rho(H) = 8$.
- By the last proposition, the 2-token graph $F_2 = F_2(H)$ has a regular partition π with quotient and quotient Laplacian matrices

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \boldsymbol{L}(F_2/\pi) = 2\begin{pmatrix} 1 & -1 & 0 \\ -2 & 5 & -3 \\ 0 & -2 & 2 \end{pmatrix}.$$

- ev$\boldsymbol{A}(F_2/\pi) = 0, \pm 4\sqrt{2}$, ev$\boldsymbol{L}(F_2/\pi) = 0, 8 \pm 2\sqrt{7}$. Thus, $\rho_A(F_2(H)) = 4\sqrt{2}$ and $\rho_2(H) = \rho_L(F_2(H)) = 8 + 2\sqrt{7}$.
- $\alpha_2(\overline{H}) = 2(n-1) - \rho_2(H) = 18 - 2\sqrt{7} > 8 = \alpha_1(\overline{H})$. Since the algebraic connectivity of $F_2(\overline{H})$ also is 8, $\alpha_1(F_2(\overline{H})) = \alpha_1(\overline{H})$, as expected.

Distance-regular graphs

Corollary

Let $\mathcal{F}$ be the family of all **distance-regular graphs** *with diameter d and the same parameters (or intersection array). Then, every graph $G \in \mathcal{F}$ has 2-token graph F_2 with the d (adjacency or Laplacian) eigenvalues of $\boldsymbol{A}(F_2/\pi)$ or $\boldsymbol{L}(F_2/\pi)$ given as before. In particular, F_2 has spectral radii $\rho_A(F_2) = \rho_A(F_2/\pi)$, and $\rho_L(F_2) = \rho_L(F_2/\pi)$ if it is bipartite.*

Distance-regular graphs

Corollary

Let $\mathcal{F}$ be the family of all **distance-regular graphs** *with diameter d and the same parameters (or intersection array). Then, every graph $G \in \mathcal{F}$ has 2-token graph F_2 with the d (adjacency or Laplacian) eigenvalues of $\boldsymbol{A}(F_2/\pi)$ or $\boldsymbol{L}(F_2/\pi)$ given as before. In particular, F_2 has spectral radii $\rho_A(F_2) = \rho_A(F_2/\pi)$, and $\rho_L(F_2) = \rho_L(F_2/\pi)$ if it is bipartite.*

Corollary

Let G be a distance-regular graph with (adjacency) eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then, 2-token graph $F_2(G)$ has some eigenvalues $\mu_0 > \mu_1 > \cdots > \mu_{d-1}$ satisfying

$$2\theta_{i+1} \leq \mu_i \leq 2\theta_i, \qquad i = 0, \ldots, d-1.$$

Distance-regular graphs

Open problem:
All the **strongly regular graphs** with the same parameters are **cospectral**. Does the same happen with all **distance-regular graphs** with the same parameters (with respect to the adjacency or Laplacian matrix)???

IWONT 2011

IWONT 2016

References

K. Audenaert, C. Godsil, G. Royle, and T. Rudolph, Symmetric squares of graphs, *J. Combin. Theory B* **97** (2007) 74–90.

C. Dalfó, E. R. van Dam, and M. A. Fiol, On perturbations of almost distance-regular graphs, *Linear Algebra Appl.* **435** (2011), no. 10, 2626–2638.

C. Dalfó, F. Duque, R. Fabila-Monroy, M. A. Fiol, C. Huemer, A. L. Trujillo-Negrete, and F. J. Zaragoza Martínez, On the Laplacian spectra of token graphs, *Linear Algebra Appl.* **625** (2021) 322–348.

C. Dalfó, M. A. Fiol, E. Garriga, A differential approach for bounding the index of graphs under perturbations, *Electron. J. Combin.* **18** (2011), no. 1, $P172$.

R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, Token graphs, *Graphs Combin.* **28** (2012), no. 3, 365–380.

C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995) 593–616.

A. Lew, Garlands method for token graphs, `https://arxiv.org/abs/2305 .02406v1`, 2023.

V. Nikiforov, The spectral radius of subgraphs of regular graphs, *Electron. J. Combin.* **14** (2007) #N20.

V. Nikiforov, Revisiting two classical results on graph spectra, *Electron. J. Combin.* **14** (2007) #R14.

References

K. Audenaert, C. Godsil, G. Royle, and T. Rudolph, Symmetric squares of graphs, *J. Combin. Theory B* **97** (2007) 74–90.

C. Dalfó, E. R. van Dam, and M. A. Fiol, On perturbations of almost distance-regular graphs, *Linear Algebra Appl.* **435** (2011), no. 10, 2626–2638.

C. Dalfó, F. Duque, R. Fabila-Monroy, M. A. Fiol, C. Huemer, A. L. Trujillo-Negrete, and F. J. Zaragoza Martínez, On the Laplacian spectra of token graphs, *Linear Algebra Appl.* **625** (2021) 322–348.

C. Dalfó, M. A. Fiol, E. Garriga, A differential approach for bounding the index of graphs under perturbations, *Electron. J. Combin.* **18** (2011), no. 1, $P172$.

R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, Token graphs, *Graphs Combin.* **28** (2012), no. 3, 365–380.

C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995) 593–616.

A. Lew, Garlands method for token graphs, `https://arxiv.org/abs/2305 .02406v1`, 2023.

V. Nikiforov, The spectral radius of subgraphs of regular graphs, *Electron. J. Combin.* **14** (2007) #N20.

V. Nikiforov, Revisiting two classical results on graph spectra, *Electron. J. Combin.* **14** (2007) #R14.

TAPADH LEAT! THANK YOU! GRÀCIES!