Cyclic connectivity of cages

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IWONT 2023, Edinburgh

Cage problem

- (k, g)*-graph* = a k-regular graph with girth g
- (k, g)*-cage* = a smallest (k, g)-graph
- $n(k, g)$ = size of a (k, g)-cage
- $n(k, g) \geq M(k, g)$ (Moore bound)

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Partial progress:

- $(\lfloor k/2 \rfloor + 1)$-connected for odd $g \geq 7$ [Balbuena, Salas, 2012]
- $(r + 1)$-connected for each even g and $r^3 + 2r^2 \leq k$ [Lin et al., '08]

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Theorem (Wang et al. 2003 + Lin et al. 2005)

Each (k, g)-cage is k-edge-connected.

Limits of connectivity

- Problem: k-regular graph is at most k-(edge-)connected

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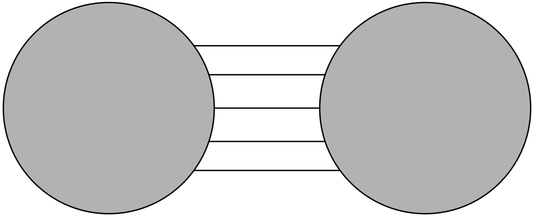
- Does a $(3, 47)$-cage look like this?



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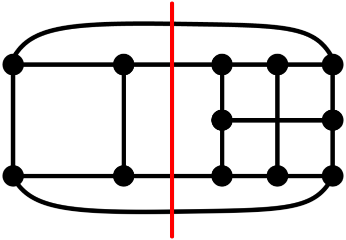

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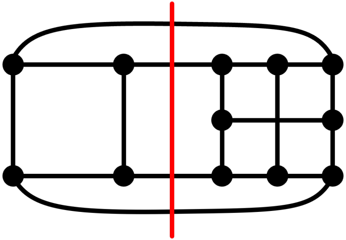

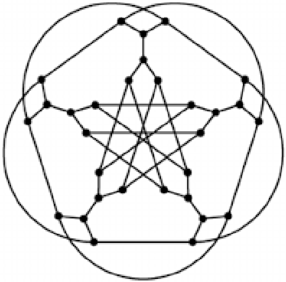

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- Non-triviality measure

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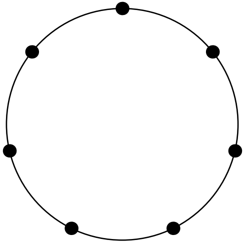

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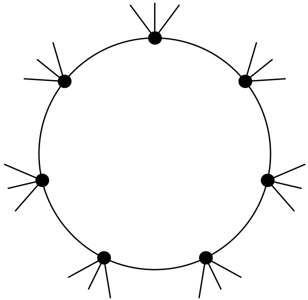

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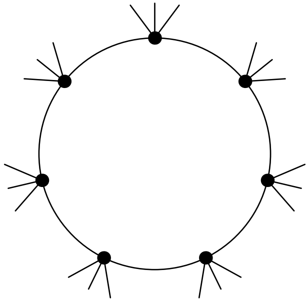

- Almost always a cycle-separating $(k-2)g$-edge-cut

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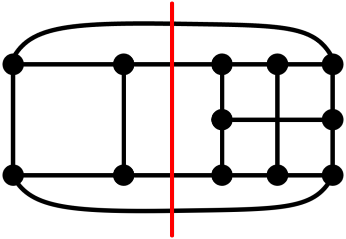
- Both true for all known cubic cages, that is for $g \leq 12$
- We prove them for some small values of k, g

Multipoles



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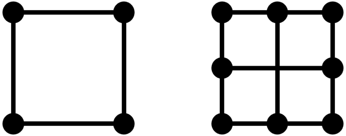

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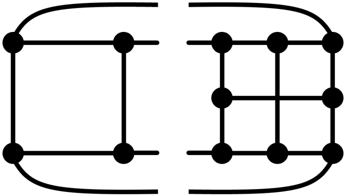

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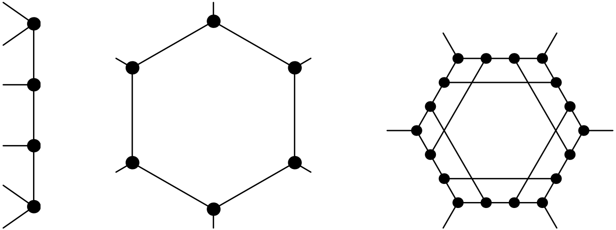

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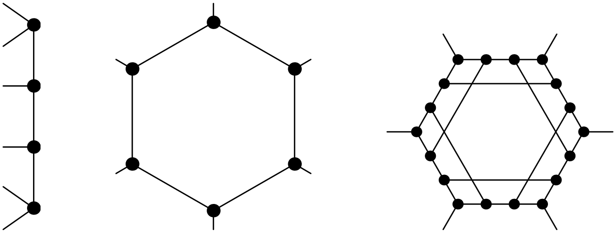

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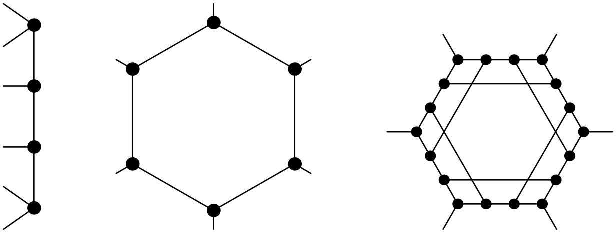

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Problem

What is the size $n(k, g, s)$ of a smallest nontrivial (k, g, s)-multipole.

Small cases: $k = 3$

$s =$	0	1	2	3	4	5	6	7	8	9
$g = 3$	4	5	4	5						
$g = 4$	6	7	6	5	6					
$g = 5$	10	11	10	9	8	7				
$g = 6$	14	15	14	13	12	11	8			
$g = 7$	24	25	24	23	22	21	20	17		
$g = 8$	30	31	30	29	28	27	26	25	22	

Obtained by hand and by the program `multigraph` [Brinkmann]

Small cases: $k = 4$

$s =$	0	2	4	6	8	10	12	14	16
$g = 3$	5	5	4	4					
$g = 4$	8	8	7	6	5				
$g = 5$	19	19	18	17	15	10			
$g = 6$	26	26	25	24	23	22	20		

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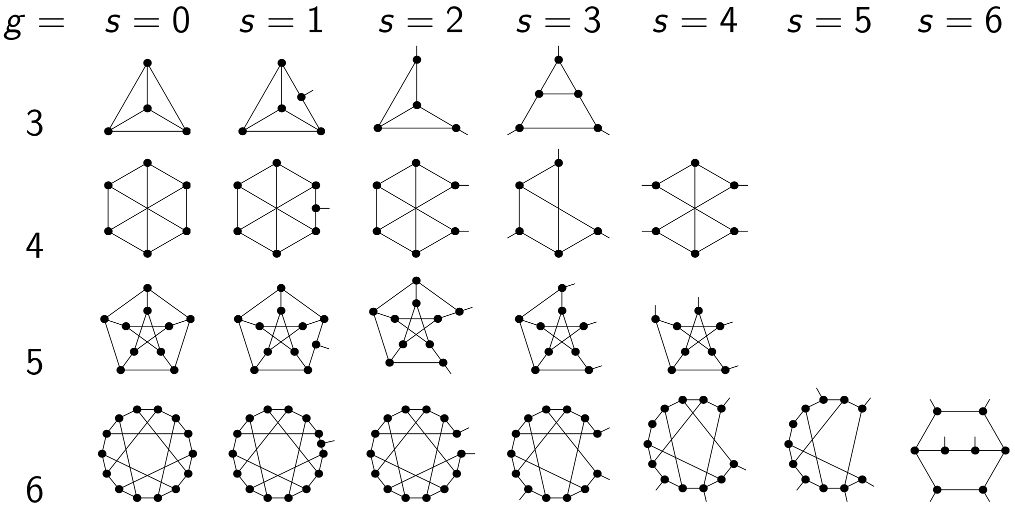

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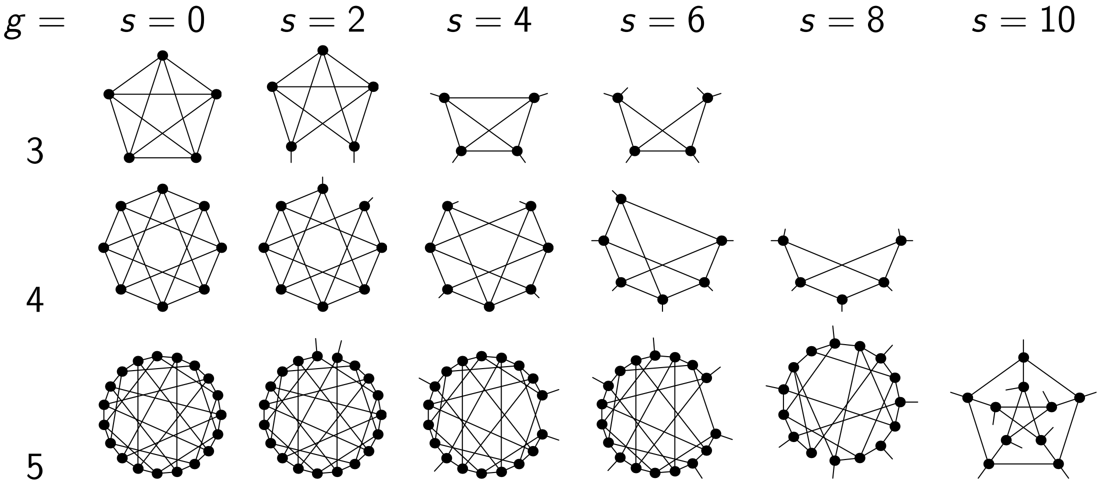

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- $n(3, g, s) \leq n(3, g) - s + 2$ for $0 < s < g$
- $n(3, g, g) \leq n(3, g) - g$ for $g \geq 6$
- Generalisation (except small cases):

$$n(k, g, s) \leq n(k, g) - \left\lfloor \frac{s-2}{k-2} \right\rfloor + \left(k \left\lfloor \frac{s-2}{k-2} \right\rfloor + s \right) \bmod 2$$

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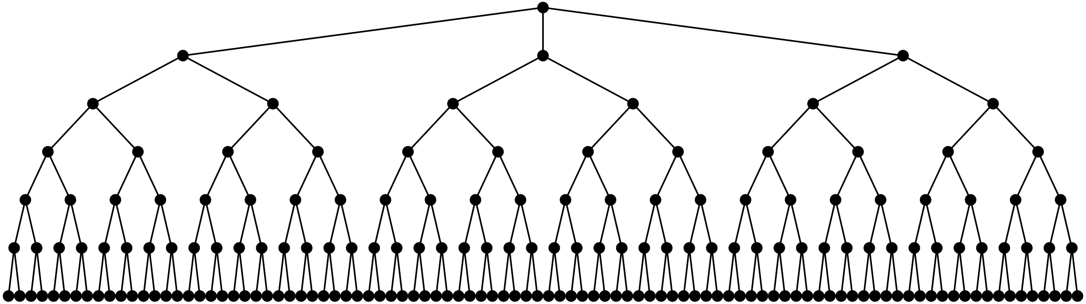

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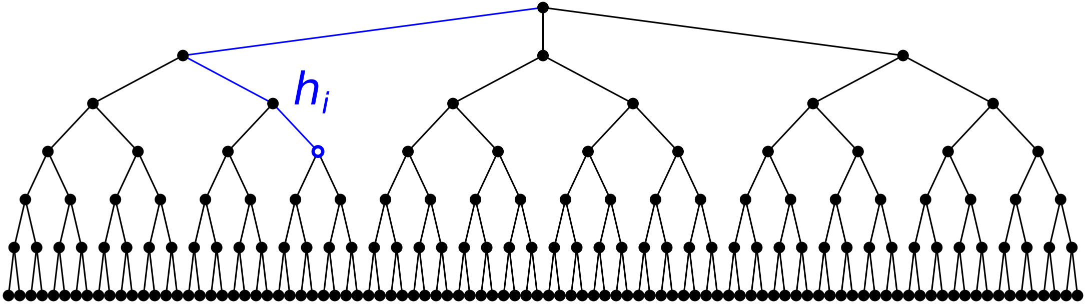

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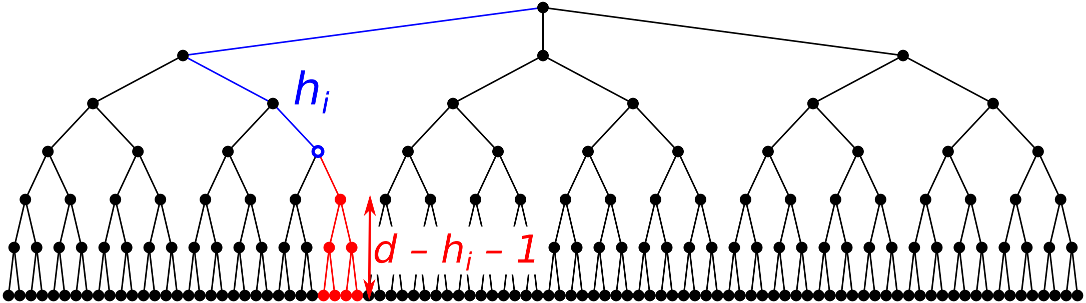

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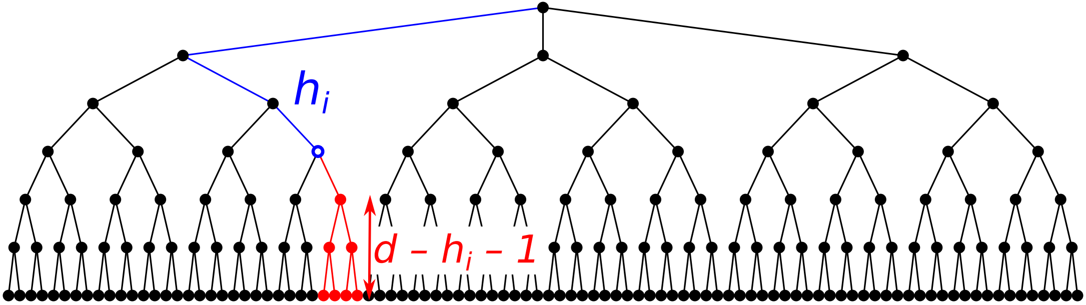

$$|G| \geq M(k,g) - \sum_{i=1}^{s} \frac{(k-1)^{d-h_i} - 1}{k-2}$$

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Some summation...

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$$n^2 - Mn + s \cdot \frac{(dk-2d-1)(k-1)^d + 1}{(k-2)^2} \geq 0$$

Summary

Let n be the order of a (k, g, s)-multipole

- For odd g:

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- Therefore, $n \leq b_1(k, g, s)$ or $n \geq b_2(k, g, s)$

Simpler bounds

$$n \leq b_1(k,g,s) \leq \frac{g^2}{2} \qquad \text{or} \qquad n \geq b_2(k,g,s) \geq M(k,g) - \frac{g^2}{2}$$

- $k = 3$ and $g \geq 11$,
- $k = 4$ and $g \geq 7$,
- $k \in \{5, 6\}$ and $g \geq 5$,
- $k \in \{7, 8, 9, 10\}$ and $g \geq 3$ and $g \neq 4$,
- $k \geq 11$ and $g \geq 3$.

Bound on $n(k, g, s)$

Theorem (Lukoťka, Máčajová, R. 2022+)

For $s \leq (k-2)g$, the order of a non-trivial (k, g, s)-multipole is:

$$n(k, g, s) \geq b_2(k, g, s) \geq M(k, g) - O(g^2)$$

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- Otherwise, $|G| \geq M(3, g)$

Results

Theorem (Lukoťka, Máčajová, R. 2022+)

Each (k, g)-graph or order less than

$$2b_2(k, g, s) = 2M(k, g) - O(g^2)$$

is cyclically $(k-2)g$-edge-connected and every cycle-separating $(k-2)g$-edge-cut separates a g-cycle.

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If a $(57, 5)$-Moore graph exists, it is cyclically 275-edge-connected

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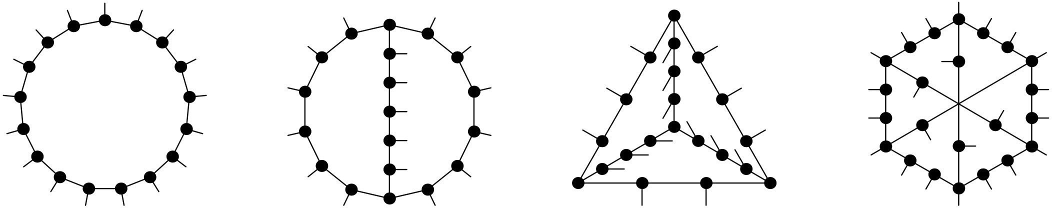

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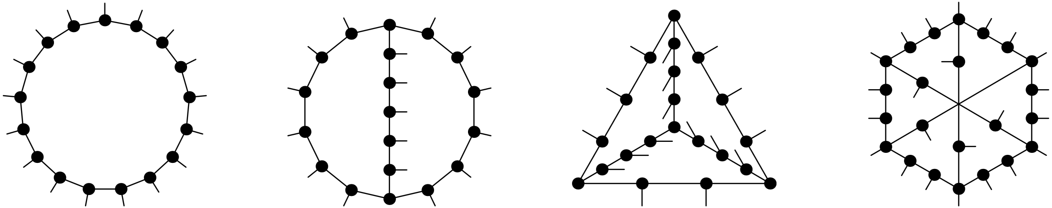

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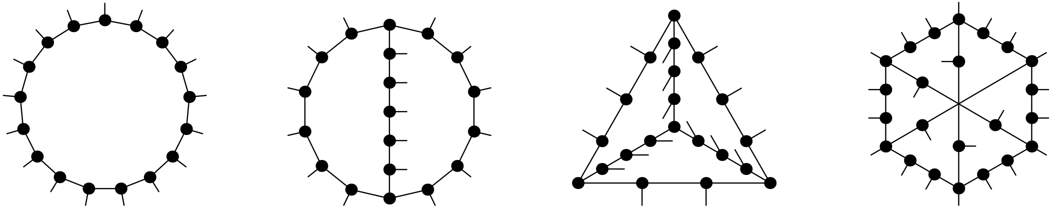

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- No other $(3, 11, 17)$-multipoles up to order $43 = b_1(3, 11, 17)$
- Thus, nothing up to order $b_2(3, 11, 17) = 51$

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- Obtain better lower bounds (close to $M(k,g)$)
- Investigate relation to cages
- Find small nontrivial (k,g,s)-multipoles using a computer
- Estimate the number of semiedges of a multipole induced by the Moore tree

$$n^2 - Mn + s \cdot \frac{(dk-2d-1)(k-1)^d+1}{(k-2)^2} \geq 0$$

$$kn^2 - (kM+s)n + Ms + 2s \cdot \frac{(dk-2d-1)(k-1)^{d+1}+k-1}{(k-2)^2} \geq 0$$

Thank you for your attention.