Cubic lifts of 2-vertex graphs in the cage problem (work in progress).

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Introduction

Overview

- The cage problem
- Voltage graphs and their lists
- Cubic lifts of 2-vertex graphs, "large"girth
- Cubic lifts of 2-vertex graphs, small girth

Disclaimer

Everything is finite and connected.

Cages

(k,g)-graphs

A (k,g)-*graph* is a k-regular graph of girth g.
Moore bound $M(k,g)$ A (k,g)-graph on n vertices has n at least
$(k(k-1)^{(g-1)/2}-2)/(k-2)$ if g is odd, and
$(2(k-1)^{g/2}-2)/(k-2)$ if g is even.

A (k,g)-graph on n vertices is
a *Moore graph* if $n = M(k,g)$,
a *cage* if n is the smallest possible value,
a *record holder* if n is the smallest known value.

Cages are known only for very restricted set of pairs (k,g).
More in the talk by Robert Jajcay.

Symetries

Many of record holders are highly symmetric (vertex-, arc-, edge-, flag- transitive/regular).

Theorem (Sabidussi) Graph is Cayley iff it has group G automorphism which has unique orbit and this orbit has size $|G|$.

Theorem Graph is a lift of an l-vertex graph iff it has group G automorphism which has exactly l orbits all of size $|G|$.

Question What is the number of 2-vertex cubic graphs?

Lifts

Lifts

Let Γ be a graph with the dart set $D(\Gamma)$. Each edge e consists of a pair of mutually inverse darts d and d^{-1}. Each dart d has an initial vertex $i(d)$ and a tail vertex $t(d) = i(d^{-1})$.

A *voltage assignment* on Γ with values in a group G is a mapping $\alpha : D(\Gamma) \to G$ such that $\forall d \in D(\Gamma) : \alpha(d^{-1}) = \alpha(d)^{-1}$.

The *lift* of *base graph* Γ (w.r.t. α) is the graph $\tilde{\Gamma}$ with

$$\begin{aligned} V(\tilde{\Gamma}) &= V(\Gamma) \times G, \\ D(\tilde{\Gamma}) &= D(\Gamma) \times G, \\ i(d,g) &= (i(d), g), \\ t(d,g) &= (t(d), g\alpha(d)), \\ (d,g)^{-1} &= (d^{-1}, g\alpha(d)). \end{aligned}$$

Examples

Cayley graphs are lifts of one vertex graphs.

Haar graphs are lifts of dipoles.

Canonical double covers are lifts of all ones voltages in $\mathbb{Z}_2$.

Each graph is its own lift.

$K_{3,3}$ is a lift of the *theta graph* Θ (the dipole with 3 edges) with voltages in $\mathbb{Z}_3$.

Petersen graph is a lift of the *dumbbell graph* o-o with voltages in $\mathbb{Z}_5$.

Why to use lifts

G is a group of automorphisms of $\tilde{\Gamma}$ which acts semiregularly with $|V(\Gamma)|$ orbits.

Various invariants (e.g., the girth) of $\tilde{\Gamma}$ can be computed from Γ and α.

Many record holders are lifts of smaller graphs.

Cubic lifts of 2-vertex graphs, "large"girth

Motivation

In 2022 G. Erskine and J. Tuite found new cubic recold holders for girth in 20-32. Their construction was based on Cayley uniform hypergraph and corresponding groups were often of the form perfect $\times$ small (a group G is *perfect* if $G = G'$, i.e., it has no nontrivial abelian quotients).

g	lower bound	pre E.T.	E.T.
20	2048	5736	12096
22	4096	16206	23328
24	8192	49608	35640
26	16384	109200	109200
28	32768	415104	368640
30	65536	1143408	806736
32	131072	3650304	1441440

Idea Let us look at cubic Haar graph with group of the form perfect $\times$ small abelian.

Results

g	lower bound	pre E.T.	E.T.	θ
18	1024	2560	2688	2640
20	2048	5736	12096	7920
22	4096	16206	23328	18480
24	8192	49608	35640	
26	16384	109200	109200	
28	32768	415104	368640	
30	65536	1143408	806736	
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32	131072	3650304	1441440	7488000

Cubic lifts of 2-vertex graphs, small girth

Motivation

In 2013 P. Potočnik, P. Spiga and G. Verret constructed all vertex-transitive cubic graph on up to 1280 vertices. They found the smallest vertex-transitive and Cayley $(3, g)$ graphs for $g \leq 16$.

Idea For small g, find the smallest $(3, g)$ graph which are lifts of 2-vertex graphs.

Some remarks

Small groups are (usually) far from perfect (e.g, they are solvable, nipotent or abelian) and there are many of them.

P.S.V. developed techniques how to deal with them.

There are neccessary condition on the rank of the voltage group.

It is possible to do induction on groups with a central involution using 2-cohomology (or GAP).

Results

g	lower bound	record	VT	Cayley	Θ	o-o
4	6	6	6	6	6	8
5	10	10	10	50	—–	10
6	14	14	14	14	14	16
7	24	24	26	30	—–	26
8	30	30	30	42	40	36
9	58	58	60	60	—–	60
10	70	70	80	96	80	96
11	112	112	192	192	—–	168
12	126	126	162	162	162	168
13	202	272	272	272	—–	448
14	258	384	406	406	406	480
15	384	620	620	864	—–	936
16	512	960	1008	1008	1008	1008
17	768	2176			—–	2530
18	1024	2560			2640	

Semiedges

When doing lifts, it is extremally useful to consider base graphs with semiedges. A *semiedge* is an edge with incident vertex on just one side. It consists of one self-inverse dart. Voltage on a semiedge is an involution (different from e_G).

There are three cubic graphs on 2 vertices with semiedges: >-<, o-<, and -0-.

(In the cage problem, multiple edges and loops in lifts disappear due to condition on girth. Semiedges have to be explicitly forbidden, otherwise the problem becomes trivial.)

More results

g	l. b.	rec.	VT	Cay.	Θ	o-o	>-<	o-<	-0-
4	6	6	6	6	6	8	8		
5	10	10	10	50	−−	10	−−		
6	14	14	14	14	14	16	16		
7	24	24	26	30	−−	26	48		
8	30	30	30	42	40	36	36		
9	58	58	60	60	−−	60	−−		
10	70	70	80	96	80	96	80		
11	112	112	192	192	−−	168	240		
12	126	126	162	162	162	168	192		
13	202	272	272	272	−−	448	784		
14	258	384	406	406	406	480	480		
15	384	620	620	864	−−	936	1344		
16	512	960	1008	1008	1008	1008	1080		
17	768	2176			−−	2530	2456		
18	1024	2560			2640		2688		

Girth 9

Voltages on semiedges are a, b on the left, and c, d on the right. The voltage on the edge can be 1. Voltages are all different.

Cycle of length 9 in $\tilde{\Gamma}$ implies a word in $1, a, b, c, d$ of length 9 which in G equals 1. Each possible word gives rise to a shorter cycle.

$1 = ababababa = a^{abab}, 1 = a$

$1 = aba1cdcd1, aba = dcdc, 1 = abaaba = dcdcdcdc$, length 8

$1 = ab1cdcdc1, ab = cdcdc = ba, 1 = abab$, length 4

$1 = a1cdcdcd1, ac = cdcdcdc = ca, 1 = a1c1a1c1$, length 8

$1 = ab1c1a1c1, ba = cac = ba$, length 8

$1 = ab1c1b1c1, ab = cbc = bc$, length 8

$1 = ab1c1a1d1, 1 = bcada, bc = ada = cb$, length 8

$1 = ab1c1b1d1, 1 = dabcb$, length 8

What next?

- Finish the computations for small girth.
- Do more thorough computations for "large"girth.
- Look at lifts of larger graphs with semiedges.

Thank you for Your attention