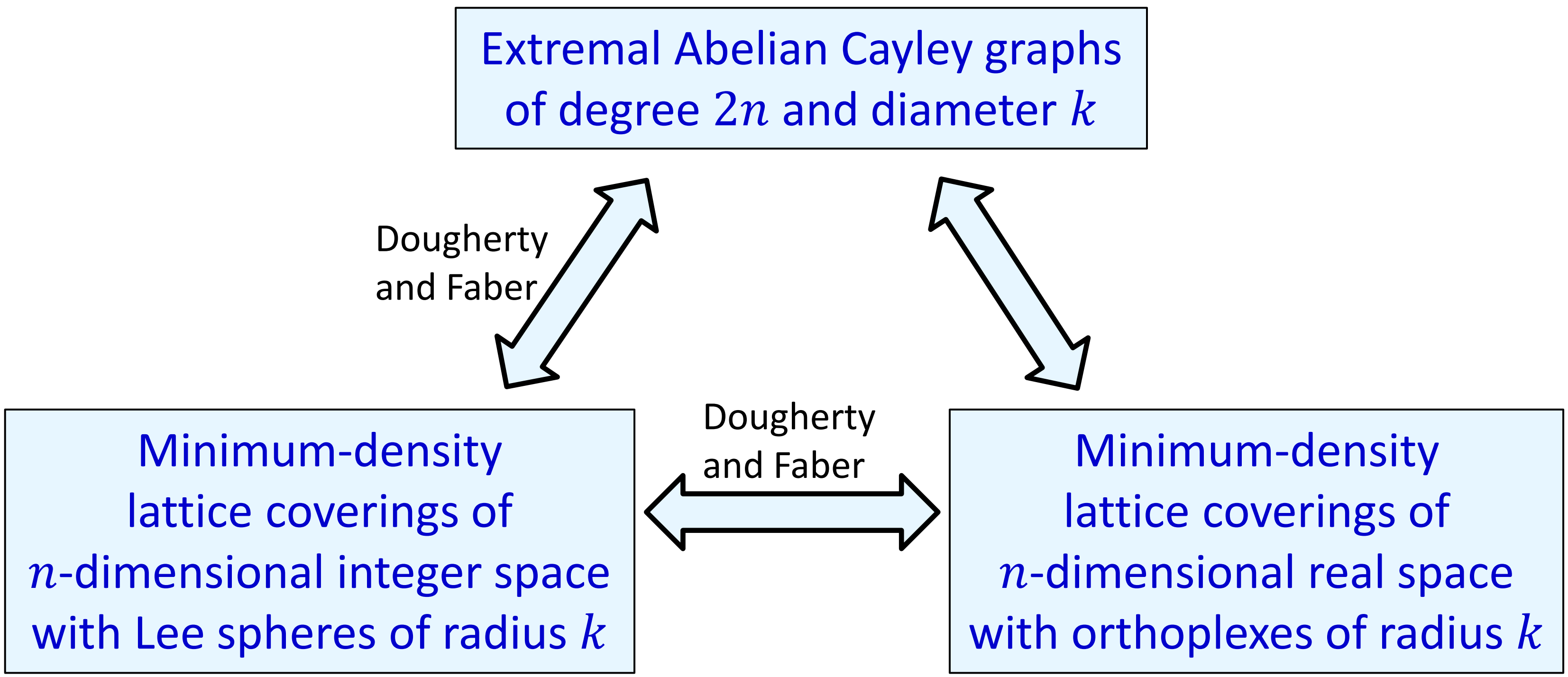
Upper bounds for the order of Abelian Cayley graph families in the degree-diameter problem

Rob Lewis, Open University, 18th July 2023

Extremal Abelian Cayley graphs and minimum-density lattice coverings of space

The degree-diameter problem for Abelian Cayley graphs is equivalent to finding minimum-density lattice coverings of real space with orthoplexes.

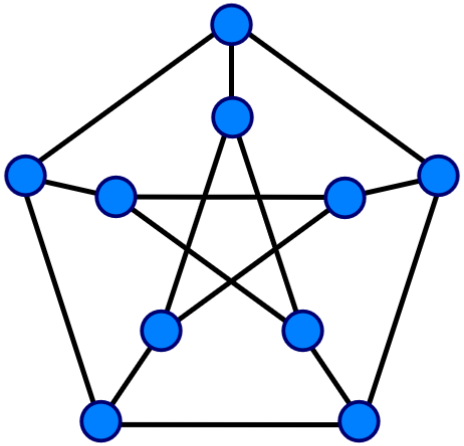


The degree-diameter problem

The degree-diameter problem is a fundamental and simple question of graph theory: given any *maximum degree* and *diameter*, what is the *largest possible order* of a graph?

Within this context, such graphs are called *extremal*.

For example, the extremal graph of maximum degree 3 and diameter 2 is the Petersen graph, with order 10:



Only *seven* graphs with degree greater than 2 are known to be extremal.

Degree, d	Diameter, k 2	3	4
3	10*	20	38
4	15		
5	24		
6	32		
7	50^		

* Petersen graph

^ Hoffman-Singleton graph

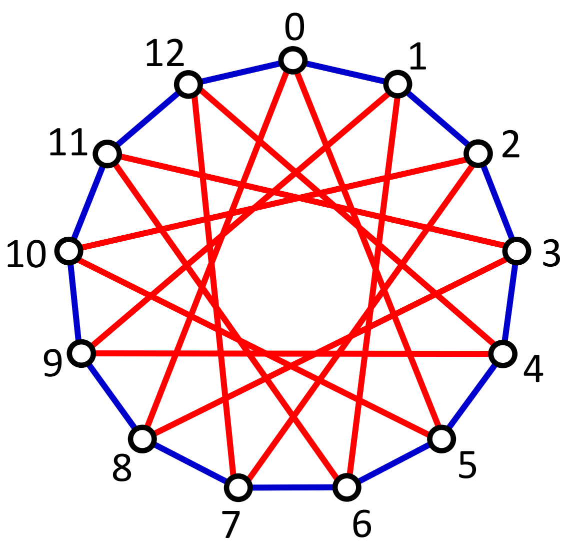
The degree-diameter problem for undirected Abelian Cayley graphs is much more accessible

For an undirected Abelian Cayley graph, each edge must have arcs in both directions, so connection elements occur in pairs such as ± 1 (except for the involution if the degree is odd).

The number of elements in the connection set determines the degree d of the graph.

The dimension n of the graph is the number of pairs of connection elements: $n = \lfloor d/2 \rfloor$.

The simplest Abelian case is a finite cyclic group, generating a *circulant graph*.



In this example, the group is $\mathbb{Z}_{13}$ and the connection set $\{\pm 1, \pm 5\}$.

The degree $d = 4$ and the dimension $n = 2$.

It has diameter $k = 2$.

With order 13, it is the extremal Abelian Cayley graph of degree 4 and diameter 2.

Extremal and largest-known circulant graphs up to degree 20 and diameter 16

- Extremal circulant graph families defined for degrees 2 to 5 and arbitrary diameter
- Largest-known families for degrees 6 to 20 and beyond

Order of extremal (yellow) and largest-known (green) circulant graphs

f	$d \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33
1	3	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
2	4	13	25	41	61	85	113	145	181	221	265	313	365	421	481	545
2	5	16	36	64	100	144	196	256	324	400	484	576	676	784	900	1024
3	6	21	55	117	203	333	515	737	1027	1393	1815	2329	2943	3629	4431	5357
3	7	26	76	160	308	536	828	1232	1764	2392	3180	4144	5236	6536	8060	9744
4	8	35	104	248	528	984	1712	2768	4280	6320	9048	12552	17024	22568	29408	37664
4	9	42	130	320	700	1416	2548	4304	6804	10320	15004	21192	29068	39032	51300	66336
5	10	51	177	457	1099	2380	4551	8288	14099	22805	35568	53025	77572	110045	152671	208052
5	11	56	210	576	1428	3200	6652	12416	21572	35880	56700	87248	128852	184424	259260	355576
6	12	67	275	819	2120	5044	10777	21384	39996	69965	117712	190392	295965	448920	662680	952985
6	13	80	312	970	2676	6256	14740	30760	57396	106120	182980	295840	476100	732744	1081860	1593064
7	14	90	381	1229	3695	9800	23304	49757	103380	196689	350700	593989	996240	1603216	2486227	3843540
7	15	96	448	1420	4292	12232	32092	68944	142516	276928	514580	908480	1550228	2566712	4013468	6155056
8	16	112	518	1788	5847	17733	45900	107748	232245	479255	924420	1702428	2982623	5209347	8476048	13588848
8	17	130	570	1954	6468	20360	57684	136512	321780	659464	1350820	2479104	4557364	7729000	13275108	21252864
9	18	138	655	2645	8425	27273	80940	208872	492776	1078280	2202955	4388640	8068383	14718984	25609955	43068508
9	19	156	722	2696	9652	31440	99420	258040	652004	1416256	3101860	6100520	11797684	21659528	38328220	66601304
10	20	171	815	3175	12396	42252	132720	371400	930184	2232648	4947880	10238745	20452920	38155632	70612644	126967008

Key: Dimension f, degree d, diameter k

Table from Combinatorics Wiki

Largest-known Abelian Cayley graphs of even degree

The largest-known Abelian Cayley graphs of any given even degree $d = 2n$, with diameter k above a critical value for each degree, belong to graph families and have order defined by a polynomial of degree n in the diameter k.

Degree $d = 2n$	Dimension n	Order
2	1	$2\,k + 1$
4	2	$2\,k^2 + O(k)$
6	3	$(32/27)\,k^3 + O(k^2)$
8	4	$(1/2)\,k^4 + O(k^3)$
10	5	$(512/3125)\,k^5 + O(k^4)$
12	6	$(32/729)\,k^6 + O(k^5)$
14	7	$(8192/823543)\,k^7 + O(k^6)$
16	8	$(1/512)\,k^8 + O(k^7)$

These leading coefficients share a common expression

$$\frac{1}{2}\left(\frac{4}{n}\right)^n.$$

For odd degree $2n + 1$, the largest-known families all have a leading coefficient twice as large:

$$\left(\frac{4}{n}\right)^n.$$

Conjecture: This is also true for extremal graph families.

Largest-known graphs and lattice coverings of lowest-known density

The order of largest-known Abelian Cayley graphs of degree $2n$

Leading term

$$\frac{1}{2}\left(\frac{4}{n}\right)^n k^n$$

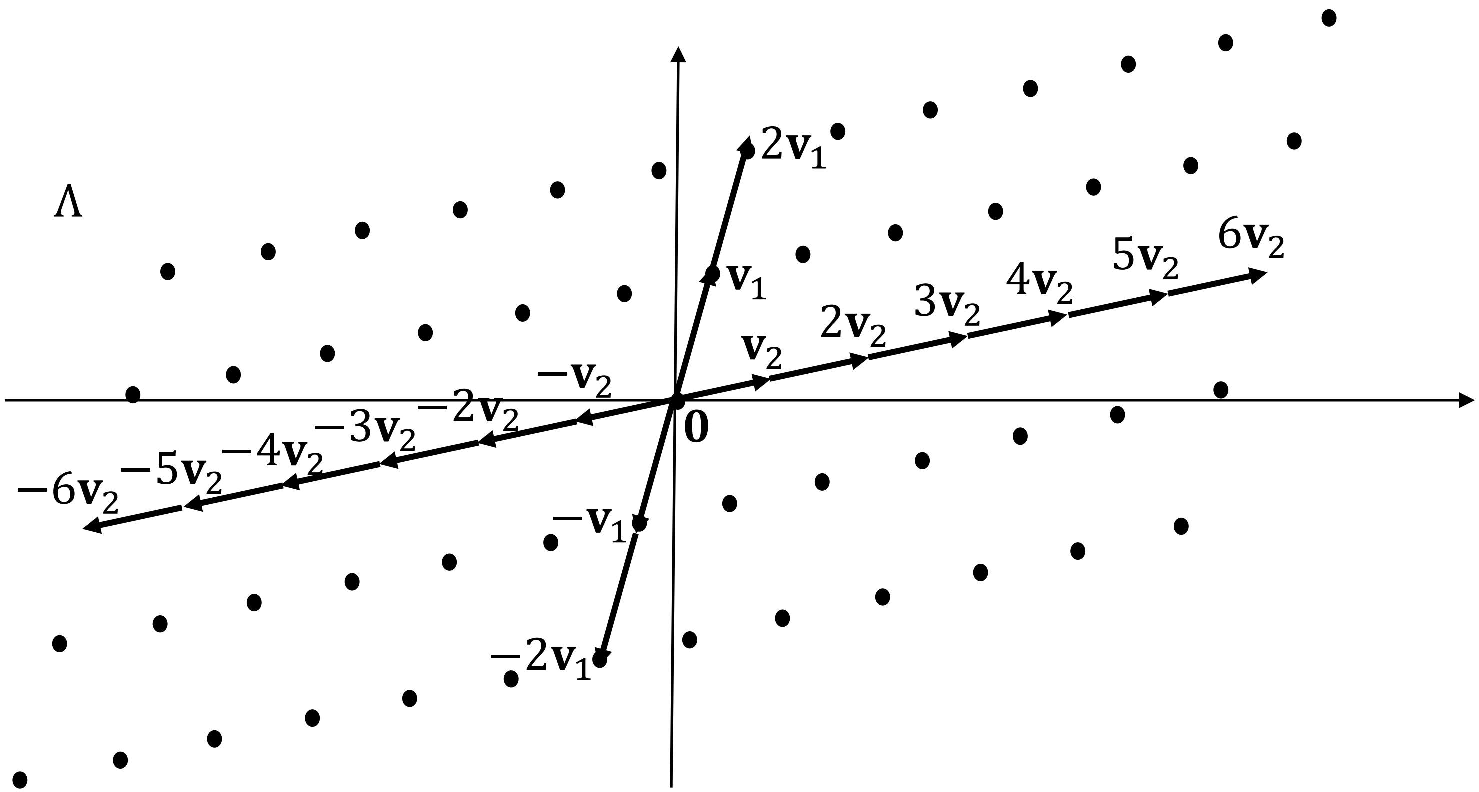
Lattice coverings of n-dimensional integer space with Lee spheres having lowest-known density

Lattice coverings of n-dimensional real space with orthoplexes having lowest-known density

What is a lattice?

A *lattice* in n-dimensional real space $\mathbb{R}^n$ is the set of all integer linear combinations of a basis of $\mathbb{R}^n$. That is, to each basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ we associate the lattice

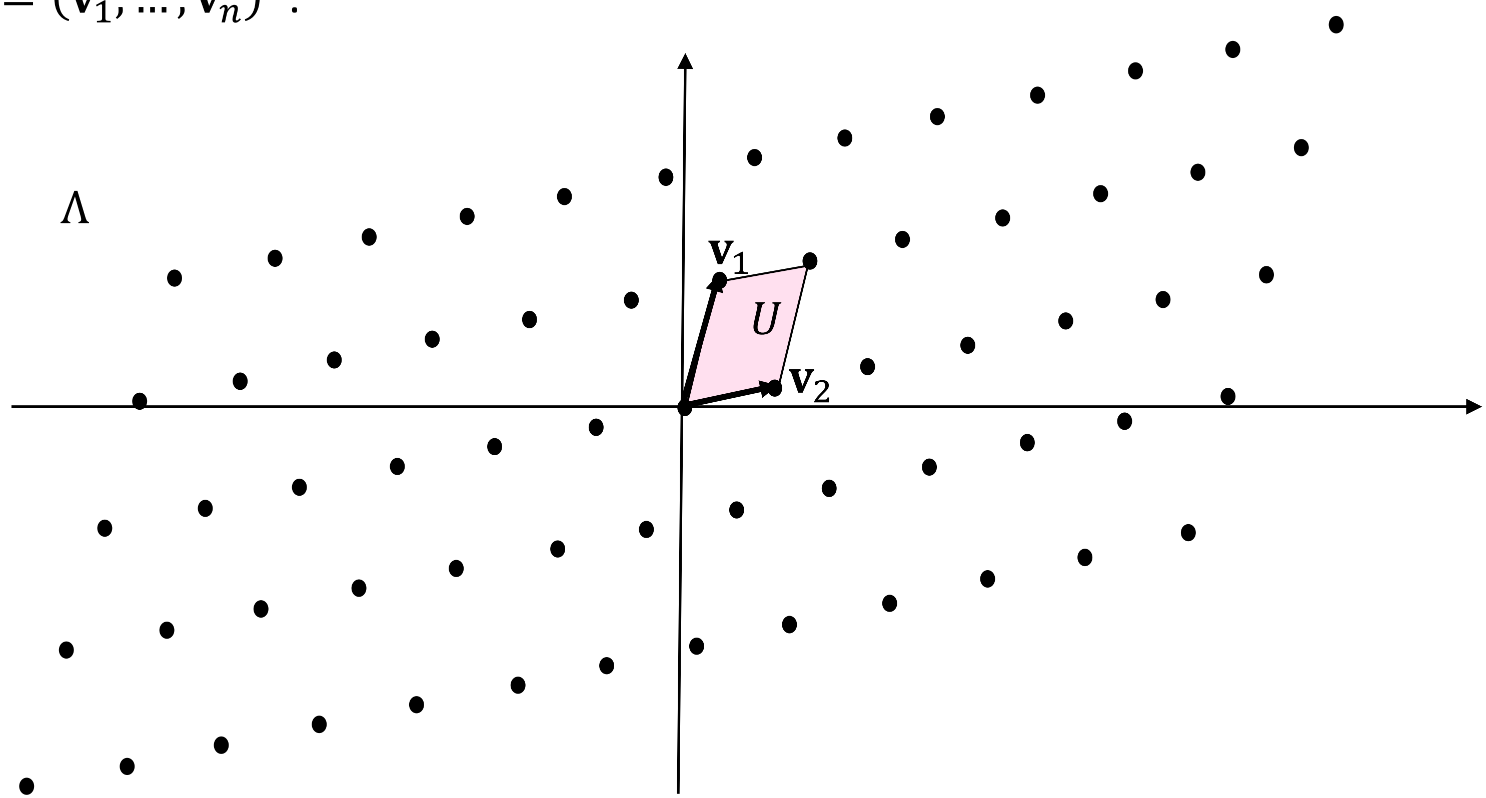
$\Lambda = \{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_{\boldsymbol{n}}: a_i \in \mathbb{Z}\}$.



A unit cell of the lattice

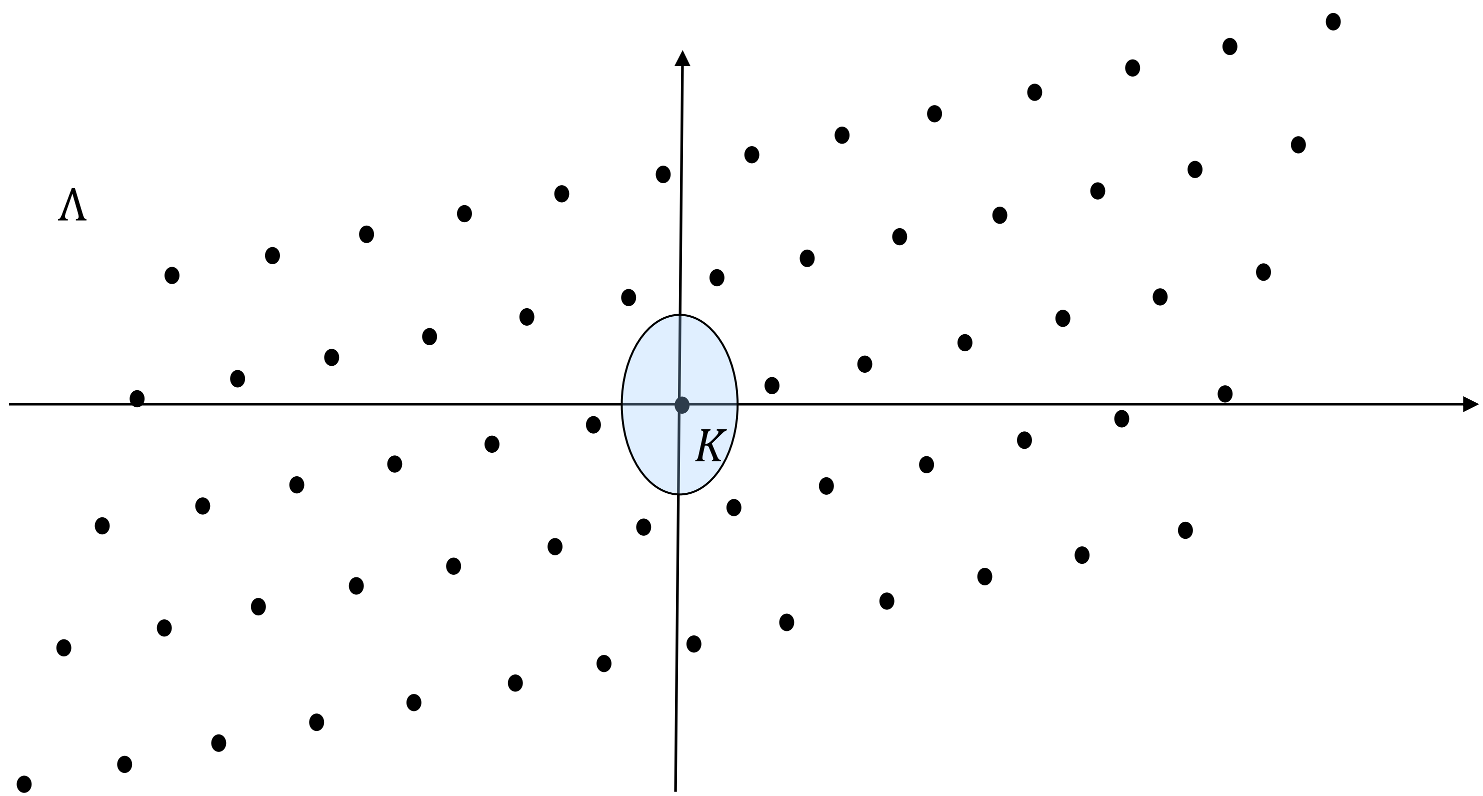
A *unit cell* of the lattice is the set of points defined by all linear combinations of a set of lattice generating vectors with coefficients between 0 and 1 inclusive. If the vectors are $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then the unit cell $U = \{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n : 0 \le a_i \le 1\}$.

The volume of the unit cell $V(U) = |\det(M)|$, where M is the lattice generator matrix, $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$.



A lattice covering $\mathcal{K}$ of a lattice Λ with a set K

Let Λ be a lattice in $\mathbb{R}^n$, and K a set of points in $\mathbb{R}^n$.

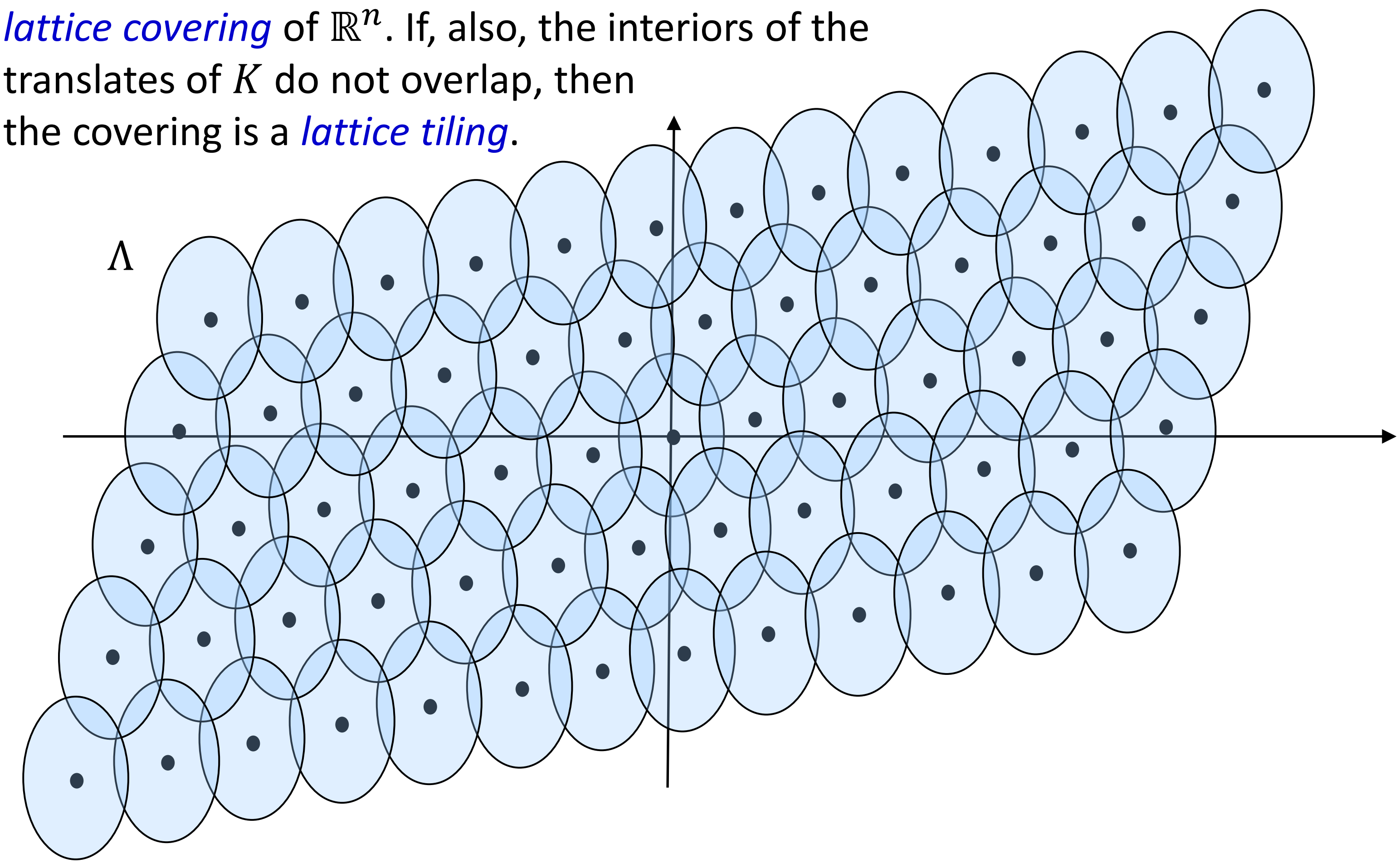


A lattice covering $\mathcal{K}$ of a lattice Λ with a set K

Let Λ be a lattice in $\mathbb{R}^n$, and K a set of points in $\mathbb{R}^n$.

Let $\mathcal{K} = \{K + \lambda : \lambda \in \Lambda\}$ be the system of translates of K by the lattice points of Λ.

If every point of $\mathbb{R}^n$ lies in at least one of the sets of the system $\mathcal{K}$, then $\mathcal{K}$ forms a *lattice covering* of $\mathbb{R}^n$. If, also, the interiors of the translates of K do not overlap, then the covering is a *lattice tiling*.



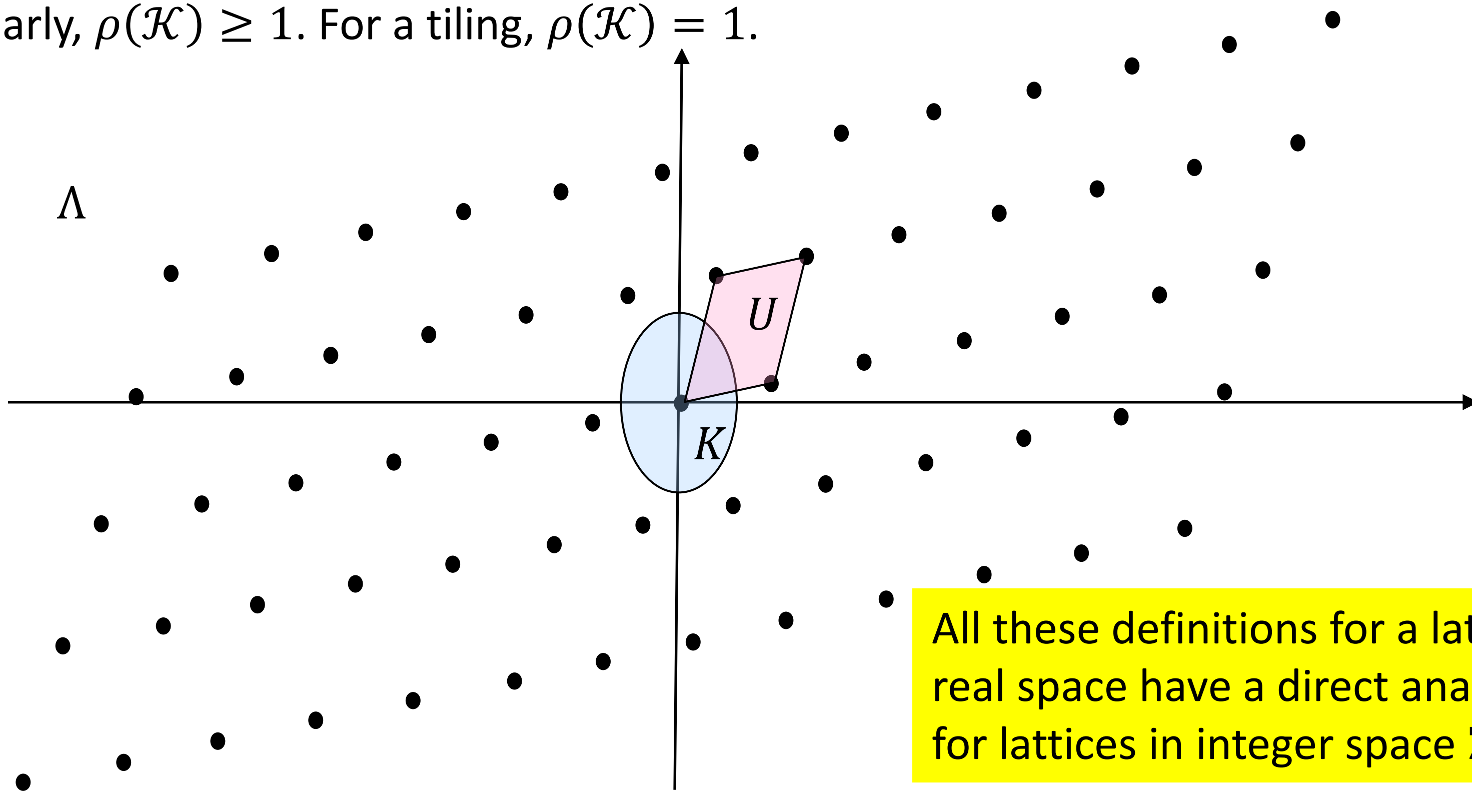
The density of a lattice covering $\mathcal{K}$ is the volume of the set K divided by the volume of a unit cell

Let Λ be a lattice in $\mathbb{R}^n$ and K a set of points in $\mathbb{R}^n$ such that $\mathcal{K} = \{K + \lambda: \lambda \in \Lambda\}$ is a lattice covering of $\mathbb{R}^n$.

Let U be a unit cell of Λ, also $V(U)$ and $V(K)$ be the volume of U and K respectively.

Then the *density* of the covering $\mathcal{K}$ is: $\rho(\mathcal{K}) = V(K)/V(U)$.

Clearly, $\rho(\mathcal{K}) \geq 1$. For a tiling, $\rho(\mathcal{K}) = 1$.



All these definitions for a lattice in real space have a direct analogue for lattices in integer space $\mathbb{Z}^n$.

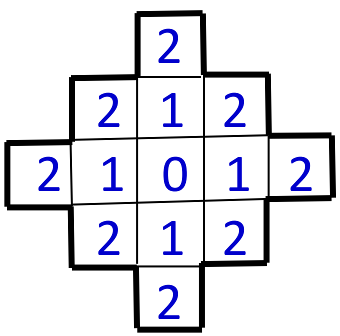
What is a Lee sphere?

In n-dimensional integer space $\mathbb{Z}^n$, a *Lee sphere* $S_{n,k}$ of radius k is the set of points distant at most k from the origin under the Manhattan norm (Rook moves):

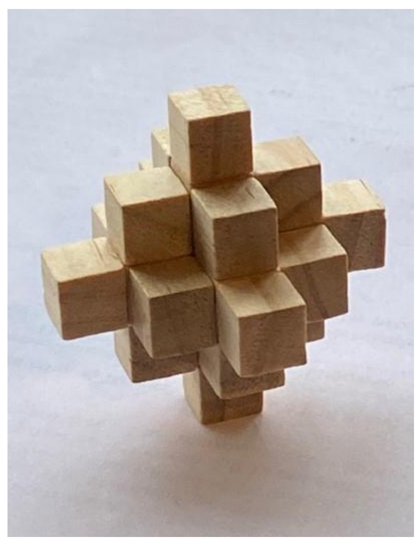
$$S_{n,k} = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n : |m_1| + \cdots + |m_n| \le k\}.$$

It may be represented visually by replacing each point of the Lee sphere by a unit cube centred on the point:

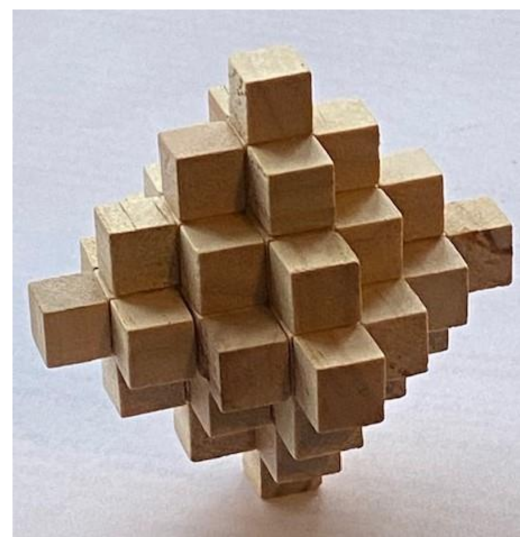
$S_{2,2}$



$S_{3,2}$



$S_{3,3}$



Equivalence of Abelian Cayley graphs and lattice coverings of integer space with Lee spheres

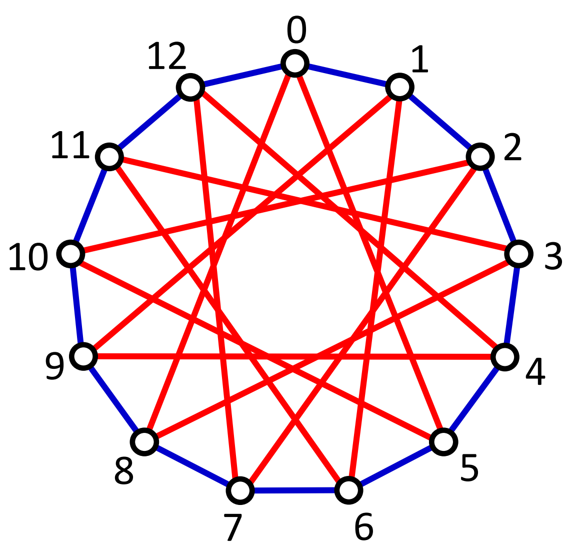
The parameters that define an Abelian Cayley graph can also vertex-label the points of a Lee sphere. Cayley graphs are vertex transitive, so the centre vertex is arbitrary.

The number of inverse pairs in the connection set determines the dimension of the graph and the Lee sphere. The diameter of the graph is equal to the radius of the Lee sphere.

Example: Abelian group $\mathbb{Z}_{13}$, connection set (self-inverse) $\{\pm 1, \pm 5\}$.

Circulant graph

Dimension 2
Diameter 2



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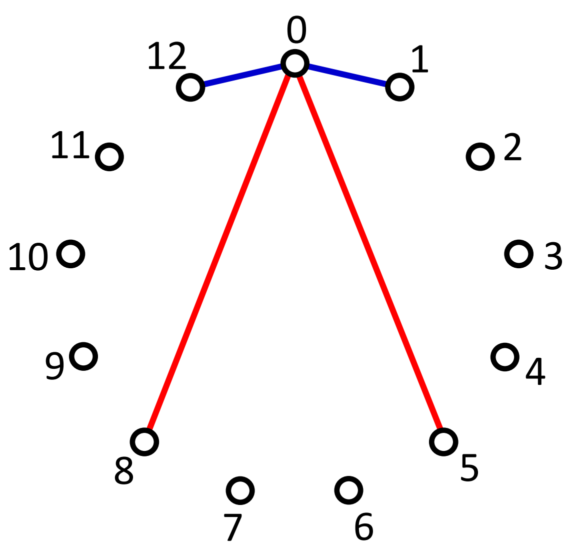
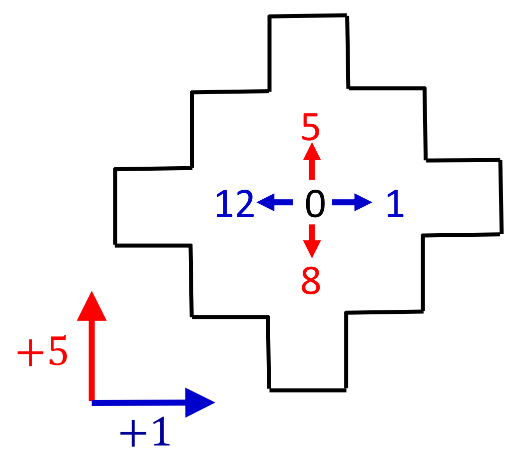


Lee sphere

Dimension 2



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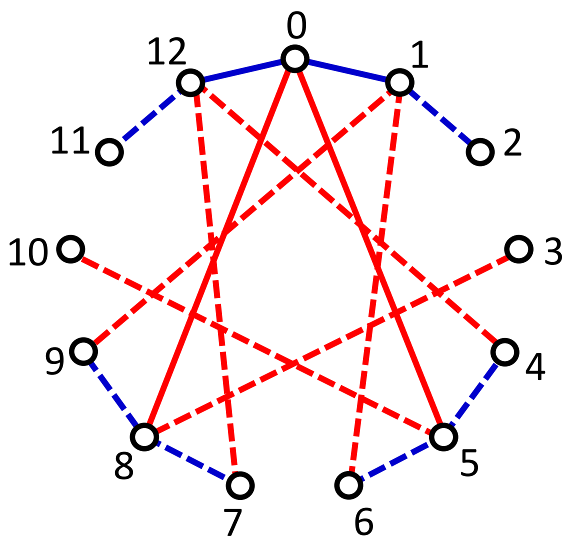
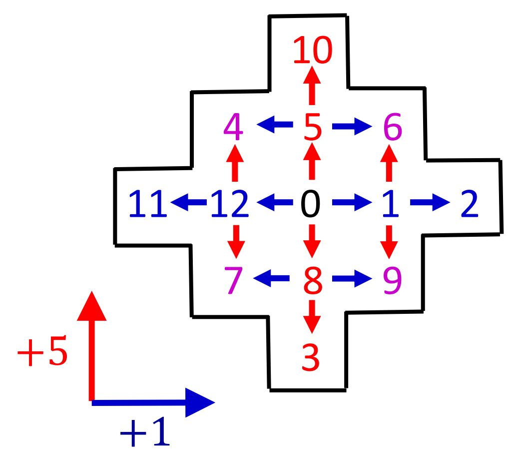
Dimension 2
Diameter 2



Lee sphere

Dimension 2
Radius 2



This equivalence provides an upper bound for the order of an Abelian Cayley graph

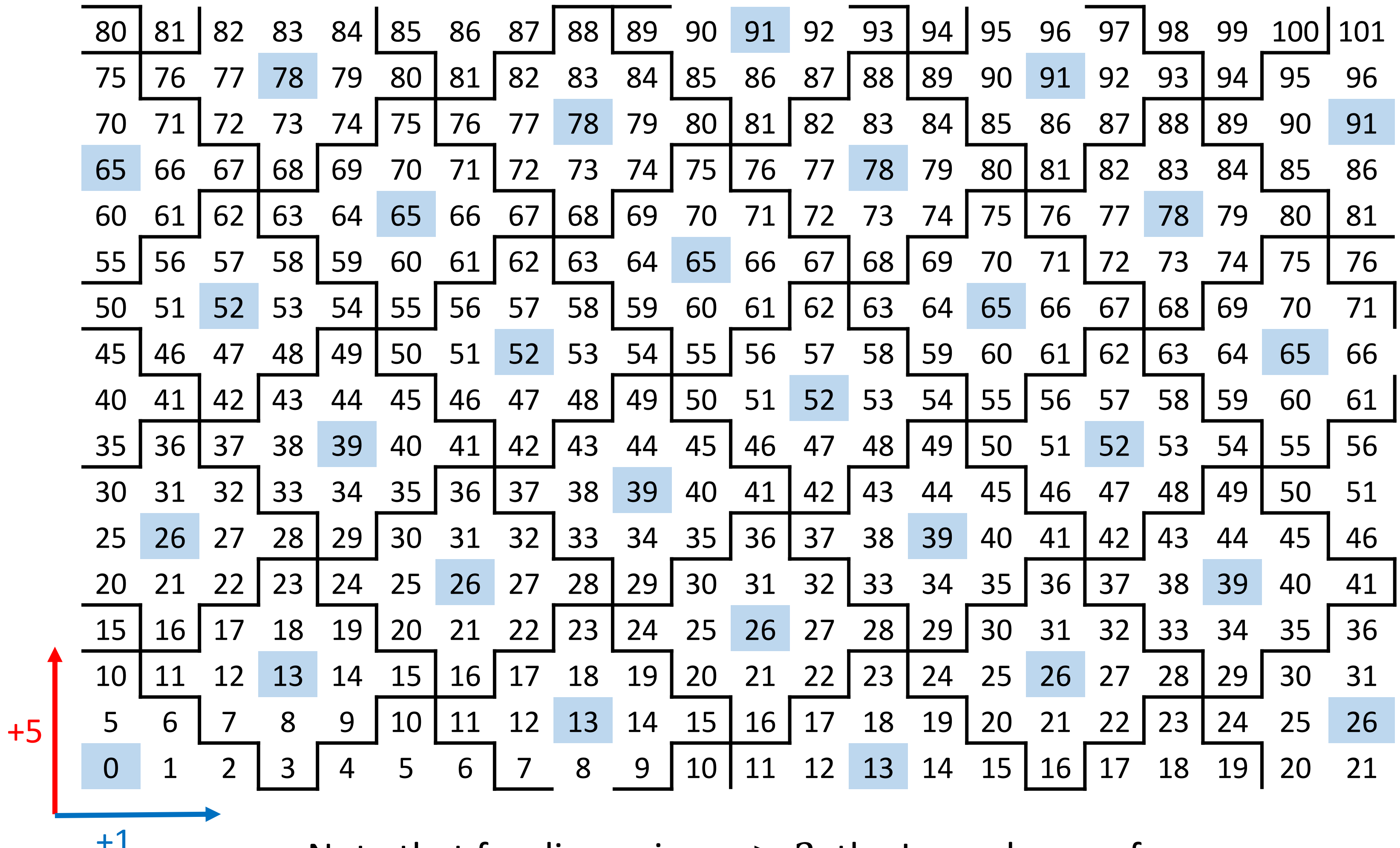
The order of an extremal Abelian Cayley graph of even degree $d = 2n$ and diameter k does not exceed the volume of an n-dimensional Lee sphere of radius k:

$$Ext(d,k) \leq |S_{n,k}| = \sum_{i=0}^{n} 2^i \binom{n}{i} \binom{k}{i} = \frac{2^n}{n!} k^n + O(k^{n-1})$$

For odd degree $d = 2n + 1$, the extremal order does not exceed the sum of the volumes of n-dimensional Lee spheres of radius k and radius $k - 1$:

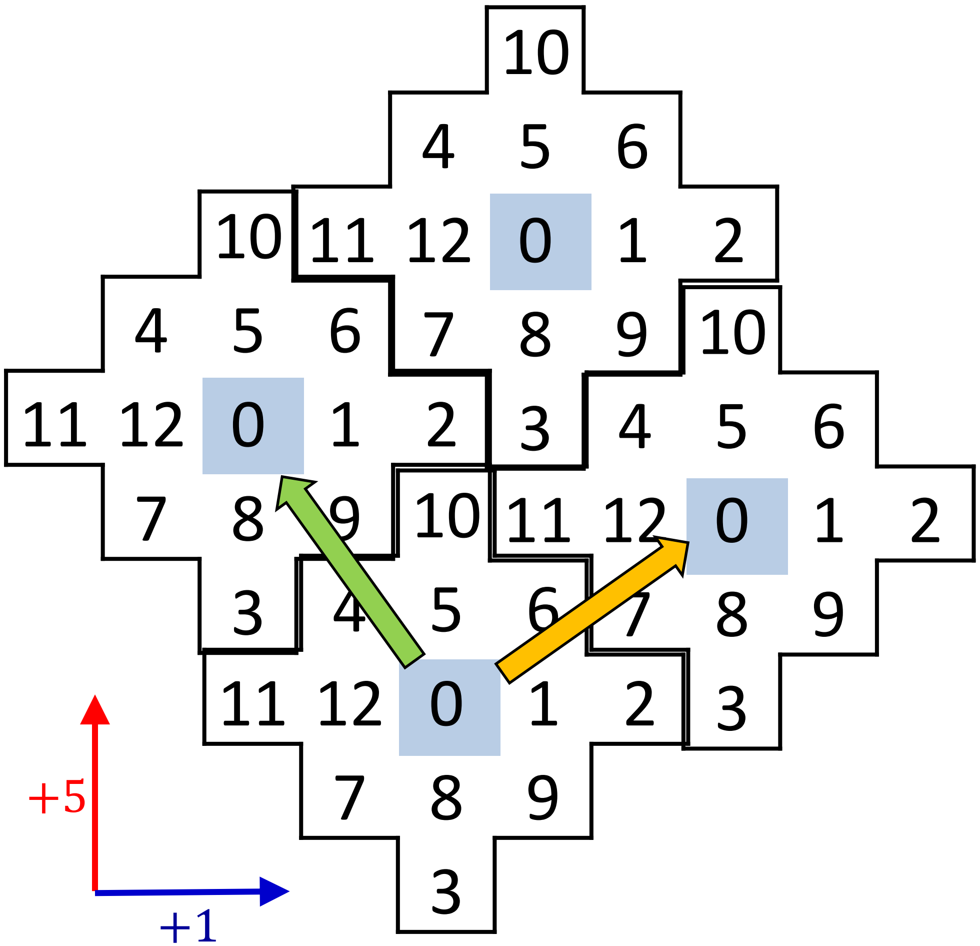
$$Ext(d,k) \leq |S_{n,k}| + |S_{n,k-1}| = \frac{2^{n+1}}{n!} k^n + O(k^{n-2})$$

These Lee spheres form a perfect lattice tiling of $\mathbb{Z}^2$.



Note that for dimension $n > 2$, the Lee spheres of any lattice covering will overlap (Golomb-Welch conjecture)

For $\mathbb{Z}_{13}$ with connection set $\{\pm 1, \pm 5\}$ the Lee spheres form a lattice tiling of $\mathbb{Z}^2$ (modulo 13)



Lattice generating vectors for diameter $k = 2$

$\mathbf{v}_1 = (3, 2)$

$\mathbf{v}_2 = (-2, 3)$

So we have the lattice generator matrix (LGM):

$$M = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

Area of the lattice unit cell, $|\det M| = 13$, the order of the graph.

Area of Lee sphere is also 13. So the covering density is 1 (tiling)

Largest-known Abelian Cayley graph families correspond to lattice coverings with density that is exponential in the dimension

Dimension n	Leading term: Upper bound	Graph order	Asymptotic covering density	
	$\frac{2^n}{n!}k^n$	$\frac{1}{2}\left(\frac{4}{n}\right)^n k^n$	$\frac{n^n}{2^{n-1}n!} \simeq \sqrt{\frac{2}{\pi n}}\left(\frac{e}{2}\right)^n \approx 1.359^n$	
1	2	2	1	Extremal, upper bound achieved
2	2	2	1	Extremal, upper bound achieved
3	$4/3$	$32/27$	1.125	
4	$2/3$	$1/2$	1.333	
5	$4/15$	$512/3125$	1.628	
6	$4/45$	$32/729$	2.205	
7	$8/315$	$8192/823543$	2.553	
8	$2/315$	$1/512$	3.251	

In each case, the lattice is an integer approximation to a standard body-centred cubic format.

Upper bounds on the order of Abelian Cayley graphs of degree d and diameter k

Moore bound — Example: $(d,k) = (10,10)$

$$1 + d\sum_{i=0}^{k-1}(d-1)^i \quad = \quad d^k \ - \ (k-2)\, d^{k-1} \ + \ O\left(d^{k-2}\right) \qquad 3{,}874{,}204{,}890 \qquad (1)$$

Abelian Cayley Moore bound ($d = 2n$)

$$\sum_{i=0}^{n} 2^i \binom{n}{i}\binom{k}{i} \quad = \quad \frac{2^n}{n!}\ k^n \ + \ \frac{2^{n-1}}{(n-1)!}\ k^{n-1} \ + \ O(k^{n-2}) \qquad 36{,}365 \qquad (2)$$

Largest-known Abelian Cayley graph families ($d = 2n$)

$$= \ \frac{1}{2}\left(\frac{4}{n}\right)^n k^n \ + \ \left(\frac{4}{n}\right)^{n-1} k^{n-1} \ + \ O(k^{n-2}) \qquad 22{,}805 \qquad (3)$$

The ratio, $(2)/(3)$, of leading coefficients gives the asymptotic density of the corresponding lattice covering for large diameter

$$\frac{n^n}{2^{n-1}n!} \qquad \simeq \quad \sqrt{\frac{2}{\pi n}}\left(\frac{e}{2}\right)^n \ \approx \ 1.359^n$$

Largest-known graphs and lattice coverings of lowest-known density

Largest-known circulant graphs correspond to lattice coverings of n-dimensional integer space with Lee spheres having an asymptotic density of about 1.359^n.

The order of largest-known Abelian Cayley graphs of degree $2n$

Leading term $\frac{1}{2}\left(\frac{4}{n}\right)^n k^n$

Density $\frac{n^n}{2^{n-1}n!} \approx 1.359^n$

Lattice coverings of n-dimensional integer space with Lee spheres having lowest-known density

Lattice coverings of n-dimensional real space with orthoplexes having lowest-known density

In real space, the analogue of the Lee sphere is the orthoplex

In n-dimensional real space $\mathbb{R}^n$, an *orthoplex* is the dual of the hypercube.
It is a highly symmetric convex body.

An orthoplex of radius r, centred at the origin, is the set of points distant at most r from the origin under the Manhattan norm:

$$C_{n,r} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq r\}.$$

From this definition, it is clear that the orthoplex is the real analogue of the Lee sphere:

$$S_{n,k} = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : |m_1| + \cdots + |m_n| \leq k\}.$$

In 2 dimensions, an orthoplex is a square diamond, and in 3, a regular octahedron.

The Voronoi cells of a body-centred cubic lattice are orthoplexes that touch at faces, truncated by hypercubes

In a lattice, the *Voronoi cell* of a lattice point is defined to be the closure of the set of all points in space that are closer to that lattice point than to any other. Thus Voronoi cells achieve a lattice tiling of space.

For an n-dimensional body-centred cubic lattice with cubes of edge length $2a$, the Voronoi cell (centred at the origin) is the polyhedron formed by the intersection of:

- an orthoplex of radius $r = na/2$ with vertices $(\pm\, na/2\,, 0, \ldots, 0)$ in all combinations.
- a hypercube of edge length $2a$ with vertices $(\pm a, \ldots, \pm a)$ in all combinations.

For dimension $n \leq 2$, the orthoplex lies fully within its hypercube.

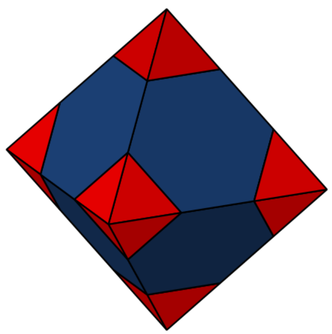
For dimension $n \geq 3$, the orthoplex is truncated by its hypercube to an increasing extent, corresponding to the density of the orthoplex covering.

The Voronoi cells of a body-centred cubic lattice are orthoplexes that touch, truncated by hypercubes

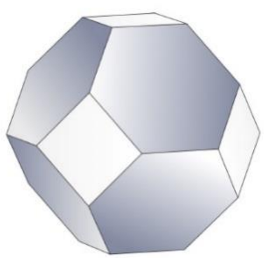
In 3-dimensions

The orthoplex
is an octahedron

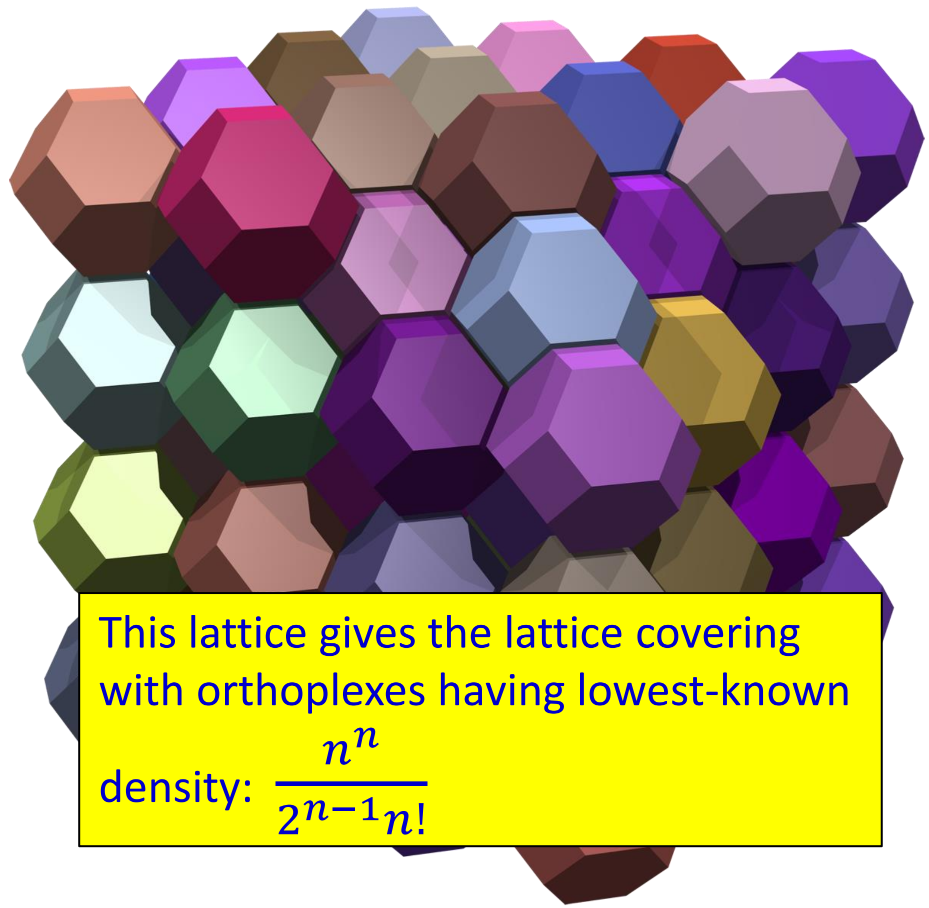
Red shows where
it is truncated
by its cube



Truncated
octahedron



Body-centred cubic lattice tiling
with truncated octahedrons



This lattice gives the lattice covering with orthoplexes having lowest-known density: $\dfrac{n^n}{2^{n-1}n!}$

Largest-known graphs are consistent with lattice coverings of lowest-known density

We have considered lattice coverings of real and integer n-dimensional space with lowest-known density, having a common asymptotic density of about 1.359^n.

The order of
largest-known Abelian Cayley
graphs of degree $2n$

Leading term
$$\frac{1}{2}\left(\frac{4}{n}\right)^n k^n$$

Density $\dfrac{n^n}{2^{n-1}n!} \approx 1.359^n$

Are they optimal?

Density $\dfrac{n^n}{2^{n-1}n!} \approx 1.359^n$

Lattice coverings of
n-dimensional integer space
with Lee spheres having
lowest-known density

Lattice coverings of
n-dimensional real space
with orthoplexes having
lowest-known density

Upper bounds $UB(n)$ for the lattice covering density $\theta_L(K)$ of an arbitrary convex body K in $\mathbb{R}^n$

From 1949 to 1958, various authors published improved upper bounds $UB(n)$ for the lattice covering density $\theta_L(K)$ of an arbitrary convex body K in $\mathbb{R}^n$.

Year	$UB(n)$	Author
1949	n^n	Hlawka
1950	3^{n-1}	Rogers
1952	e^n	Bambah & Roth (for certain symmetric bodies)
1958	2^n	Rogers
1958	1.8774^n	Rogers

All these results are exponential in the dimension.

As they all exceed 1.359^n, they are consistent with the conjecture that the body-centred cubic lattice is the optimal lattice for a lattice covering with orthoplexes.

Upper bounds $UB(n)$ for the lattice covering density $\theta_L(K)$ of an arbitrary convex body K in $\mathbb{R}^n$

Since 1959, Rogers and other authors have published upper bounds $UB(n)$ for the lattice covering density $\theta_L(K)$ of an arbitrary convex body K in $\mathbb{R}^n$ that are sub-exponential in the dimension.

Year	$UB(n)$	Author
1959	$n^{\log_2 \log_e n + c}$	Rogers
1964	$n^{\log_2 n + c\log_2 \log_2 n}$	Rogers (weaker, but easier to prove)
1985	$cn(\log_e n)^{1+\log_2 e}$	Gritzmann (for certain symmetric bodies)
2020	cn^2	Ordentlich, Regev & Weiss

These results contradict the conjecture that the body-centred cubic lattice covering with an orthoplex (with asymptotic density 1.359^n) is optimal.

They imply that much larger Abelian Cayley graphs exist.
However, none of these proofs is constructive.

All four sub-exponential upper bounds depend in part on theorems by Rogers

The proof of both of Rogers' theorems depend on a hierarchy of other theorems.

The proof by Gritzmann is a refinement of Rogers' proofs.

Ordentlich, Regev & Weiss's proof takes a different approach but also depends on a theorem in Rogers' proof.

The result in Rogers' 1964 book is presented as Theorem 5.8

The key to Rogers' approach involves the definition of a *hypercylinder* inscribed within the convex body K.

This hypercylinder is formed as the Cartesian product $H \times C$ of a convex body H in $(n-m)$-dimensional space and a hypercube C in m-dimensional space, where $m = [\log_2 n + \log_2 \log_2 n + 1]$.

There appear to be inconsistencies in the proof of Roger's theorem that relate to the edge length of the hypercube

The proof of Theorem 5.8 depends on Theorem 5.6,
and the proof of Theorem 5.6 includes iteration of Theorem 5.5.

An inconsistency within Theorem 5.6

In the statement of Theorem 5.6, the hypercube has arbitrary edge length, as required in support of the main theorem, Theorem 5.8.

However, the proof assumes the edge length has a fixed value of 1.

An inconsistency between Theorems 5.5 and 5.6

The proof of Theorem 5.6 invokes Theorem 5.5 with the assumption that the edge length of the hypercube is 1.

However, in the statement of Theorem 5.5, the edge length has the value 2.

Is the statement of Rogers' Theorem 5.8 valid?

It appears that the *proof* of Rogers' Theorem 5.8 may be invalid.

But perhaps the *statement* of the theorem remains valid?

Orthoplexes are highly symmetric convex bodies, arguably the most symmetric polyhedra alongside hypercubes.

So if Theorem 5.8 were correct, then it should be relatively straightforward to construct a lattice covering with orthoplexes that has sub-exponential density.

But this has not been achieved during the 64 years since Rogers' work was published.

So perhaps the body-centred cubic lattice is optimal for orthoplexes?

Open question in two equivalent formats

What is the order of extremal Abelian Cayley graph families of dimension n?

- As for largest-known families, with order defined by a polynomial of degree n in the diameter k with leading term $\frac{1}{2}\left(\frac{4}{n}\right)^{n} k^{n}$, corresponding to lattice covering densities of 1.359^{n}?
- Or significantly larger, closer to the Abelian Cayley Moore bound, corresponding to lattice covering densities below cn^{2} for some constant c?

What is the minimum-density lattice covering of $\mathbb{R}^{n}$ with orthoplexes?

- A body-centred cubic lattice, with lattice covering density of 1.359^{n}, corresponding to extremal Abelian Cayley graphs with order defined by a polynomial with the same leading term as for largest-known graphs?
- Or with significantly lower density, below cn^{2} for some constant c?

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