All McKay-Miller-Širán graphs are lifts of dipoles

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Introduction to McKay-Miller-Širáň (MMS) graphs

In 1998, McKay, Miller and Širáň defined an infinite family of graphs H_q where $q > 2$ is a prime power.

The construction is motivated by the degree-diameter problem.

These graphs have diameter 2, degree $(3q - \delta)/2$ and order $2q^2 = \frac{8}{9}(d + \delta/2)^2$ where $q = 4k + \delta$, $\delta \in \{1, 0, -1\}$.

For $\delta = 1$ they are vertex-transitive and non-Cayley. For $q \geq 7$, $\delta = -1$ and $\delta = 0$ they are not vertex-transitive but they are still highly symmetric.

H_5 is the Hoffman-Singleton graph which meets the Moore bound. H_4 is one of the graphs with degree 6, diameter 2 and order 32. This is known to be the largest possible size with those parameters.

Constructions of MMS graphs

In the original paper H_q are constructed as lifts of $K_{q,q}$.

In 2001, Šiagiová constructed graphs for $q = 4k + 1$ as lifts of dipoles.

In 2004, Hafner described them geometrically by using incidence graphs of finite biaffine planes and used this description to obtain their automorphism groups.

Using ideas from both Šiagiová and Hafner, we will show that all McKay-Miller-Širáň graphs are lifts of dipoles.

Voltage graph and their lifts

We need:

Base graph: Γ is a directed graph and $D(\Gamma)$ is the set of its arcs (darts). Multiple edges, loops and semiedges are allowed.

Voltage group: G is a finite group.

Voltage assignment: A function $\alpha : D(\Gamma) \to G$ that satisfies the condition: for each pair of opposite arcs e, e^{-1} we have $\alpha(e^{-1}) = \alpha(e)^{-1}$.

Voltage graph and their lifts

Given these three things we can construct the graph Γ^{α} called the lift of Γ by G or the derived graph.

Its set of vertices is $V(\Gamma) \times G$ and its set of arcs is $D(\Gamma) \times G$.

The adjacency in the lift is defined as follows: If e is an arc connecting u to v, then (e, g) connects the vertex (u, g) to the vertex $(v, g\alpha(e))$.

If the original graph Γ is undirected, then its lift is undirected as well because (e, g) and $(e^{-1}, g\alpha(e))$ are mutually reverse arcs.

The function $p : \Gamma^{\alpha} \to \Gamma$ defined as $p(v, g) = v$ is a regular covering projection.

Examples of lifts

Each Cayley graph is a lift of a graph with one vertex, loops and semiedges. The voltages on a loop are a pair of different inverse elements in the generating set while the voltage on a semiedge is an element from the generating set, whose order is 2.

The Petersen graph is a lift of a dipole with one edge and two loops, one on each vertex. The voltage group is $\mathbb{Z}_5$. The voltages on loops are ± 1 and ± 2, while the voltage on the edge is 0.

The bipartite double cover is a lift by the group $\mathbb{Z}_2$ where all arcs are assigned the voltage 1.

Theorem concerning lifts and graph automorphisms

This is a theorem that will be used later to prove our result.

Graph Γ is a lift with voltages in a group G if and only if there exists a semiregular action of G on vertices of Γ.

Original construction of MMS graphs

Let $q > 2$ be a prime power of the form $q = 4k + \delta$ where $\delta \in \{-1, 0, 1\}$, let F denote the field $GF(q)$ and let ξ be a primitive element of $(F^*, \cdot)$.

Base graph is the complete bipartite graph $K_{q,q}$ with either k loops attached at each vertex when $\delta = \pm 1$ or $2k$ semiedges attached at each vertex when $\delta = 0$.

Label the vertices with elements of $\mathbb{Z}_2 \times F$.

The voltage group is the additive group of F.

Original construction of MMS graphs

The voltage on arc going from $(0, x)$ to $(1, y)$ is defined as the product xy.

The voltages on loops or semiedges are defined using the following sets:

$\delta = 1$: $\quad X = \{1, \xi^2, \ldots, \xi^{2k-2}\}$; $\quad X' = \{\xi, \xi^3, \ldots, \xi^{2k-1}\}$

$\delta = -1$: $\quad X = \{1, \xi^2, \ldots, \xi^{2k-2}\}$; $\quad X' = \{1, \xi, \xi^3, \ldots, \xi^{2k-3}\}$

$\delta = 0$: $\quad X = \{1, \xi^2, \ldots, \xi^{4k-2}\}$; $\quad X' = \{\xi, \xi^3, \ldots, \xi^{4k-1}\}$

For any vertex $(0, x)$ we assign the elements of X to the attached loops/semiedges in any 1 to 1 way.

Similarly, for any vertex $(1, y)$ we assign the elements of X' to the attached loops/semiedges in any 1 to 1 way.

The associated lift is the McKay-Miller-Širáň graph H_q.

Šiagiová's dipole construction

Let q be a prime power of the form $q = 4k + 1$, let F denote the field $GF(q)$ and let ξ be a primitive element of $(F^*, \cdot)$. Let $p(x)$ be a quadratic polynomial over F.

The base graph contains just two vertices u and v, q edges between them and k loops attached to each vertex.

The voltage group is $F \times F$ with addition as the operation, i.e vectors of length 2 over F.

Šiagiová's dipole construction

The voltages on q arcs from u to v are of the form $(x, p(x))$ for $x \in F$.

The voltages on loops are defined using the following sets. These are the same as X, X' in the original construction for $\delta = 1$.

$X = \{1, \xi^2, \ldots, \xi^{2k-2}\}; \quad X' = \{\xi, \xi^3, \ldots, \xi^{2k-1}\}$

To all the loops attached to u assign voltages of the form $(0, x)$ where $x \in X$.

To all the loops attached to v assign voltages of the form $(0, x')$ where $x' \in X'$.

Theorem (Šiagiová 2001)

The associated lift is isomorphic to the graph H_q.

Hafner's results

Now we will discuss Hafner's work on the family of McKay-Miller-Širáň graphs.

First, let's quickly introduce some concepts from affine geometry.

Affine and biaffine planes

Let $q > 2$ be a prime power and let F denote the field $GF(q)$. The classical affine plane of order q has points (x, y) where $x, y \in F$ and lines defined by equations of the form $ax + by = c$ where $a, b, c \in F$. These lines split into $q + 1$ parallel classes.

If we remove a class of lines, then we obtain a structure called the biaffine plane. We will remove the class of vertical lines of the form $x = c$.

For both the affine plane and the biaffine plane we can construct an incidence graph in the usual way.

Collineations of the classical affine plane

A bijective map between two planes that transforms collinear points into collinear points is called a collineation.

Examples of collineations in the classical affine plane are translations, linear transfromations and field automorphisms.

Theorem

Every collineation in the classical affine plane can be uniquely written as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x^{\rho} \\ y^{\rho} \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

If the field automorphism ρ is equal to identity, then we call f an affine transformation (linear transformations together with a translation).

Collineations of the biaffine plane

These are those collineations of the underlying affine plane which preserve the parallel class of lines $x = c$.

All translations and field automorphisms meet this criteria but only some linear transformations.

The general form is

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x^\rho \\ y^\rho \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

When working with the incidence graph of the biaffine plane, there are also graph automorphisms which exchange the vertices belonging to points and lines. These automorphism are called correlations and they can exist because in the biaffine plane there equal number of points and lines.

Hafner's construction

Let $q > 2$ be a prime power, let F denote the field $GF(q)$ and let ξ be a primitive element of $(F^*, \cdot)$.

Also define the sets X and X' as before.

Let $V_q = \mathbb{Z}_2 \times F \times F$ be the vertex set and define the edges as:

$(0, x, y)$ is adjacent to $(1, m, c)$ if and only if $y = mx + c$.

This construction is precisely the incidence graph of the biaffine plane over F.

The vertex $(0, x, y)$ corresponds to the point (x, y) while the vertex $(1, m, c)$ corresponds to the line $y = mx + c$.

Hafner's construction

To obtain MMS graphs we need to define additional edges within partitions:

$(0, x, y)$ is adjacent to $(0, x, y')$ if and only if $y - y' \in X \cup -X$;

$(1, m, c)$ is adjacent to $(1, m, c')$ if and only if $c - c' \in X' \cup -X'$.

Theorem (Hafner 2004)

The resulting graph is isomorphic to H_q.

Hafner's results about automorphism groups

Let $q = p^n > 2$ be a power of prime p.

The graph H_3 is vertex transitive and its automorphism group has order 216.

The graph H_4 is vertex transitive and its automorphism group has order 1920.

The graph H_5 (the Hoffman–Singleton graph) is vertex transitive, its affine automorphism group has order 2000 while the full automorphism group has order 252000.

If $q = 4k - 1$ and $q > 3$ the automorphism group of H_q is not transitive. It is transitive on point vertices and on line vertices and has order $2(q-1)q^3$; all automorphisms are induced by affine transformations of the point vertices.

If $q = 4k$ and $q > 4$ the automorphism group of H_q is not transitive. It is transitive on point vertices and on line vertices and has order $(q-1)q^3$; all automorphisms are induced by affine transformations of the point vertices.

If $q = 4k + 1$ and $q > 5$ the automorphism group of H_q is transitive. Its order is $n(q-1)^2q^3$; all automorphisms are induced by collineations and correlations of the underlying biaffine plane.

Our results

Šiagiová's simpler construction is defined only for MMS graphs with $q = 4k + 1$. It would be convenient to extend it to all MMS graphs.

According to the previously mentioned theorem, we need to find an automorphism group that acts semiregularly on the vertices of MMS graphs.

H_q has $2q^2$ vertices and we want a base graph with just 2 vertices. It follows that the automorphism group should have order q^2.

Using Hafner's description of MMS graphs, we need a collineation group of order q^2 that acts semiregularly on both points and lines of the biaffine plane.

Some subgroups of $Aut(H_q)$

From results of Hafner it follows that all MMS graphs have an automorphism group of order q^3 comprised of collineations of the following type:

$$f(x) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} x + \begin{pmatrix} b \\ c \end{pmatrix}$$

This is basically the Heisenberg group.

An obvious subgroup of order q^2 is the subgroup of all translations. But it doesn't work since a non-trivial translation can map a line to itself.

What does meet our criteria is a group consisting of collineations of the form:

$$f(x) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} x + \begin{pmatrix} a \\ c \end{pmatrix}$$

A group with semiregular action

Theorem

Let $q > 2$ be a prime power and let $F = GF(q)$. For $a, c \in F$ let $[a, c]$ be the collineation

$$f(x) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} x + \begin{pmatrix} a \\ c \end{pmatrix}.$$

Then these collineations form a group

$$G_q = \{[a, c];\ a, c \in F\}$$

and it acts semiregularly on the vertices of the graph H_q as group of automorphisms with 2 orbits: points and lines.

Corollary

Each McKay-Miller-Širáň graph H_q is a lift of a dipole by the group G_q.

The structure of G_q

The group G_q has different structure depending on whether q is odd or even.

Theorem

G_q is abelian for all q.

If $q = p^n$ for some odd prime p then $G_q \cong (F \times F, +) \cong (\mathbb{Z}_p)^{2n}$.

If $q = 2^n$, then $G_q \cong (\mathbb{Z}_4)^n$.

Voltage assignments on dipoles

From the correspondence between semiregular actions and voltage assignments we can obtain the voltage assignments on dipole that lift to H_q.

Voltages on loops or semiedges attached to the first vertex u are $[0, x]$ for $x \in X$.

Voltages on loops or semiedges attached to the second vertex v are $[0, x']$ for $x' \in X'$.

Voltages on edges going from u to v are $[a, a^2]$ where $a \in F$.

References

[1] P. R. Hafner.
Geometric realisation of the graphs of McKay–Miller–Širáň.
Journal of Combinatorial Theory, Series B, 90(2):223–232, 2004.

[2] J. H. Kwak and R. Nedela.
Graphs and their coverings.
Lecture Notes Series 17, 2007.

[3] B. D. McKay, M. Miller, and J. Širáň.
A note on large graphs of diameter two and given maximum degree.
Journal of Combinatorial Theory, Series B, 74(1):110–118, 1998.

[4] J. Šiagiová.
A note on the McKay–Miller–Širáň graphs.
Journal of Combinatorial Theory, Series B, 81(2):205–208, 2001.

Thank you for your attention.