On the algebra of token graphs

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Two parts

On the Laplacian spectra of token graphs

On the algebra of token graphs

Abstract

Given a graph $G = (V, E)$ and an integer $k \in [1, n-1]$, its token graph $F_k(G)$ has vertices corresponding to the k-subsets of V, and two vertices are adjacent when its symmetric difference are the end-vertices of an edge in E.

In this talk, we describe some properties of the Laplacian matrices of $F_k(G)$ and $F_k(\overline{G})$. In particular, we study the closed relationship between the algebra of the pair $(F_k(G), F_k(\overline{G}))$ with the Bose-Mesner algebra of the Johnson graph $J(n, k)$.

Graphs, spectra, and orthogonal polynomials

Given a graph G with adjacency matrix $\boldsymbol{A}$ and spectrum

$$\operatorname{sp} \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\},$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$.
The *predistance polynomials* $p_0, p_1, \ldots, p_d$, are a sequence of orthogonal polynomials, $\deg p_i = i$, with respect to the scalar product

$$\langle f, g \rangle_A = \frac{1}{n} \operatorname{tr}(f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i),$$

normalized in such a way that

$$\|p_i\|_A^2 = p_i(0).$$

The Laplacian predistance polynomials

Let G be a graph on n vertices, with Laplacian matrix $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ and spectrum

$$\operatorname{sp} \boldsymbol{L} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}.$$

where $\lambda_0 (= 0) < \lambda_1 < \cdots < \lambda_d$.
The Laplacian predistance polynomials $q_0, q_1, \dots, q_d$ are a sequence of orthogonal polynomials, $\operatorname{dgr} q_i = i$, with respect to the scalar product

$$\langle f, g \rangle_L = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i),$$

normalized in such a way that

$$\|q_i\|_L^2 = q_i(0).$$

In both cases,

- $\|1\|_A = \|1\|_L = 1.$
- To find them, apply Gram-Schmidt to $1, x, x^2, \ldots, x^d$ and normalize accordingly.

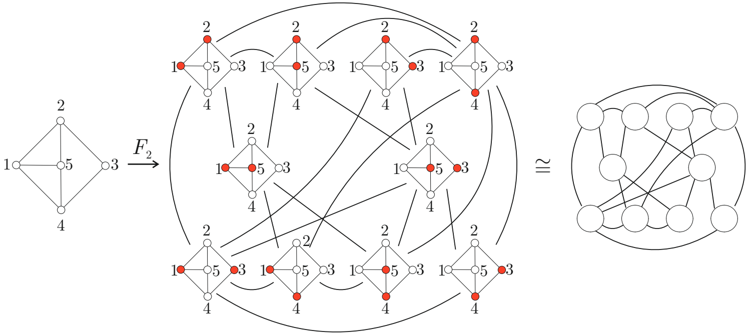
Why ‘predistance’ polynomials?

Token graphs

Let G be a (simple) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \leq k \leq n$, the **k-token graph** $F_k(G)$ of G is the graph in which

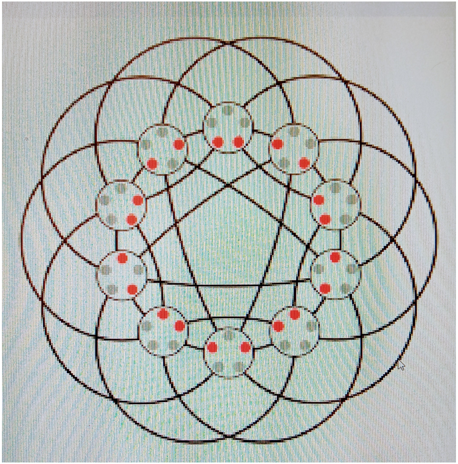
- the **vertices** of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G,
- two configurations are **adjacent** whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex.

An example of 2-token graph



- $F_0(G) := \{u\}$.
- $F_1(G) = G$.
- $F_k(G) = F_{n-k}(G)$.
- $F_k(K_n) = J(n,k)$ (The Johnson graph).
- $J(n,1)$?, $J(4,2)$?, $J(5,2)$?,...

The Johnson graph $J(5,2) = F_2(K_5)$

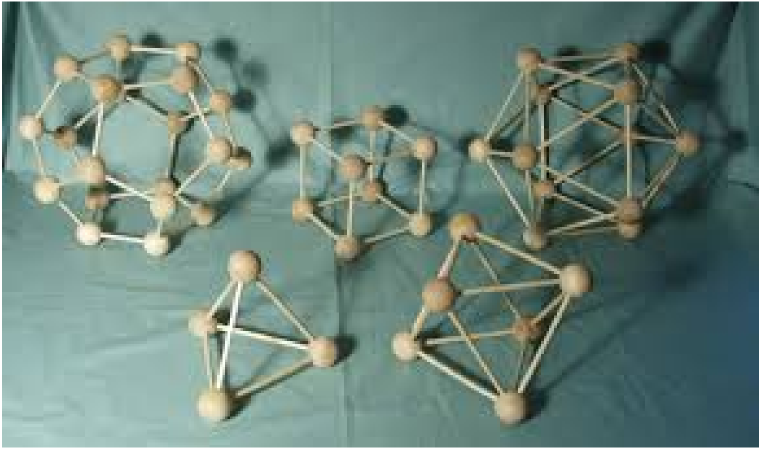


Distance-regular graphs

A graph G with diameter d is distance-regular if, for any pair of vertices u, v, the intersection parameters $a_i(u) = |G_i(u) \cap G(v)|$, $b_i(u) = |G_{i+1}(u) \cap G(v)|$, and $c_i(u) = |G_{i+1}(u) \cap G(v)|$, for $i = 0, 1, \ldots, d$, only depend on the distance $\operatorname{dist}(u, v) = i$.
Then, we have the intersection array

$$\iota(G) = \left\{ \begin{array}{ccccc} - & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & - \end{array} \right\} \qquad (1)$$

Distance-regular graphs



Another chracterization

A graph Γ with diameter d and distance matrices

$$\boldsymbol{A}_0(=I), \boldsymbol{A}_1(=\boldsymbol{A}), \ldots, \boldsymbol{A}_d,$$

is distance-regular if and only if there exists sequence of (orthogonal) polynomials $p_0, p_1, \ldots, p_d$ such that

$$\boldsymbol{A}_i = p_i(\boldsymbol{A}), \qquad i = 0, 1, \ldots, d.$$

The Bose-Mesner algebra

Let G be a graph with diameter D, distance matrices $\boldsymbol{A}_0(=\boldsymbol{I}), \boldsymbol{A}_1(=\boldsymbol{A}), \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D$, and $d+1$ distinct eigenvalues. Consider the vector spaces

$$\mathcal{A} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\},$$
$$\mathcal{D} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D\},$$

with dimensions $d+1$ and $D+1$, respectively.

- $\mathcal{A}$ is an algebra with the ordinary product of matrices, known as the adjacency algebra of G.
- $\mathcal{D}$ is an algebra with the entrywise (or Hadamard product) of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = \boldsymbol{X}_{uv}\boldsymbol{Y}_{uv}$, called the distance $\circ$-algebra of G.

Theorem

Let G, $\mathcal{A}$, and $\mathcal{D}$ as above. Then, G is distance-regular if and only if

$$\mathcal{A} = \mathcal{D},$$

so that $\mathcal{A}$ is an algebra with both, the ordinary product and the Hadamard product of matrices, and it is known as the Bose-Mesner algebra associated to G.

Johnson graphs again

- If G is the complete graph K_n, then $F_k(K_n) \simeq J(n,k)$, is the **Johnson graph**.
- The Johnson graph $J(n,k)$, with $k \leq n-k$ is a distance-regular graph with degree $k(n-k)$, diameter $d = k$, and intersection parameters

$$b_j = (k-j)(n-k-j); \quad c_j = j^2, \quad j = 0, 1, \ldots, d.$$

- The Laplacian spectrum of $J(n,k)$ is

$$\lambda_j = j(n+1-j), \quad m_j = \binom{n}{j} - \binom{n}{j-1}, \quad j = 0, 1, \ldots, k.$$

For instance, the Laplacian eigenvalues of $J(n,4)$ are

$$0, \quad n, \quad 2(n-1), \quad 3(n-2), \quad 4(n-3).$$

Some properties of Johnson graphs

- $J(n,k)$ is isomorphic to $J(n,n-k)$
- Any pair of vertices are at distance j, with $0 \leq j \leq k$, if and only if they share $k-j$ elements in common.
- The Johnson graph $J(n,k)$ is maximally connected, that is, $\kappa = k(n-k)$.

The (n, k)-binomial matrix

Given integers n and $k \leq n$, the $(n; k)$-**binomial matrix** is an $\binom{n}{k} \times n$ matrix whose rows are the characteristic vectors of the k-subsets of $[n] = \{1, \ldots, n\}$ in a given order.
For instance, for $n = 4$ and $k = 2$,

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

If $\boldsymbol{J}$ is the all-1 matrix,

$$\boldsymbol{B}^\top \boldsymbol{B} = \binom{n-2}{k-1} \boldsymbol{I} + \binom{n-2}{k-2} \boldsymbol{J}.$$

The Laplacian spectra of token graphs

Theorem (Dalfó, Duque, Fabila-Monroy, F., Huemer, Trujillo-Negrete, Zaragoza, 2021)

Let G be a graph with Laplacian matrix $\boldsymbol{L}_1$. Let $F_k = F_k(G)$ be its token graph with Laplacian $\boldsymbol{L}_k$. Then, the following holds:

- $\boldsymbol{B}\boldsymbol{L}_1 = \boldsymbol{L}_k\boldsymbol{B}$,
- $\boldsymbol{L}_1 = (\boldsymbol{B}^\top\boldsymbol{B})^{-1}\boldsymbol{B}^\top\boldsymbol{L}_k\boldsymbol{B} = \frac{1}{\binom{n-2}{k-1}}\boldsymbol{B}^\top\boldsymbol{L}_k\boldsymbol{B}$,
- *The column space (and its orthogonal complement) of $\boldsymbol{B}$ is $\boldsymbol{L}_k$-invariant.*
- *The characteristic polynomial of $\boldsymbol{L}_1$ divides the characteristic polynomial of $\boldsymbol{L}_k$. Thus, $\operatorname{sp}\boldsymbol{L}_1 \subseteq \operatorname{sp}\boldsymbol{L}_k$.*
- *If $\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_1$, then $\boldsymbol{B}\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_k$.*
- *If $\boldsymbol{u}$ is a λ-eigenvector of $\boldsymbol{L}_k$ such that $\boldsymbol{B}^\top\boldsymbol{u} \neq \boldsymbol{0}$, then $\boldsymbol{B}^\top\boldsymbol{u}$ is a λ-eigenvector of $\boldsymbol{L}_1$.*

The Laplacians of a graph and its complement

Proposition (DDFFHTZ, 2021)

Let $G = (V, E)$ *be a graph on* $n = |V|$ *vertices, and let* $\overline{G}$ *be its complement. For a given* k*. Then the Laplacian matrices of their token graphs* $\boldsymbol{L}_k = \boldsymbol{L}(F_k(G)$ *and* $\overline{\boldsymbol{L}}_k = \boldsymbol{L}(F_k(\overline{G})$ *commute*

$$\boldsymbol{L}_k \overline{\boldsymbol{L}}_k = \overline{\boldsymbol{L}}_k \boldsymbol{L}_k.$$

In general, this does NOT hold for the respective adjacency matrices.

Corollary (by a theorem of Frobenius)

The $\binom{n}{k}$ *eigenvalues of* $\boldsymbol{L}_k$ *and* $\overline{\boldsymbol{L}}_k$*, can be matched up as* $\lambda_i \leftrightarrow \overline{\lambda}_i$ *in such a way that the* n *eigenvalues of any polynomial* $p(\boldsymbol{L}_k, \overline{\boldsymbol{L}}_k)$ *in the two matrices is the multiset of the values* $p(\lambda_i, \overline{\lambda}_i)$*.*

Moreover, since the Laplacian matrix of $J(n,k)$ is $\boldsymbol{L}_J = \boldsymbol{L}_k + \overline{\boldsymbol{L}}_k$, $\boldsymbol{L}_J$ commutes with both $\boldsymbol{L}$ and $\overline{\boldsymbol{L}}$, and every eigenvalue λ_J of $J(n,k)$ is the sum of one eigenvalue λ of $F_k(G)$ and one eigenvalue $\overline{\lambda}$ of $F_k(\overline{G})$.

Pairing eigenvalues

Proposition

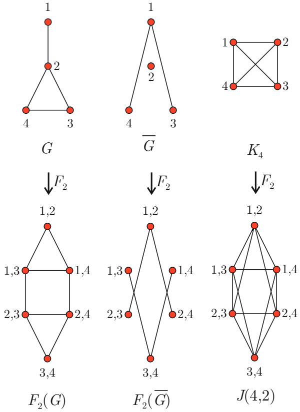
Let $\boldsymbol{L}_k$ and $\overline{\boldsymbol{L}}_k$ be the Laplacian matrices of $F_k(G)$ and $F_k(\overline{G})$, respectively. For $j = 0, 1, \ldots, k$,

- *Let $\lambda_j = j(n+1-j)$ and $m_j = \binom{n}{j} - \binom{n}{j-1}$ be the eigenvalues and multiplicities of $J(n,k)$.*
- *Let $\lambda_{j1} \leq \lambda_{j2} \leq \cdots \leq \lambda_{jm_j}$ be the eigenvalues in $\operatorname{sp} F_j(G) \setminus \operatorname{sp} F_{j-1}(G)$.*
- *Let $\overline{\lambda}_{j1} \geq \overline{\lambda}_{j2} \geq \cdots \geq \overline{\lambda}_{jm_j}$ be the eigenvalues in $\operatorname{sp} F_j(\overline{G}) \setminus \operatorname{sp} F_{j-1}(\overline{G})$.*

Then,

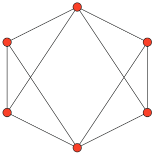
$$\lambda_{jr} + \overline{\lambda}_{jr} = \lambda_j \quad r = 0, 1, \ldots, m_j \tag{2}$$

Example 1

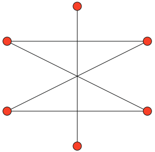


Spectrum	ev G	ev $\overline{G}$	ev Johnson
$\mathrm{sp}(F_0) = \mathrm{sp}(K_1)$	0	0	0
$\mathrm{sp}(F_1) - \mathrm{sp}(F_0)$	1	3	4
	3	1	4
	4	0	4
$\mathrm{sp}(F_2) - \mathrm{sp}(F_1)$	3	3	6
	5	1	6

Example 2



G



$\overline{G}$

Spectrum	ev G	ev $\overline{G}$	ev Johnson
$\mathrm{sp}(F_0) = \mathrm{sp}(K_1)$	0	0	0
$\mathrm{sp}(F_1) - \mathrm{sp}(F_0)$	2	4	6
	4	2	6
	4	2	6
	4	2	6
	6	0	6
$\mathrm{sp}(F_2) - \mathrm{sp}(F_1)$	4	6	10
	4	6	10
	6	4	10
	6	4	10
	6	4	10
	8	2	10
	8	2	10
	8	2	10
	10	0	10
$\mathrm{sp}(F_3) - \mathrm{sp}(F_2)$	4	8	12
	8	4	12
	8	4	12
	10	2	12
	10	2	12

Lemma

Let $\boldsymbol{B}$ be the (n,k)-binomial matrix, and let $\boldsymbol{A}_0(=\boldsymbol{I}), \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ be the distance matrices of the Johnson graph $J(n,k)$, $k \le n-k$. Then

$$\boldsymbol{M} = \boldsymbol{B}\boldsymbol{B}^\top = \sum_{i=0}^{k-1} (k-i)\boldsymbol{A}_i.$$

Corollary

The matrices $\boldsymbol{M}$, $\boldsymbol{L}_k$, $\overline{\boldsymbol{L}}_k$, and $\boldsymbol{L}_J$ commute with each other.

On the algebra of token graphs

Theorem

Let G and $\overline{G}$ be a graph and its complement on n vertices. For some $k \leq n/2$, let $\boldsymbol{L}_k$ and $\overline{L_k}$ be Laplacian matrices of the token graphs $F_k(G)$ and $F_k(\overline{G})$, respectively.

Let $\mathcal{L}(G) = \mathbb{R}[\boldsymbol{L}_k, \overline{\boldsymbol{L}}_k]$ be the $\mathbb{R}$-vector space of the $\binom{n}{k} \times \binom{n}{k}$ matrices $M_n(\mathbb{R})$ generated by $\boldsymbol{L}_k$ and $\overline{\boldsymbol{L}}_k$. Then, the following hold:

(i) *$\mathcal{L}(G)$ is a unitary commutative algebra.*

(ii) *The Bose-Mesner algebra of the Johnson graph $J(n, k)$ is a subalgebra of $\mathcal{L}(G)$.*

Theorem (cont.)

(iii) *The dimension of $\mathcal{L}(G)$ is the number of different pairs $(\lambda_{jr}, \overline{\lambda}_{jr})$, for $j = 0, \ldots, k$ and $r = 0, \ldots, m_j$, defined in Proposition 2.*

(iv) *If $\dim(\mathcal{L}(G)) = r$, then there exists a (non-unique) matrix such that*

$$\{\boldsymbol{I}, \boldsymbol{R}, \boldsymbol{R}^2, \ldots, \boldsymbol{R}^r\} \quad \textit{and} \quad \{\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_r\},$$

(where the $\boldsymbol{E}_i$'s are the idempotents of $\boldsymbol{R}$) are bases of $\mathcal{L}(G)$.

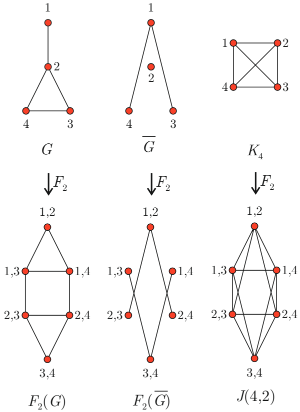
From (ii) and (iii), notice that

$$k + 1 \leq \dim(\mathcal{L}(G)) \leq \binom{n}{k}.$$

(In fact Gerstenhaber (1961) as well as Motzkin and Taussky-Todd (1955) proved independently that the variety of a commuting pair of matrices $\boldsymbol{A}$, $\boldsymbol{B}$ is irreducible, so that its dimension is also bounded above by the size of the matrices)

The pair $(\boldsymbol{L}_k, \overline{\boldsymbol{L}}_k)$ generates an algebraic variety (that is, the collection of all common eigenvectors shared by the two matrices).

Example 1

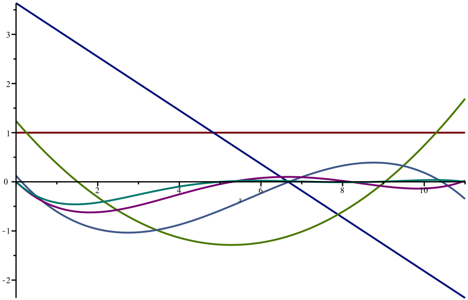


ev $\boldsymbol{L}_2$	ev $\overline{\boldsymbol{L}_2}$	ev $\boldsymbol{L}_J$	ev $\boldsymbol{R} = 2\boldsymbol{L}_2 + \overline{\boldsymbol{L}_2}$
0	0	0	**0**
1	3	4	**5**
3	1	4	**7**
4	0	4	**8**
3	3	6	**9**
5	1	6	**11**

Then, for every matrix $\boldsymbol{M} \in \mathcal{L}(G)$, there exists a polynomial $p \in \mathbb{R}^6[x]$ such that $p(\boldsymbol{R}) = \boldsymbol{M}$. In particular $H_L(\boldsymbol{R}) = \boldsymbol{J}$.

The Laplacian predistance polynomials of $\boldsymbol{R}$ are

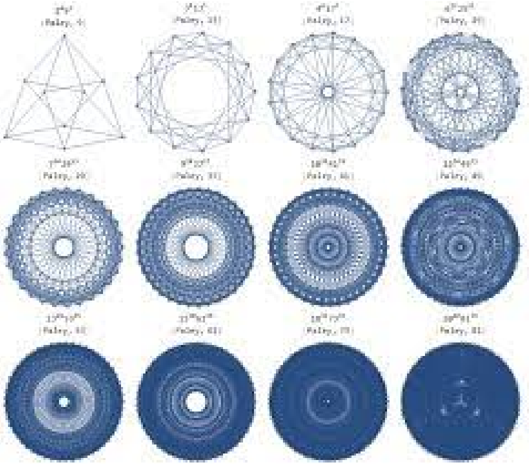
$$p_0(x) = 1,$$
$$p_1(x) = \frac{1}{11}(-6x + 40),$$
$$p_2(x) = \frac{1}{9097}(825x^2 - 8700x + 11250),$$
$$p_3(x) = \frac{1}{205747676}(-2691885x^3 + 46442340x^2 - 193990839x + 25428060),$$
$$p_4(x) = \cdots$$
$$p_5(x) = \cdots$$



Problems and (possible) future work

- Use this algebra in the context of codes or designs.
- Prove (or disprove) that all distance-regular graphs with the same parameters have cospectral 2-token (symmetric square) graphs.
- What about $\boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{L}_3 = \boldsymbol{L}_J$?
- Consider the case when G and $\overline{G}$ are strongly regular or self-complementary graphs.

Payley graphs



Bibliography

1. K. Audenaert, C. Godsil, G. Royle, and T. Rudolph, Symmetric squares of graphs, *J. Combin. Theory B* **97** (2007) 74–90.
2. C. Dalfó, F. Duque, R. Fabila-Monroy, M. A. Fiol, C. Huemer, A. L. Trujillo-Negrete, F. J. Zaragoza Martínez, On the Laplacian spectra of token graphs, submitted, 2020, `https://arxiv.org/abs/2012.00808`.
3. R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, Token graphs, *Graphs Combin.* **28** (2012), no. 3, 365–380.
4. V. Nikiforov, The spectral radius of subgraphs of regular graphs, *Electron. J. Combin.* **14** (2007), #N20.

Thanks for your attention