Directed and mixed graphs with order close to the Moore bound

James Tuite
Joint work with Grahame Erskine, Geoffrey Exoo

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Motivation: girth problem for undirected graphs

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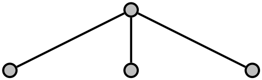

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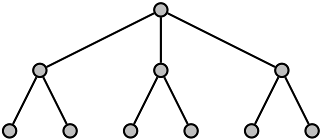

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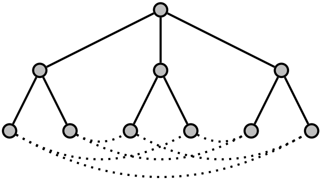

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So the lower bound (*Moore bound*) for graphs of girth 5 is $d^2 + 1$.
This is only achieved if we can connect the vertices in the last level so that the graph has girth 5.

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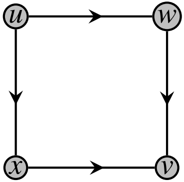

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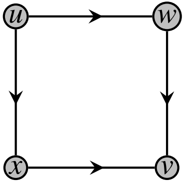
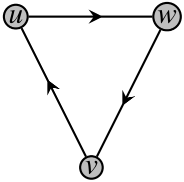


This digraph is not 3-geodetic.



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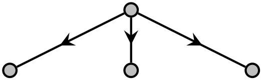

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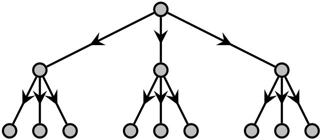

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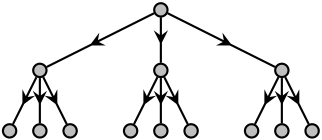

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This is impossible for any d, k except in the trivial cases $d = 1$ or $k = 1$ (same argument as Bridges and Toueg)

Definitions

The order of G will be $M(d,k)+\epsilon$, where ϵ is the excess of G.

A k-geodetic digraph with minimum out-degree d and excess ϵ is a $(d,k;+\epsilon)$-digraph.

A smallest possible k-geodetic digraph with out-degree $\geq d$ a geodetic cage.

The set of vertices that are at distance $> k$ from u is the outlier set $O(u)$ of u.

Results from the last IWONT

Theorem Sillasen

There are no diregular k-geodetic digraphs with degree $d = 2$ and excess $\epsilon = 1$.

Theorem Miller, Miret, Sillasen

All digraphs with excess one are diregular.

Theorem Miller, Miret, Sillasen

There are no $(d, k; +1)$-digraphs for $k = 2, 3, 4$.

The outlier function

Lemma

The outlier function is a digraph automorphism!

Proof

Define a matrix P by setting the (u, v)-entry equal to one iff $o(u) = v$. Let A be the adjacency matrix of G. Counting paths,

$$I + A + A^2 + \cdots + A^k = J - P.$$

G can be shown to be diregular, so A and J commute. Therefore $AP = PA$, so o is an automorphism.

Do geodetic cages exist?

The vertices of $P(d,k)$ are all permutations of length k from the alphabet $[d+k]$. We set $x_0x_1\dots x_{k-1} \rightarrow x_1x_2\dots x_{k-1}x_k$ iff $x_k \notin \{x_1, x_2, \dots, x_{k-1}\}$.

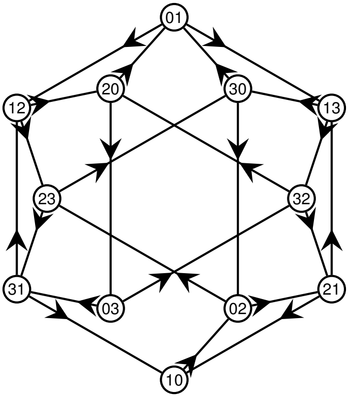

Figure 1: $P(2,2)$

Cages for $d = 2, k = 2$

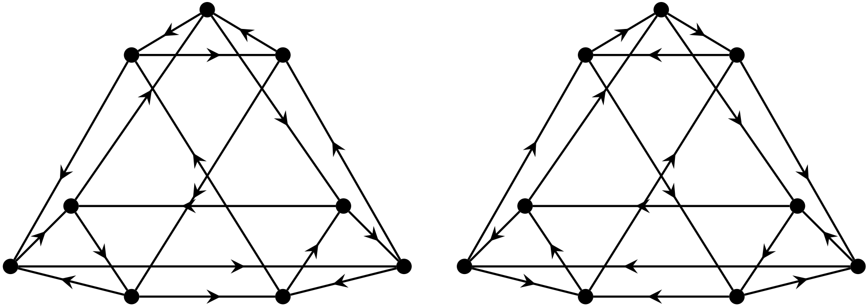

Figure 2: The two $(2, 2)$-geodetic-cages

Cages with $d = 2, k = 3$

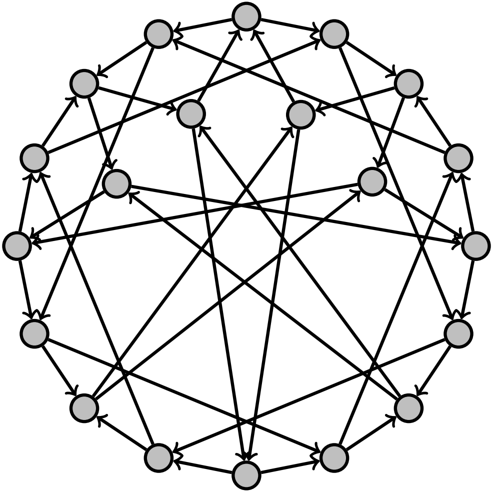

Figure 3: $d = 2, k = 3, \epsilon = 5$

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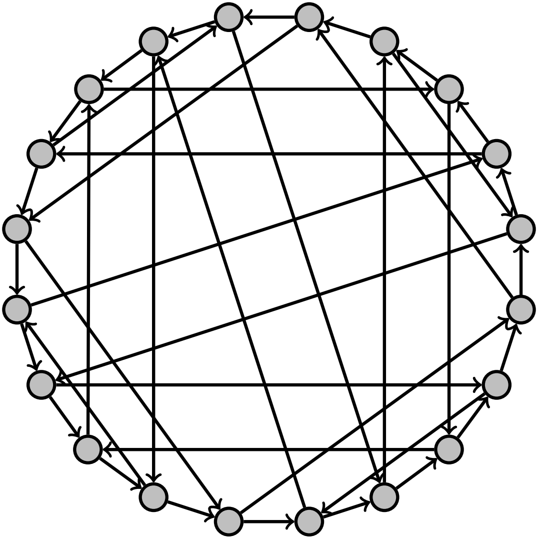
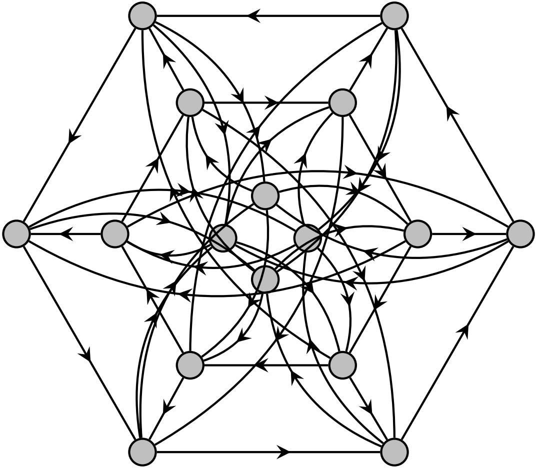


Figure 4: $d = 2, k = 3, \epsilon = 5$

The unique cage with $d = 3, k = 2$



What we know now

Theorem JT

There are no $(2,k;+2)$-digraphs for $k \geq 3$ (diregular or otherwise).

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The structure of $(3,k;+1)$-digraphs is 'highly constrained' (i.e. we know the permutation structure and there are divisibility conditions). There are no 'involutary' $(3,k;+1)$-digraphs. (N.B. This approach also settles the nonexistence of $(d,2;+1)$-digraphs for $3 \leq d \leq 7$).

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Question

Where next for digraphs?

Diregularity of $(d, 2; +2)$-digraphs

Let G be a 2-geodetic digraph with minimum out-degree d and excess $\epsilon = 2$. We can assume that $d \geq 4$. Let $S = \{v \in V(G) : d^-(v) < d\}$ and $S' = \{v' \in V(G) : d^-(v') > d\}$.

Definition

The *deficiency* $\sigma^-(v)$ of a vertex $v \in S$ is $\sigma^-(v) = d - d^-(v)$ and the *surplus* $\sigma^+(v')$ of a vertex $v' \in S'$ is $d^-(v') - d$.

Our aim is to show that the 'divergence' from diregularity is confined to a small range of values. We measure this divergence using the total deficiency

$$\sigma = \sum_{v \in S} \sigma^-(v) = \sum_{v' \in S'} \sigma^+(v').$$

We set $\psi(u) = \sum_{v \in N^-(u)} \sigma^-(v)$.

Facts

Fact one

The largest possible in-degree is $d + 2$. Each vertex $v' \in S'$ is the outneighbour of at least $\sigma^+(v')$ vertices of any outlier set $O(u)$.

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As there are at most d out-neighbours of any outlier set, $\sigma \leq 2d$.

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Fact

If $\sigma < 2d$, then the largest possible in-degree is $d+1$ and $\sigma = |S'|$.

Proof: Suppose that $d^-(v') = d+2$. There are at least $1 + d + d^2 + 2(d+1) - \sigma$ vertices that can reach v' by paths of length ≤ 2. Therefore

$$n = 3 + d + d^2 \geq d^2 + 3d + 3 - \sigma,$$

so $\sigma \geq 2d$.

Facts

Fact

$\sigma \geq d - 1$.

Proof: Let $v' \in S'$. If $d^-(v') = d + 2$ we are done, so take $d^-(v') = d + 1$. Counting paths of length ≤ 2 again, Thus

$$n = 3 + d + d^2 \geq 1 + d + d^2 + (d + 1) - \psi(v') = 2 + 2d + d^2 - \psi(v').$$

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Fact

If $\sigma \leq 2d - 2$, then S' is an independent set.

Proof: if $N^-(v') \cap S' \neq \emptyset$, then by the previous fact v' has deficiency $\geq d - 1$ in $N^-(v')$ and $\geq d - 1$ in $N^{-2}(v')$.

Facts

Fact

For $d+1 \leq \sigma \leq 2d-2$ or $\sigma = d-1$, no vertex $v' \in S'$ is an outlier.

Proof: Suppose that $O(u) = \{v', x\}$ for $v' \in S'$. Each vertex of S' is an out-neighbour of at least one vertex of $O(u) = \{v', x\}$. For this range of σ, S' is independent, so $S' \subseteq N^+(x)$ and hence $\sigma \leq d$.

Finishing the proof

Fact

The total deficiency takes one of the following seven values:

$$\sigma \in \{d-1, d, d+1, d+2, 2d-2, 2d-1, 2d\}.$$

Proof: suppose that $d+1 \leq \sigma \leq 2d-3$ and fix $v' \in S'$. Every vertex in S' has surplus one. v' is not an outlier, so every vertex of G is contained once in $T_{-2}(v')$. Also S' is independent, so $S' - \{v'\} \subset N^{-2}(v')$.

Suppose that a vertex u has ≥ 2 vertices of S' as in-neighbours. Then we would have $\sigma \geq 2d-2$, contradicting our assumption. Therefore each vertex in $N^{-}(v')$ has at most one in-neighbour in S'. As there are $d+1$ in-neighbours of v' and every vertex in S' has surplus one, it follows that $\sigma \leq d+2$.

Mixed graphs

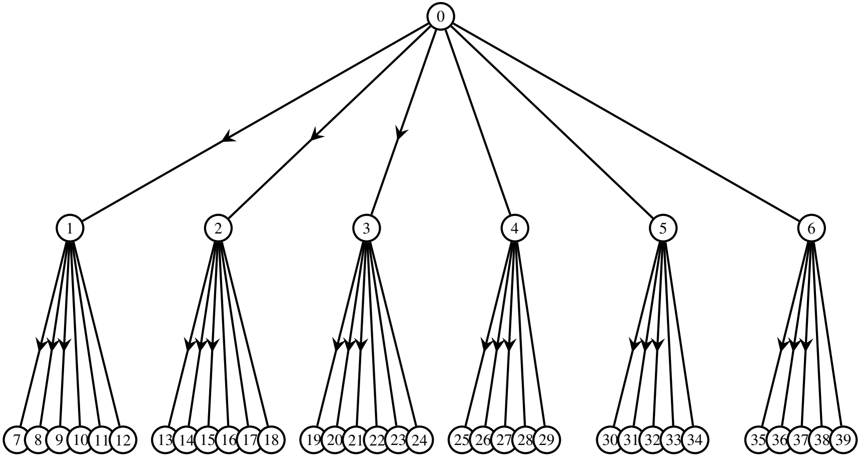

Figure 5: The Moore tree for $z = 3, r = 3, k = 2$

Geodecity problem for mixed graphs

A mixed graph is k-geodetic if for any ordered pair of vertices (u, v) there is at most one non-backtracking mixed walk of length $\leq k$ from u to v.

The undirected degree of a vertex u is the number of edges incident to u. The directed out-degree (in-degree) is the number of arcs from (to) u.

What is the smallest order of a k-geodetic mixed graph with undirected degree r and directed out-degree z? A minimum such graph is a mixed cage. We can prove that cages exist by truncation.

A k-geodetic mixed graph with undirected degree r, directed out-degree z and excess ϵ is an $(r, z, k; +\epsilon)$-graph.

The mixed Moore bound

$$M(r,z,k) = A\frac{u_1^{k+1}-1}{u_1-1} + B\frac{u_2^{k+1}-1}{u_2-1},$$

where $v = (z+r)^2 + 2(z-r) + 1$ and

$$u_1 = \frac{z+r-1-\sqrt{v}}{2}, u_2 = \frac{z+r-1+\sqrt{v}}{2}$$

$$A = \frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}}, B = \frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}},$$

Mixed Moore graphs with $k = 2$

The mixed Moore bound for $k = 2$ is $(r + z)^2 + z + 1$.

Theorem Nguyen, Miller, Gimbert, 2007

There are no mixed Moore graphs with $k \geq 3$.

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Theorem, after López & Miret

Let G be a totally regular $(r, z, 2, +1)$-graph. Then either:

- $r = 2$,
- $4r + 1 = c^2$ for some $c \in \mathbb{N}$ and $c|(16z^2 - 24z + 25)$, or
- $4r - 7 = c^2$ for some $c \in \mathbb{N}$ and $c|(16z^2 + 40z + 9)$.

Total regularity

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Definition

A mixed graph is totally regular if there exist r and z such that every vertex has undirected degree r and directed in- and out-degree z.

Theorem

Almost mixed Moore graphs are out-regular with undirected degree r and directed out-degree z.

Question

Are almost mixed Moore graphs totally regular?

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$S = \{v \in V(G) : d^{\rightarrow}(v) < z\}, S' = \{v' \in V(G) : d^{\rightarrow}(v') > z\}$.
$N^{+}(u) = \{w \in V(G) : u \sim w$ or $u \rightarrow w\}$,
$N^{-}(u) = \{w \in V(G) : u \sim w$ or $w \rightarrow u\}$.

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For all $u \in V(G)$ *we have* $S \subseteq N^+(r(u))$ *and* $S' \subseteq r(N^+(u))$.

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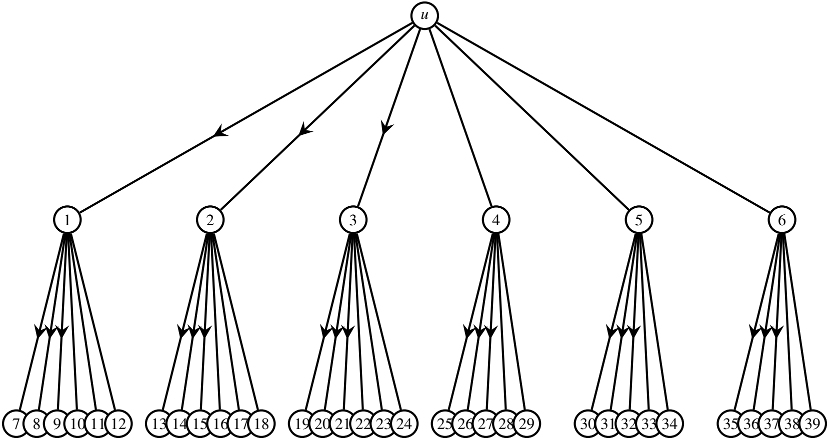

Figure 6: The labelled Moore tree for

Corollary

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All vertices in S have directed in-degree $z - 1$.

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As the average directed in-degree is z,

$$\sum_{v' \in S'} (d^{\rightarrow}(v') - z) = \sum_{v \in S} (z - d^{\rightarrow}(v)) = |S|.$$

There are $r + z$ vertices in S

Lemma

$|S| = r + z$.

Proof.

Let $v \in S$. By the preceding lemma $d^{\rightarrow}(v) = z - 1$. We obtain an upper bound for $|V(G)|$ by assuming that $S' \subseteq N^{-}(v)$.

$$|V(G)| \leq 1 + r + (z-1) + r(r-1) + rz + (z-1)(r+z) + |S|.$$

Rearranging, $|S| \geq r + z$. Combined with $S \subseteq N^{+}(r(u))$, we see that $|S| = r + z$. □

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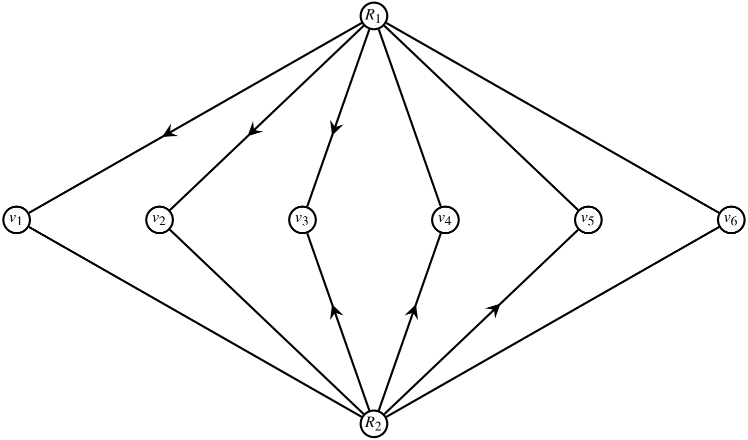
Corollary

$S = N^{+}(r(u))$ for all $u \in V(G)$.

The undirected degree

Theorem

The undirected degree of G is two.



There are exactly two repeats

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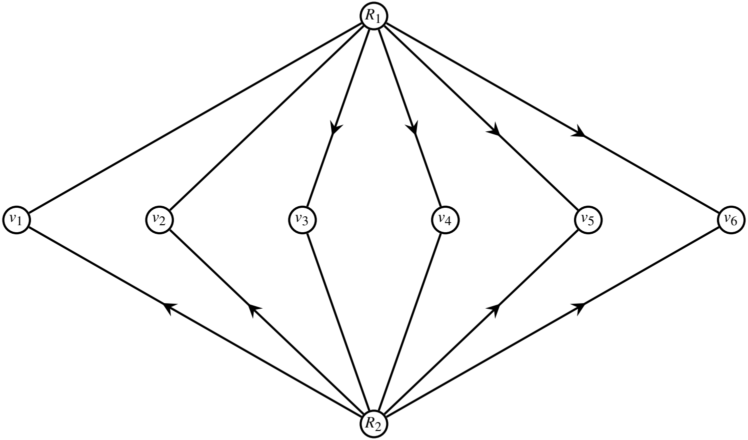

Completing the argument

Now is we count paths we obtain the equation

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The matrix $J + P$ has spectrum $\{n+1, -1, 0^{(n-2)}\}$, so $I + A + A^2$ has spectrum $\{n+3, 1, 2^{(n-2)}\}$.

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As the trace of A is zero, the sum of these eigenvalues must be zero. However, this is impossible!

A graph with excess one

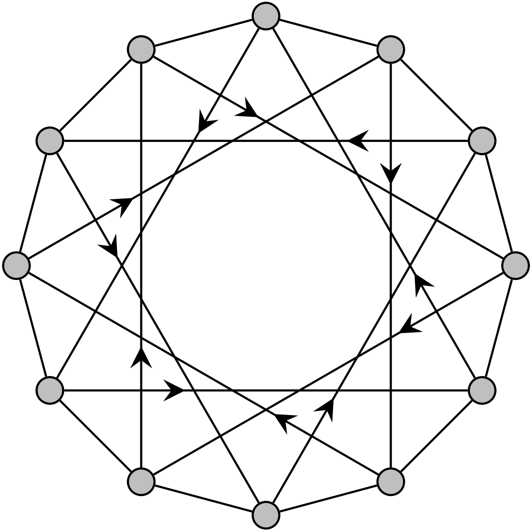

Figure 7: The unique mixed graph with $r = 2, z = 1, k = 2$ and excess $\epsilon = 1$.

What do we know about excess one?

Theorem

Let G be a totally regular $(r, z, 2, +1)$-graph. Then either
$r = 2$
$4r + 1 = c^2$ for some $c \in \mathbb{N}$ and $c|(16z^2 - 24z + 25)$ or
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Theorem JT

Any $(r, z, k; +1)$-graph with $k = 2$ or $z = 1$ is totally regular.

A counting argument

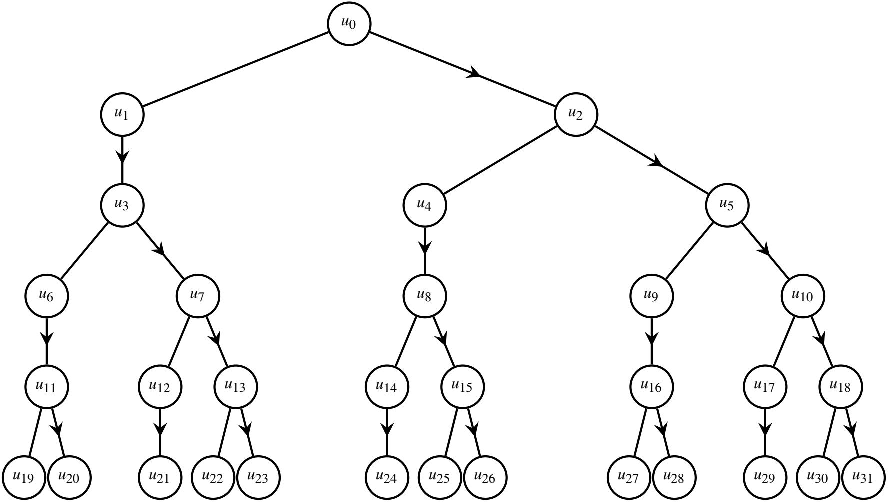

Figure 8: The Moore tree for $r = z = 1$ and $k = 5$.

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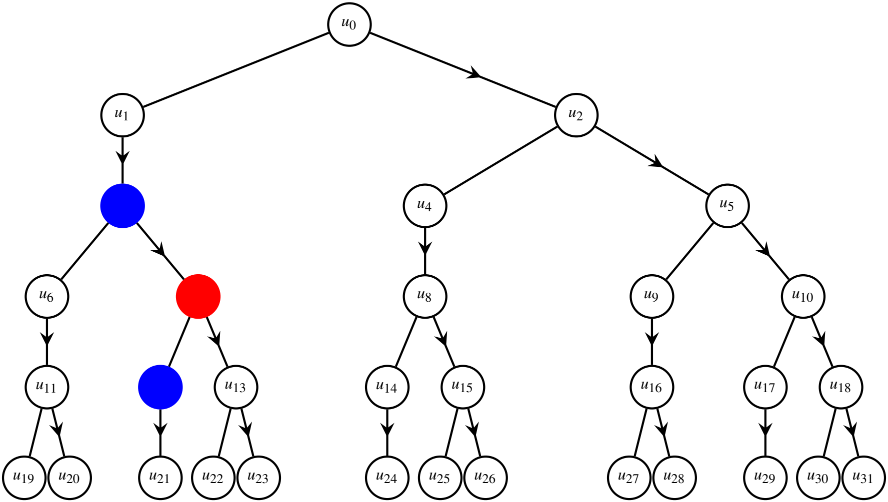

Figure 9: The Moore tree for $r = z = 1$ and $k = 5$.

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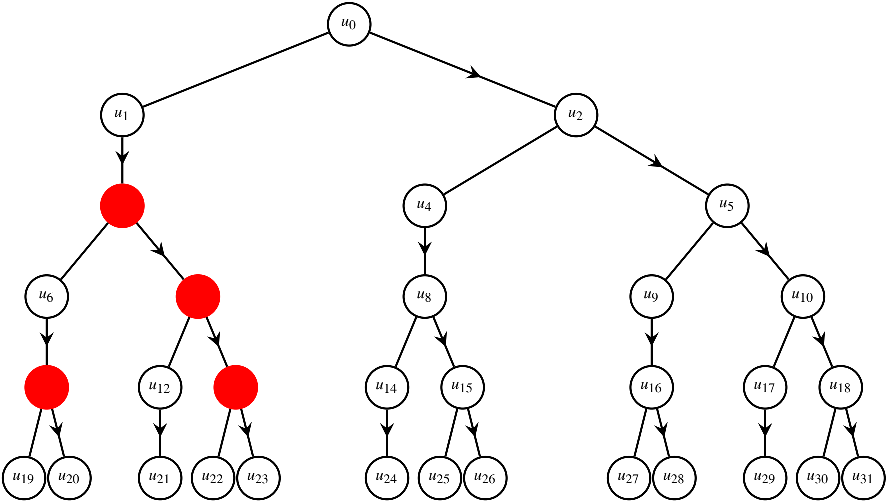

Figure 10: The Moore tree for $r = z = 1$ and $k = 5$.

Result

Theorem

For $k \geq 3$, the excess ϵ of a totally regular $(r, z, k; +\epsilon)$-graph satisfies

$$\epsilon \geq \frac{r}{\phi}\left[\frac{\lambda_1^{k-1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1} - 1}{\lambda_2 - 1}\right] = \frac{A(k)}{z},$$

where

$$\phi = \sqrt{(r + z - 1)^2 + 4z},$$

$$\lambda_1 = \frac{1}{2}(r + z - 1 + \phi)$$

and

$$\lambda_2 = \frac{1}{2}(r + z - 1 - \phi).$$

All $(r, z, k; +1)$-graphs

Theorem

The excess of any (r, z, k)-cage satisfies

$$\epsilon \geq \frac{rz}{(2r+3z)\phi}\left[\frac{\lambda_1^{k-1}-1}{\lambda_1-1} - \frac{\lambda_2^{k-1}-1}{\lambda_2-1}\right].$$

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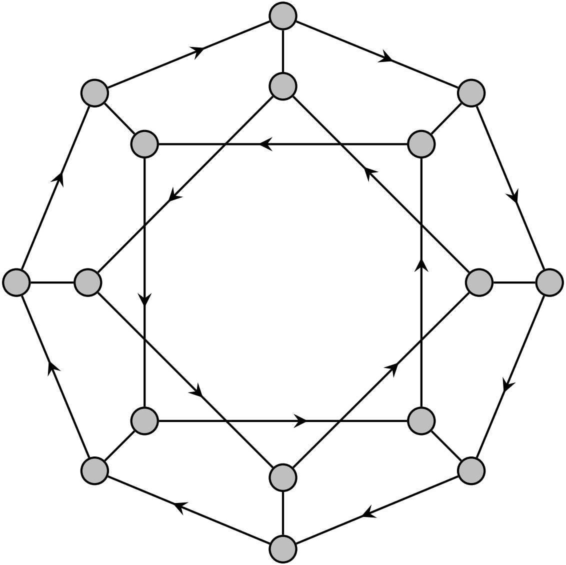
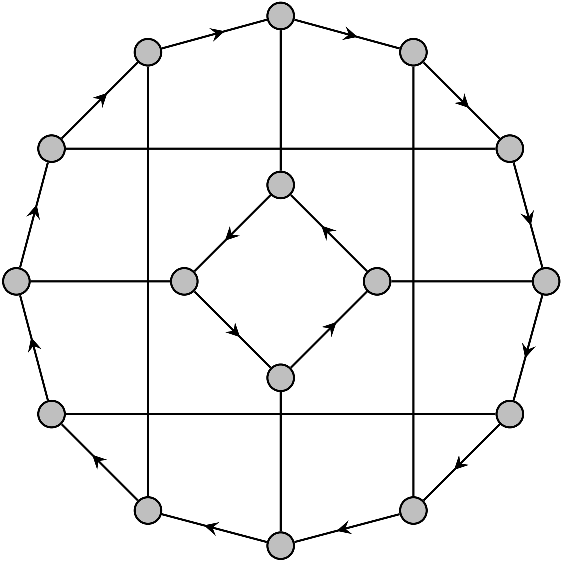
Result

With (quite a lot) more work, this implies that any $(r, z, k; +1)$-graph has $k = 2$.

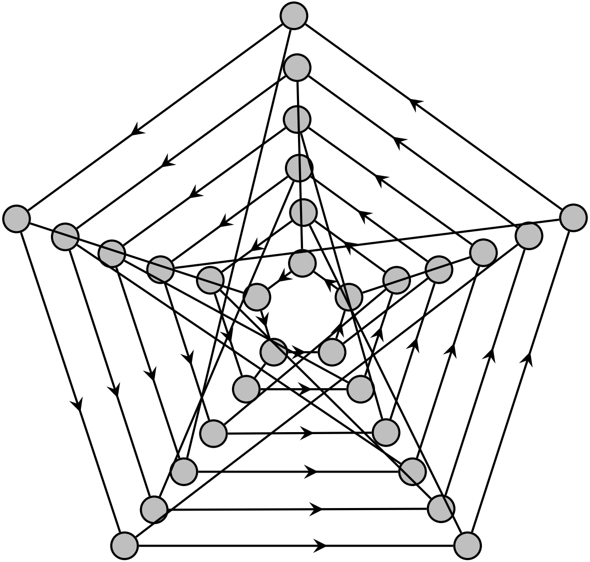
Cages for $r = z = 1, k = 3$



Cages for $r = z = 1, k = 3$



Cage for $r = z = 1, k = 4$



Application to degree/diameter

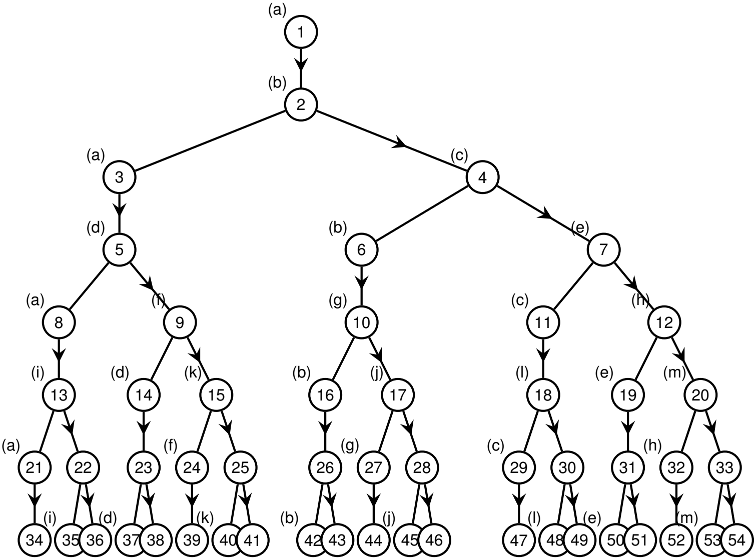


Figure 11: The chain decomposition for $k = 8$

Application to degree/diameter

Theorem

For $k \geq 1$ we have

$$\delta(k+6) = \delta(k) + F_{k-1} + F_{k+4},$$

where $F_0 = F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$ is the Fibonacci sequence.

Some open problems

- Are geodetic cages totally regular/diregular?
- Are $(d, 2; +2)$- and $(2, k; +3)$-digraphs diregular?
- Are there $(r, z, k; -1)$-graphs with $k \geq 3$? In particular, are they totally regular?
- Are $(r, z, k; +2)$-graph totally regular?
- Find smallest arc-transitive $(d, k; +\epsilon)$-digraphs.
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Thank you!

References

- Sillasen, A.A. On k-geodetic digraphs with excess one. Electron. J. Graph Theory Appl. 2 (2) (2014), 150-159.
- Miller, M., Miret, J. M., & Sillasen, A. A. On digraphs of excess one. Discrete Appl. Math. 238 (2018), 161-166.
- Tuite, J. On diregular digraphs with degree two and excess two. Discrete Appl. Math. 238 (2018), 167-174.
- Tuite, J. Digraphs with degree two and excess two are diregular. Discrete Math. 342 (5) (2019), 1233-1244.
- Tuite, J. On diregular digraphs with degree two and excess three. Discrete Appl. Math. 266 (2019), 331-339.
- Tuite, J. & Erskine, G. On total regularity of mixed graphs with order close to the Moore bound. Graphs Comb. (35) (2019), 1253–1272.
- Tuite, J. & Erskine, G. On networks with order close to the Moore bound. Graphs Comb. 38 (143) (2022).