Cycle regularity of cubic vertex-transitive graphs

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JOINT WORK WITH

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Combinatorial homogeneity vs. symmetry

Graphs may possess different types of combinatorial homogeneity.

- regularity: all vertices have the same number of neighbours;
- strong regularity: every edge lies on the sam number of triangles and the same for the complement.
- walk regularity: the number of closed walks (of every fixed length) starting at a vertex does not depend on the vertex.

Combinatorial homogeneity follows from transitivity.

- vertex-transitivity $\Rightarrow$ regularity and walk regularity
- edge-transitivity of the graph and its complement $\Rightarrow$ strong-regularity.

HOWEVER:

Transitivity (almost) never follows from homogeneity.

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  (because it is a cycle).
- Platonic solids: If X is a *regular convex polyhedron* (constant valence, all faces congruent to a fixed regular polygon), then $\mathrm{Aut}(X)$ acts transitively on vertices, edges, faces, flags.
- Maps of type $\{6, 3\}$ on orientable surface are vertex-transitive.

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Cycle regularity

- For $k \in \mathbb{N}$, $e \in E(\Gamma)$:

$$c_k(e) = \text{ number of } k\text{-cycles that pass through } e.$$

- Γ is k-cycle regular if $c_k(e)$ is constant for all $e \in E(\Gamma)$.
- If Γ is k-cycle regular for all k, then it is cycle-regular.

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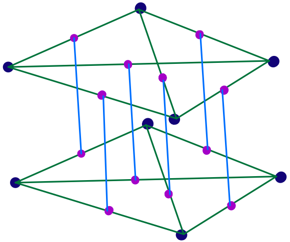
Of course not

A small counterexample (provided to us by Royle and McKay):



20 vertices, 2 orbits on edges, 2 orbits on vertices,
3-regular (cubic), girth $= 6$.

A question of Fouquet and Hahn

Question (Fouquet and Hahn, 2001):
Is every cycle-regular and vertex-transitive also edge-transitive?

NO! Answered by Marston Conder and Jin-Xin Zhou (JCTB'23):

Theorem. The line graph of a cubic locally-2-arc-transitive not vertex-transitive graph is cycle-regular, vertex-transitive but not edge-transitive.

Corollary. There are infinitely many tetravalent cycle-regular, vertex-transitive but not edge-transitive graphs of girth 3.

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So, we have tetravalent examples. Can one find examples of valence 3?

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$$\text{Cycle-regularity} \overset{?}{\Longrightarrow} \text{edge-transitivity}.$$

First reaction: SURELY NOT!

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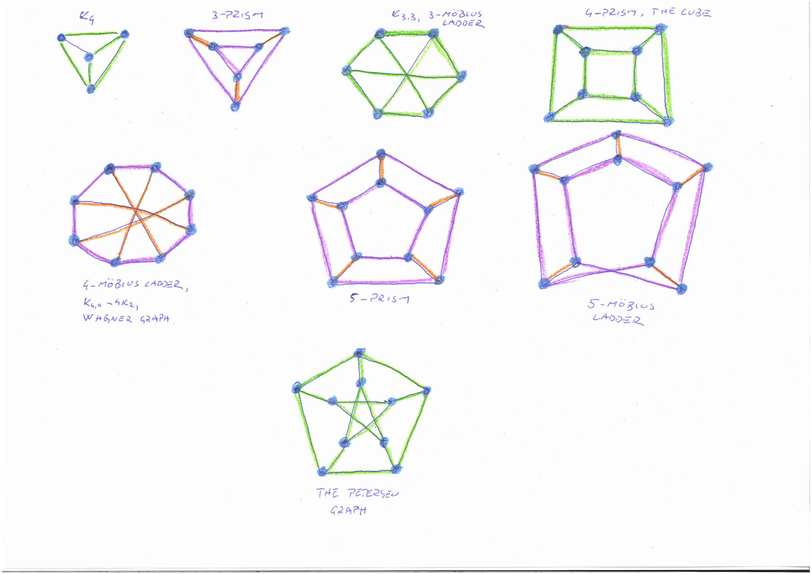

Intermezzo: Census of cubic vertex-transitive graphs

- All cubic vertex-transitive graphs up to order 1280 are known. (Spiga, Verret, PP)
- There are $111\,360$ of them.
- Only 482 of them are edge-transitive.
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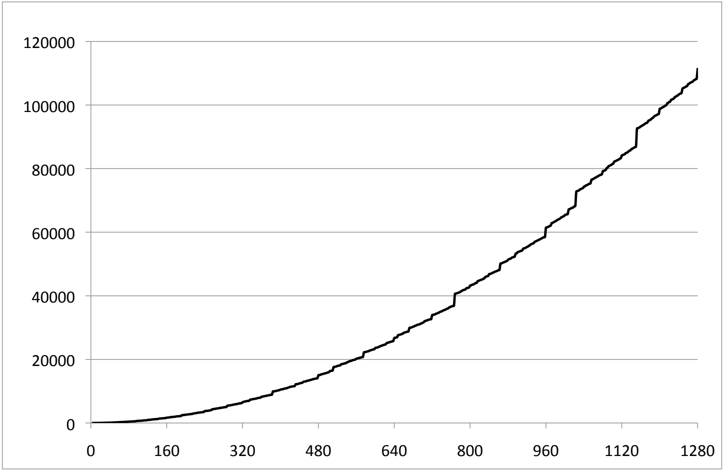
The number of cubic vertex-transitive graphs



Current status

Theorem 1. Every cubic vertex-transitive cycle-regular graph on at most 1280 vertices is edge-transitive.

Theorem 2. Every cubic (vertex-transitive) cycle-regular graph of girth at most 5 is edge-transitive.

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Theorem 2: Girth at most 5

As observed independently by many authors:

Theorem. Let Γ be a cubic graph of girth $g \leq 5$. If Γ is g-cycle-regular, then:

- $g = 3$ and $\Gamma \cong K_4$;
- $g = 4$ and $\Gamma \cong K_{3,3}$ or the cube Q_3;
- $g = 5$ and Γ is either the Petersen graph or the Dodecahedron.

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The starting point is the following classification:

Theorem. (P., Vidali, 2022) Let Γ be a cubic vertex-transitive graph of girth 6. If Γ is 6-cycle regular, then:

- $c_6(e) = 8$ and Γ is the Heawood graph (on 14 vertices),
- $c_6(e) = 6$ and Γ is the Möbius-Kantor graph (on 16 vertices),
- $c_6(e) = 4$ and Γ is
  - the Pappus graph (on 18 vertices); or
  - the Desargues (on 20 vertices),
- $c_6(6) = 2$ and Γ underlies a map of type $\{6, 3\}$ on the torus.

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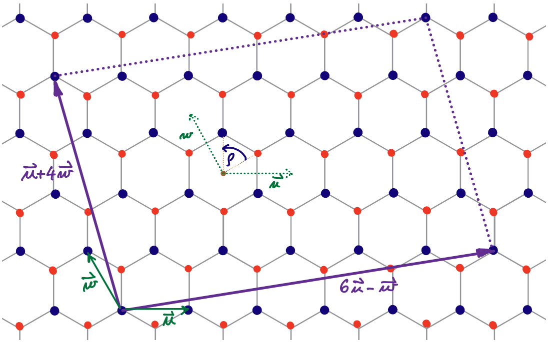
Lifting to the Euclidean plane

- Γ ... cubic graph, embedded onto torus T as a $\{6, 3\}$-map.
- Consider the universal covering projection $\wp\colon \mathbb{R}^2 \to T$.
- Γ lifts to a hexagonal tessellation of $\mathbb{R}^2$.
- Fibres are orbits of some group of translations H.



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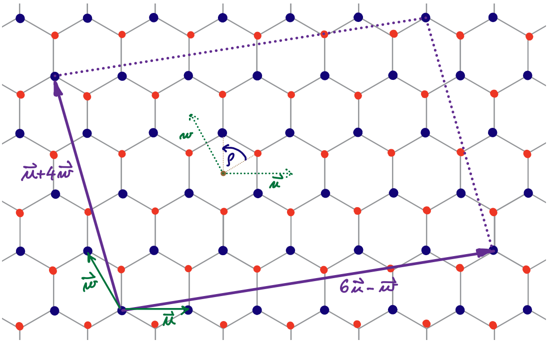

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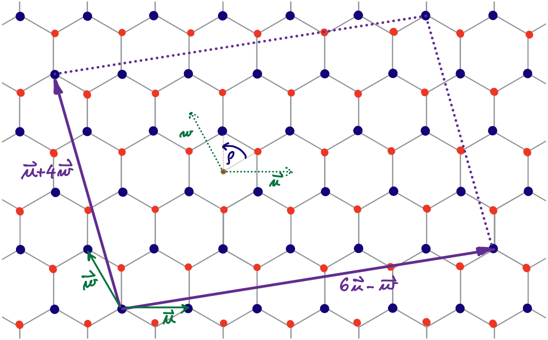

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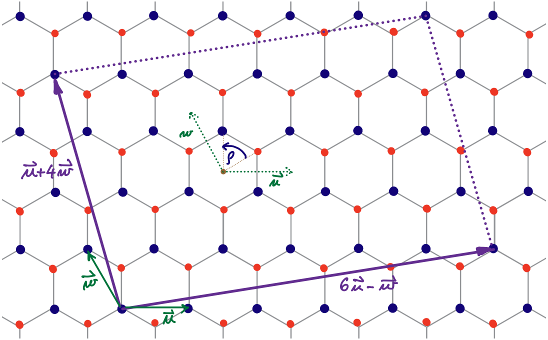

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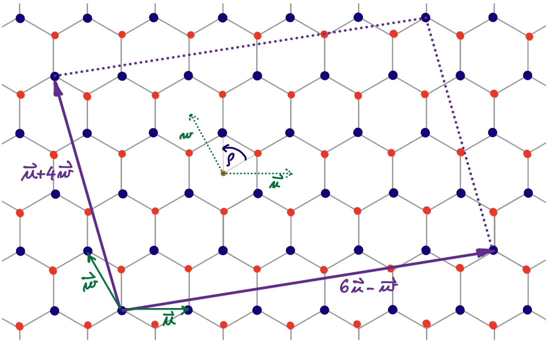
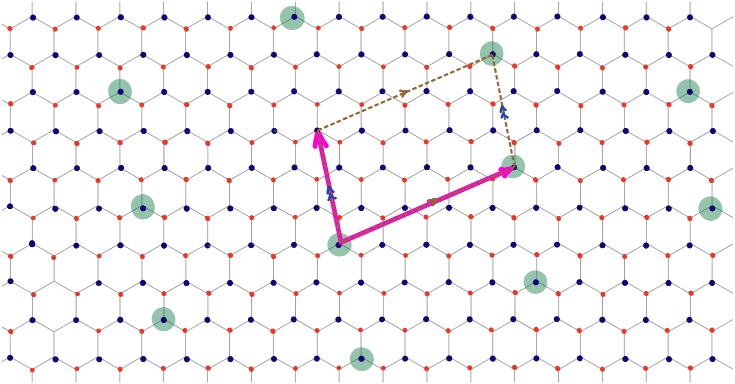


Trivial vs. non-trivial cycles



Cycles in Γ are of two types:

- Trivial: Lift to cycles in $\mathcal{H}$.
- Non-trivial: Lift to paths between vertices in the same fibre.

Short cycles

Important parameter:

$$\begin{aligned} d_{\min} &= \text{shortest distance between two vertices in } L \\ &= \text{length of shortest non-contractible cycle in } \Gamma. \end{aligned}$$

Observations:

- Every cycle of length $< d_{\min}$ is trivial.
- Γ is ℓ-cycle-regular for all $\ell < d_{\min}$.
- Critical cycle lengths: $\ell = d_{\min}$ and $\ell = d_{\min} + 2$.
- In particular, we need to consider the number of ℓ-paths, $\ell = d_{\min}$ and $d_{\min} + 2$ between two vertices in a fibre.
- First is easy. The second involves solving:

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$$L(n,k) = L(n-1,k-1) + L(n-1,k) + \binom{n-1}{k-2} + \binom{n-1}{k+1}.$$

Short cycles

Important parameter:

$$\begin{aligned} d_{\min} &= \text{shortest distance between two vertices in } L \\ &= \text{length of shortest non-contractible cycle in } \Gamma. \end{aligned}$$

Observations:

- Every cycle of length $< d_{\min}$ is trivial.
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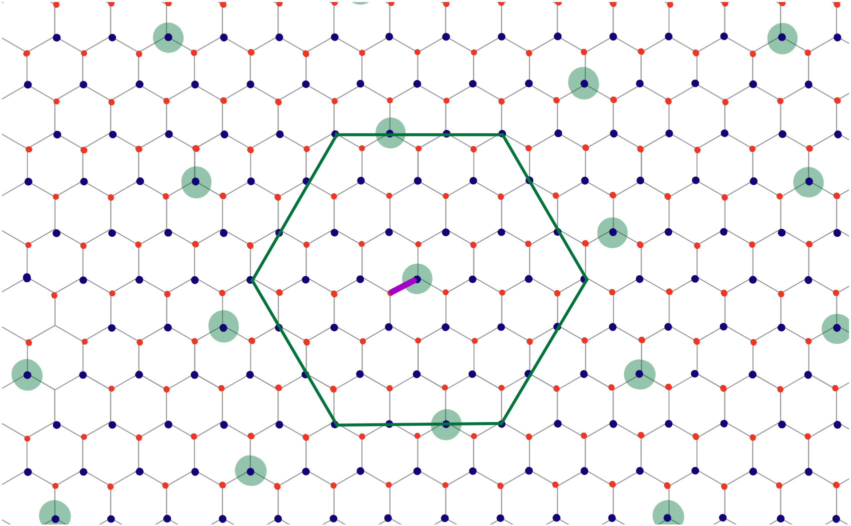
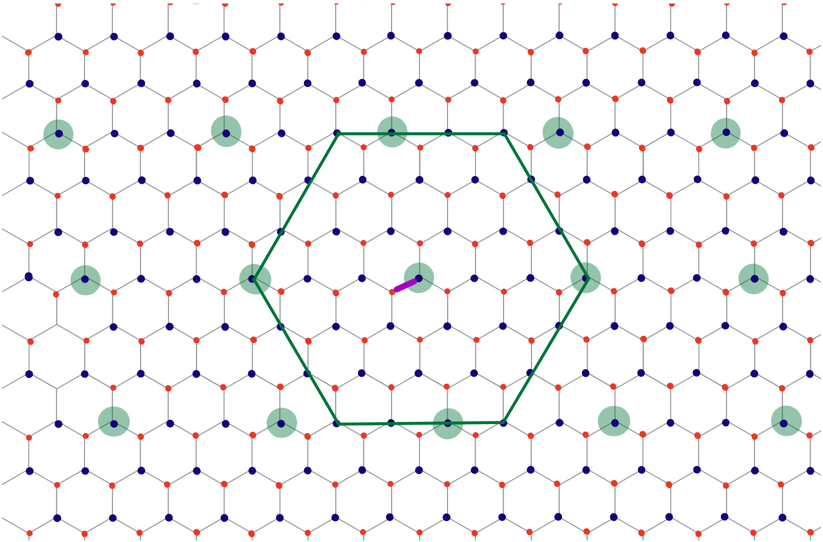
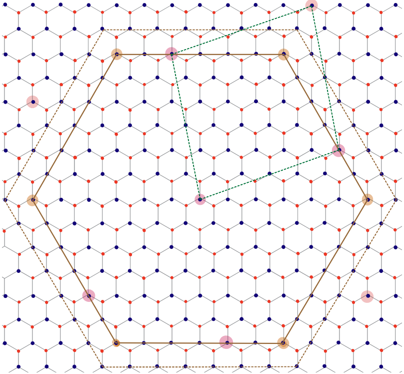
$|D| = 2$



$$|D| = 4$$



$d_{\min}$-cycle regular examples



- Infinite family of graphs $\Gamma(s)$, $s \geq 1$, that are k-cycle regular for all $k \leq \frac{3}{2}\sqrt{|V|}$.
- Smallest example: $\Gamma(s) =$ Möbius-Kantor graph (which is edge-transitive).

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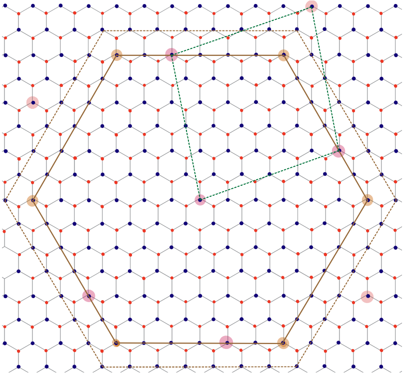

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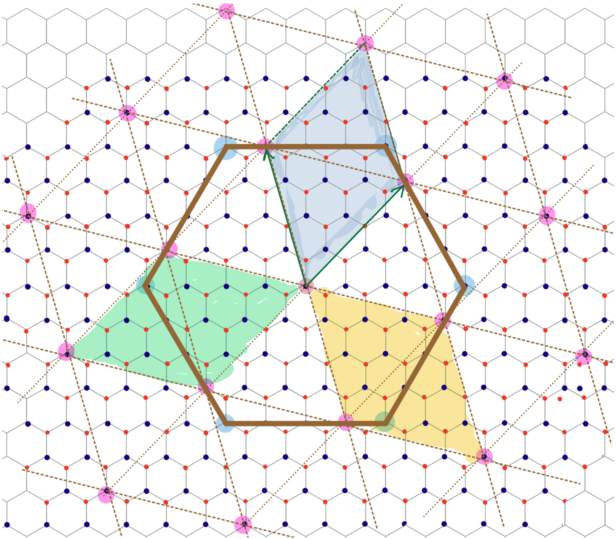
Symmetric case. $|D| = 6$



Conclusion

Within the realm of cubic vertex-transitive graphs:

- We proved: If girth ≤ 5, then:

  cycle-regular $\Rightarrow$ edge-transitive.

  - Up to girth 5, there are only finitely many cubic VT graphs with every edge on the same number of girth cycles — they are all edge-transitive.

- We also proved: If girth $= 6$, then: cycle-regular $\Rightarrow$ edge-transitive.

  - Reduction to maps with faces of length 6;
  - Analysis of the maps.

Higher girth

Girth 7 (still cubic VT):

- It seems that reduction to maps is possible:

  Computational data: if every edge on the same number of 7 cycles, then either Coxeter graph or a $\{7, 3\}$-map.

- An easy group theory argument: Every vertex-transitive $\{7, 3\}$-map is edge-transitive.

Girth $= 8$:

- Reduction to maps might be possible
- No idea how to deal with maps.