Mathematical analysis of nonlinear material response: viscoelasticity

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Outline

Dynamics of microstructure in solids

The viscoelastic model

Quasistatic case

Contributions

Strain-limiting viscoelasticity

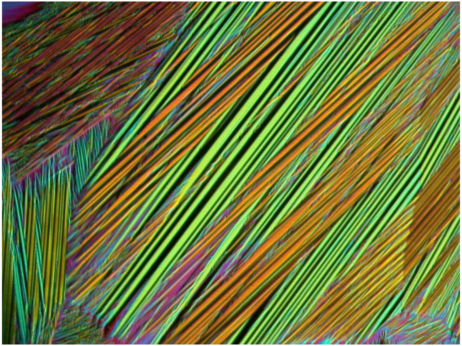
Implicit constitutive modelling

Strain-rate type models & Contributions

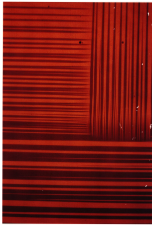
Stress-rate type models & Contributions

Dynamics of microstructure in solids

Martensitic phase transformations



Microstructure in CuZnAl (M.Morin)



Needles in CuAlNi (Chu & James)

Complex patterns in CuAlNi (Cui & James)

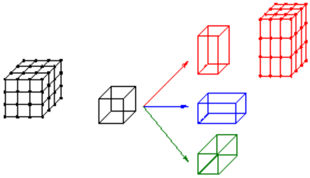
Martensitic phase transformations

Martensitic transformations involve a change of shape of the crystal lattice of some alloy at a critical temperature, e.g. cubic to tetragonal;

$\theta > \theta_c$

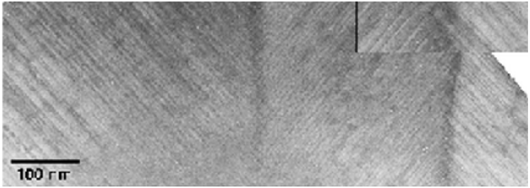
cubic

austenite



$\theta < \theta_c$

three tetragonal variants of martensite



Microtwins in $Ni_{65}Al_{35}$ (Boullay & Schryvers)

The Model

The equation of nonlinear viscoelasticity is

$$y_{tt} - \text{Div } DW(Dy) - \text{Div } S(Dy, Dy_t) = 0$$

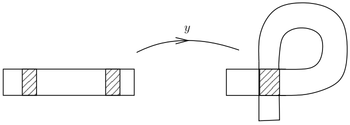
where

$$\begin{cases} F = Dy(x,t),\ F_{i\alpha} = \frac{\partial y_i}{\partial x_\alpha} \ \text{deformation gradient} \\ W : M^{3\times 3} \to [0,\infty] \ \text{stored-energy function} \\ T_R(Dy, Dy_t) = DW(Dy) + S(Dy, Dy_t) \\ \qquad \text{Piola-Kirchhoff stress tensor.} \end{cases}$$

The Problem

Constitutive assumptions:

- $\det Dy > 0$ for $x \in \Omega$
- $W(Dy) \to \infty$ as $\det Dy \to 0$
  $W(Dy) \to \infty$ as $|Dy| \to \infty$



locally invertible but not globally invertible

- **Frame-indifference**:
  - $W(R\,F) = W(F)$ for all $R \in \mathrm{SO}(3)$, $F \in M^{3\times 3}$
  - $S(Dy, Dy_t) = Dy\,G(U, U_t)$
    - G is a symmetric matrix-valued function
    - $U = (Dy^T Dy)^{1/2}$

Quasistatic Case - 1D

In one space dimension we have

$$y_{tt} = \big(\sigma(y_x) + S(y_x, y_{xt})\big)_x, \;\; x \in (0,1), \; t \in [0,T].$$

The quasistatic equation takes the form

$$\Big(\sigma(y_x) + S(y_x, y_{xt})\Big)_x = 0,$$

where $\sigma = W'$, $x \in (0,1)$, $t \in [0,T]$. Boundary conditions:

- $y(0,t) = 0$, $(\sigma + S)(1,t) = 0$ (one end stress free)
- $y(0,t) = 0$, $y(1,t) = \mu$, $\mu > 0$ constant (both ends fixed)

Quasistatic case in 1D

We will consider the quasistatic case when $S(y_x, y_{xt}) = y_{xt}$. We have

$$\big(\sigma(y_x) + y_{xt}\big)_x = 0.$$

Using the boundary conditions and putting $p = y_x$ we get

$$\begin{cases} p_t(x,t) = -\sigma(p(x,t)) + \displaystyle\int_0^1 \sigma(p(y,t))\,dy, \ x \in (0,1) \\ p(x,0) = p_0(x) > 0 \text{ a.e.} \\ \displaystyle\int_0^1 p_0(x)\,dx = \mu. \end{cases}$$

Our contributions

1. Well-posedness when W is λ-convex [1]
   - using global upper and lower bounds to pass to the limit
2. Equivalence of the theory with that of gradient flows
   - following Brézis for the analysis of the gradient flow equation.
   - following the metric gradient approach. [2]

[1] J. M. Ball, Y. Şengül, J. Dynam. Differential Equations, 27 (3), 405-442, 2015.

[2] A. Mielke, C. Ortner, Y. Şengül, SIAM J. Math. Anal. 46 (2), 1317-1347, 2014.

Strain-limiting viscoelasticity

Example 1

An explicitly constituted material:

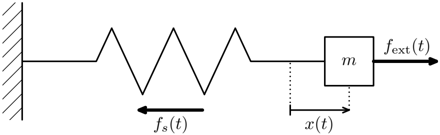


Figure: Applying an external force puts the system in motion

We can write the constitutive specification for the spring as

$$f_s = g(x) \quad \underset{\text{(linear spring)}}{\Longrightarrow} \quad f_s = kx,\ k \text{ spring constant.}$$

One then writes the balance of linear momentum and use this relation to get an ODE in terms of the displacement.

Example 2

An implicitly constituted material:

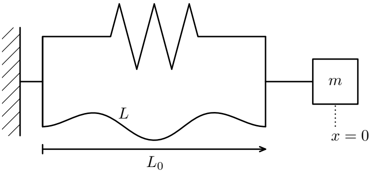


Figure: A mass-spring-wire system in its equilibrium

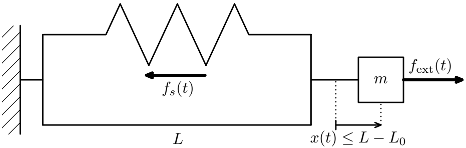
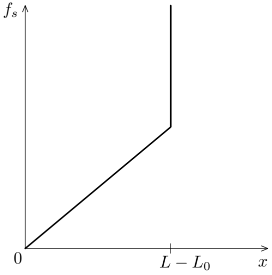


Figure: Applying an external force puts the system in motion

- The wire of maximal length L cannot break whatever force is applied to it.
- The extension of the spring is limited to L.
- Once the maximal length L is obtained, no change in the position occurs.



In this case it is much more sensible to prescribe an implicit relation between the force and the displacement as

$$g(f_s, x) = 0.$$

We are interested in [3] class of implicit models defined through

$$G(\mathbf{T}, \mathbf{B}) = 0.$$

Isotropy leads to

$$\begin{aligned} G(\mathbf{T}, \mathbf{B}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 + \alpha_5 (\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) \\ + \alpha_6 (\mathbf{T}^2 \mathbf{B} + \mathbf{B}\mathbf{T}^2) + \alpha_7 (\mathbf{T}^2 \mathbf{B}^2 + \mathbf{B}^2 \mathbf{T}^2) = 0, \end{aligned}$$

where α_i depend on the invariants

$$\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{B}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{B}^2, \mathrm{tr}\mathbf{T}^3, \mathrm{tr}(\mathbf{T}\mathbf{B}), \mathrm{tr}(\mathbf{T}^2\mathbf{B}), \mathrm{tr}(\mathbf{T}\mathbf{B}^2), \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2).$$

Here: $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green stretch tensor.

[3] Y. Şengül, Discrete Contin. Dyn. Syst. S, 14 (1), 57-70, 2021.

Under the assumption

$$\max_{x,t} \|\nabla \mathbf{u}\| \ll 1,$$

the linearization of the explicit model $\mathbf{T} = G(\mathbf{B})$ gives

$$\mathbf{T} = \mathbf{C}\boldsymbol{\epsilon}$$

where $\mathbf{C}$ is a fourth order tensor not depending on $\boldsymbol{\epsilon}$. Hence, there is no way of justifying nonlinear elastic models involving a linearized strain if one starts with a Cauchy elastic material.

Here: $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the linearized strain

On the other hand, for the implicit subclass, the smallness assumption allows us to replace the CauchyGreen tensor

$$\mathbf{B} \quad \text{by} \quad \mathbf{I} + 2\boldsymbol{\epsilon} \quad \text{in} \quad \mathbf{B} = \mathcal{F}(\mathbf{T})$$

so that we obtain

$$\boldsymbol{\epsilon} = \tilde{\mathcal{F}}(\mathbf{T}),$$

which is a nonlinear relationship between the linearized strain and the stress.

Strain-rate type models

We are interested in the viscoelastic version with the strain-rate dependance. As a subclass of the general implicit constitutive relations of the form $G(\mathbf{T}, \mathbf{B}, \mathbf{D}) = 0$, we have

$$\gamma \mathbf{B} + \nu \mathbf{D} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2,$$

where γ and ν are nonnegative constants.

Here: $\mathbf{L} = \nabla \mathbf{v}$ is the velocity gradient, $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the symmetric part of $\mathbf{L}$.

Linearizing the strain we get

$$\gamma\boldsymbol{\epsilon} + \nu\boldsymbol{\epsilon}_t = \beta_0\mathbf{I} + \beta_1\mathbf{T} + \beta_2\mathbf{T}^2,$$

where $\boldsymbol{\epsilon}_t = \partial\boldsymbol{\epsilon}/\partial t$ is the linearized counterpart of $\mathbf{D}$ and β_i depend on $\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3$.

In general one can write

$$\gamma\boldsymbol{\epsilon} + \nu\boldsymbol{\epsilon}_t = g(\mathbf{T}).$$

Our contributions in 1-D

1. Travelling wave solutions
   - by deriving the equation $T_{xx} + \nu T_{xxt} = g(T)_{tt}$, and studying different forms of g [4]
   - considering the arctangent type nonlinearity [5]
2. The Cauchy problem
   - local-in-time existence of solutions [6]
   - global existence [7]

[4] H. A. Erbay, Y. Şengül, Int. J. Nonlinear Mech., 77, 61-68, 2015.

[5] Y. Şengül, Appl. Engin. Science, 7, 100058, 2021.

[6] H. A. Erbay, A. Erkip, Y. Şengül, J. Diff. Eqns., 269, 9720-9739, 2020.

[7] Y. Şengül, In: Espanol, M. et al. Research of Mathematics of Material Science, Vol. 31. Association for Women in Mathematics Series Springer, 319-332, 2022.

Our contributions in 3-D

1. Higher-dimensional problem

   - global-in-time existence of weak solutions with periodic boundary conditions [8]
   - existence and uniqueness of weak solutions with Dirichlet boundary conditions [9]

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[8] M. Bulíček, V. Patel, E. Süli, Y. Şengül, Commun. Pure Appl. Anal., 20 (5), 1931-1960, 2021.

[9] M. Bulíček, V. Patel, E. Süli, Y. Şengül, SIAM J. Math. Anal., 54 (6), 6186-6222, 2022.

Stress-rate type models

To model the stress-rate type viscoelastic fluids within the context of implicit constitutive theories, we must consider the relation

$$G(\mathbf{T}, \dot{\mathbf{T}}, \mathbf{B}) = 0.$$

Furthermore, we will restrict our attention to the case where the Cauchy-Green stretch tensor is given as a nonlinear function of the stress and its time derivative, namely,

$$\mathbf{B} = H(\mathbf{T}, \dot{\mathbf{T}}).$$

Under the assumption of isotropic materials, we have

$$\begin{aligned}\mathbf{B} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\dot{\mathbf{T}} + \alpha_3\mathbf{T}^2 + \alpha_4\dot{\mathbf{T}}^2 + \alpha_5(\mathbf{T}\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}) \\ + \alpha_6(\mathbf{T}^2\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}^2) + \alpha_7(\dot{\mathbf{T}}^2\mathbf{T} + \mathbf{T}\dot{\mathbf{T}}^2) + \alpha_8(\mathbf{T}^2\dot{\mathbf{T}}^2 + \dot{\mathbf{T}}^2\mathbf{T}^2)\end{aligned}$$

with the scalar functions $\alpha_i, i = 0, \ldots, 8$, depending on the invariants

$$\mathrm{tr}\mathbf{T}, \mathrm{tr}\dot{\mathbf{T}}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\dot{\mathbf{T}}^2, \mathrm{tr}\mathbf{T}^3, \mathrm{tr}\dot{\mathbf{T}}^3, \mathrm{tr}(\mathbf{T}\dot{\mathbf{T}}), \mathrm{tr}(\mathbf{T}^2\dot{\mathbf{T}}), \mathrm{tr}(\dot{\mathbf{T}}^2\mathbf{T}), \mathrm{tr}(\mathbf{T}^2\dot{\mathbf{T}}^2).$$

Assuming

$$\max_{x,t} \|\nabla \mathbf{u}\| \ll 1, \ \max_{x,t} \|\nabla \mathbf{v}\| \ll 1,$$

as well as the convective terms in the expression of the material time derivative of $\mathbf{T}$ can be neglected we obtain the relation

$$\boldsymbol{\epsilon} = H(\mathbf{T}, \mathbf{T}_t).$$

A subclass that is linear with respect to $\mathbf{T}_t$ is

$$\boldsymbol{\epsilon} = h(\mathbf{T}) - \gamma(\mathbf{T})\mathbf{T}_t$$

where $h(\cdot)$ and $\gamma(\cdot)$ are nonlinear functions of the Cauchy stress $\mathbf{T}$.

Our contributions

1. Introduction of a thermodynamically consistent model [10]
   - as well as deriving the corresponding partial differential equation as $T_{xx} + \nu T_{ttt} = g(T)_{tt}$ and comparing it with the strain-rate model
2. Travelling wave solutions [11]
   - solving the corresponding ODE for the travelling wave variable numerically with different choices of the nonlinearity

[10] H. A. Erbay, Y. Şengül, Z. Angew. Math. Phys., 71:94, 2020.

[11] E. Duman Y. Şengül, Advances in Continuous and Discrete Models, to appear.

Additional References

1. K. R. Rajagopal, *On implicit constitutive theories*, Appl. Math., 48, 279-319, 2003.
2. K. R. Rajagopal, *On a new class of models in elasticity*, J. Math. Comput. Appl., 15, 506-528, 2010.
3. K. R. Rajagopal, *On the nonlinear elastic response of bodies in the small strain range*, Acta. Mech., 225, 1545-1553, 2014.
4. K.R. Rajagopal, G. Saccomandi, *Circularly polarized wave propagation in a class of bodies defined by a new class of implicit constitutive relations*, Z. Angew. Math. Phys. 65, 1003-1010, 2014.
5. M. Bulíček, J. Málek, K.R. Rajagopal, E. Süli, *On elastic solids with limiting small strain: modelling and analysis*, EMS Surv. Math. Sci.,1(2), 283-332, 2014.
6. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.
7. Y. Şengül, *Well-posedness of dynamics of microstructure in solids*, PhD thesis, University of Oxford, 2010.