Linear stability analysis of overdetermined problems

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Shape Optimisation and Geometric Spectral Theory

ICMS, Edinburgh

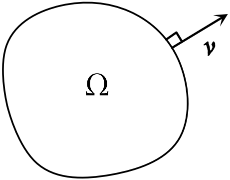
September 20–23, 2022

Introduction

Serrin's overdetermined problem

ODP

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\dfrac{\partial u}{\partial \nu} = \dfrac{1}{n} & \text{on } \partial\Omega. \end{cases}$$



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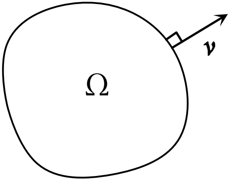
Question What shape of a bounded $\Omega \subset \mathbb{R}^n$ admits the solvability of ODP?



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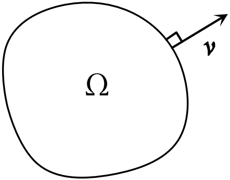

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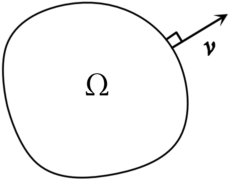

Variational Structure

ODP is the Euler-Lagrange equation of maximizing

$$\Omega \mapsto T(\Omega) = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega u\, dx\right)^2}{\int_\Omega |\nabla u|^2\, dx}$$

under the volume constraint $|\Omega| =$ const. for $g = 0$.

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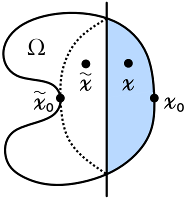

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$$\begin{cases} \Delta(\tilde{u} - u) = 0 & \text{in } \Omega_\lambda, \\ \tilde{u} - u \ge 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

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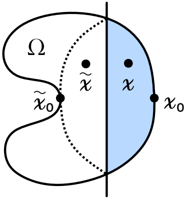

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$$d(u) \equiv 0 \;\Leftrightarrow\; D^2u = \lambda I \;\Leftrightarrow\; u = \frac{\lambda}{2}|x - \xi|^2 + C \text{ if } \nabla u(\xi) = 0.$$

$$P(u) := \frac{|\nabla u|^2}{2} + \frac{u}{n} \quad \text{satisfies} \quad \begin{cases} \Delta P = d(u) \ge 0 & \text{in } \Omega, \\ P = \text{const.} & \text{on } \partial\Omega. \end{cases}$$

$$\Rightarrow \text{ Either } \boxed{P \equiv \text{const.}} \text{ or } P < \text{const.} \;\Rightarrow\; 0 = \Delta P = d(u).$$

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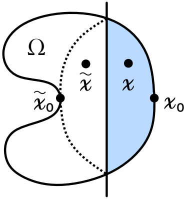

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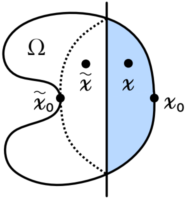

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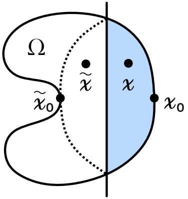

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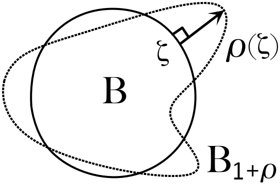
Domain deviation $\rho \in C(\partial B)$:

$$\partial B_{1+\rho} = \{(1+\rho(\zeta))\zeta \mid \zeta \in \partial B\}$$



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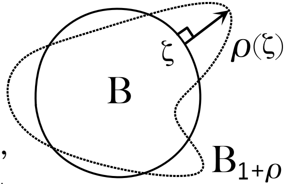

Neumann deviation:

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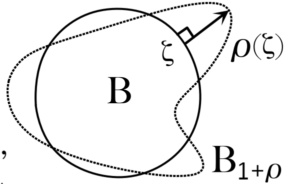

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$$\|\rho\|_{L^\infty} \leq C \left| \log \|g\|_{C^1(\partial B_{1+\rho})} \right|^{-1/n} \quad \text{(up to translation)}$$

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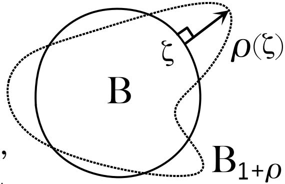

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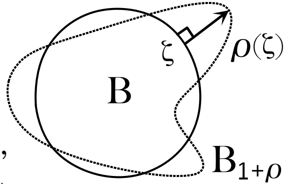

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- Ciraolo, Magnanini, Vespri ('16): *Quantitative Harnack inequality*

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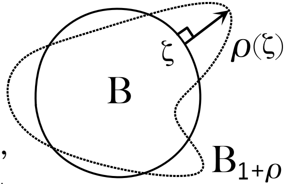

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where $\theta_2 = 1$, $\theta_3 = 1 - \varepsilon$, $\theta_n = 4/(n+1)$ $(n \geq 4)$.

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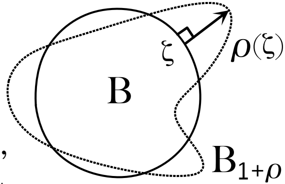

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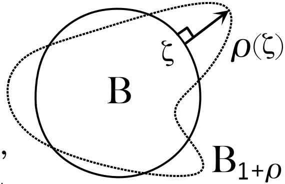

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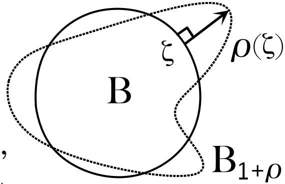

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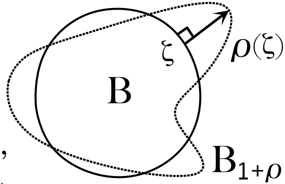

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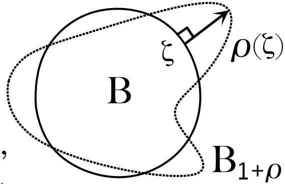

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- Gilsbach and O. ('21) and O. ('22): *Implicit Function Theorem*

"$\|\rho\|_{C^{2+\alpha}} \leq C \|g\|_{C^{1+\alpha}}$" (Main result)

Result and Proof

Main result: *Existence & Optimal stability*

Due to translational invariance of ODP, we set

$$\begin{aligned} h^{k+\alpha}(\partial B) &= \overline{C^\infty(\partial B)}^{C^{k+\alpha}} \\ &= \langle x_1, \ldots, x_n \rangle \oplus h_\perp^{k+\alpha}(\partial B), \end{aligned}$$

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Theorem (Existence)

There exist $\varepsilon, \delta > 0$ such that, for any $g_1 \in h_\perp^{3+\alpha}$ with $\|g_1\|_{h^{3+\alpha}} < \delta$, there is a unique $(\rho, g_2) \in h^{3+\alpha} \times K$ with $\|\rho\|_{h^{3+\alpha}} + \|g_2\|_K < \varepsilon$ s.t.

1. *ODP with $g = g_1 + g_2$ is solvable in $\Omega = B_{1+\rho}$;*
2. *The barycenter of $B_{1+\rho}$ is the origin.*

Main result: *Existence & Optimal stability*

Due to translational invariance of ODP, we set

$$h^{k+\alpha}(\partial B) = \overline{C^\infty(\partial B)}^{C^{k+\alpha}}$$
$$= \langle x_1, \dots, x_n \rangle \oplus h_\perp^{k+\alpha}(\partial B),$$

where $h_\perp^{k+\alpha}(\partial B)$ is the L^2-orthogonal complement of $K = \langle x_1, \dots, x_n \rangle$.

Theorem (Existence)

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Theorem (Stability)

Moreover, there is a constant $C > 0$ such that

$$\|\rho\|_{h^{2+\alpha}(\partial B)} + \|g_2\|_K \leq C \|g_1\|_{h^{1+\alpha}(\partial B)}.$$

Proof: *Reformulation & Linear analysis*

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Our problem is equivalent to finding a zero point $\rho \in h^{2+\alpha}(\partial B)$ of

$$F(\rho, g) = \theta_\rho^* \left[\frac{\partial u_\rho}{\partial \nu_\rho} \right] + \frac{1}{n} + g \in h^{1+\alpha}(\partial B),$$

$$\left(\begin{array}{l} u_\rho \in h^{2+\alpha}(\overline{B_{1+\rho}}) : \text{ solution to } \begin{cases} -\Delta u_\rho = 1 & \text{in } B_{1+\rho}, \\ u_\rho = 0 & \text{on } \partial B_{1+\rho}, \end{cases} \\ \theta_\rho^* \in \mathrm{Isom}\left(h^{1+\alpha}(\partial B_{1+\rho}), h^{1+\alpha}(\partial B)\right) : \text{ pull-back operator.} \end{array} \right)$$

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- $F(\rho, g) = 0 \;\Leftrightarrow\;$ ODP is solvable in $\Omega = B_{1+\rho}$.
- $F(0,0) = \partial_\nu \left(\frac{1 - |x|^2}{2n}\right) + \frac{1}{n} = 0$.

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3. $\partial_\rho F \in C(h^{3+\alpha} \times h^{1+\alpha}, \mathcal{L}(h^{2+\alpha}, h^{1+\alpha}))$ *(Extended operator).*

Proof: *Implicit Function Theorem with Derivative Loss*

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After eliminating the degeneracy, we have

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But this is not sufficient to make the successive approximation converge:

$$\rho_{j+1} = \Phi(\rho_j) := \rho_j - \partial_\rho F(0,0)^{-1} F(\rho_j, g),$$

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Indeed, Φ is a contraction only in a nbd of 0 in $(h^{3+\alpha}, \|\cdot\|_{h^{2+\alpha}})$:

$$\|\Phi(\rho) - \Phi(\tilde\rho)\|_{h^{2+\alpha}} \le \frac{1}{2}\|\rho - \tilde\rho\|_{h^{2+\alpha}} \;\; (\|\rho\|_{h^{3+\alpha}}, \|\tilde\rho\|_{h^{3+\alpha}} \ll 1),$$

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Now the limit $\rho = \lim \rho_j \in h^{2+\alpha}$ satisfies $F(\rho, g) = 0$ in $h^{1+\alpha}$.

In fact, if $g \in h^{3+\alpha}$, then the solution $\rho = \rho(g) \in h^{3+\alpha}$ is unique and

$$\|\rho\|_{h^{2+\alpha}} = \|\Phi(\rho)\|_{h^{2+\alpha}} \le \|\Phi(\rho) - \Phi(0)\|_{h^{2+\alpha}} + \|\Phi(0)\|_{h^{2+\alpha}}$$
$$\le \frac{1}{2}\|\rho\|_{h^{2+\alpha}} + \|\partial_\rho F(0,0)^{-1}\|_{\mathcal{L}(h^{1+\alpha}, h^{2+\alpha})}\|g\|_{h^{1+\alpha}}.$$

Summary

Main result

- Existence & (local) uniqueness:

$$g_1 \in h_\perp^{3+\alpha} \mapsto (\rho, g_2) \in h^{3+\alpha} \times K.$$

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$$\|\rho\|_{h^{2+\alpha}} + \|g_2\|_K \leq C\|g_1\|_{h^{1+\alpha}}.$$

Remark

- The same argument applies to other overdetermined problems.

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THE COFFEE IS READY!