Shape optimization of a Thermal insulation problem

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Shape optimisation and geometric spectral theory
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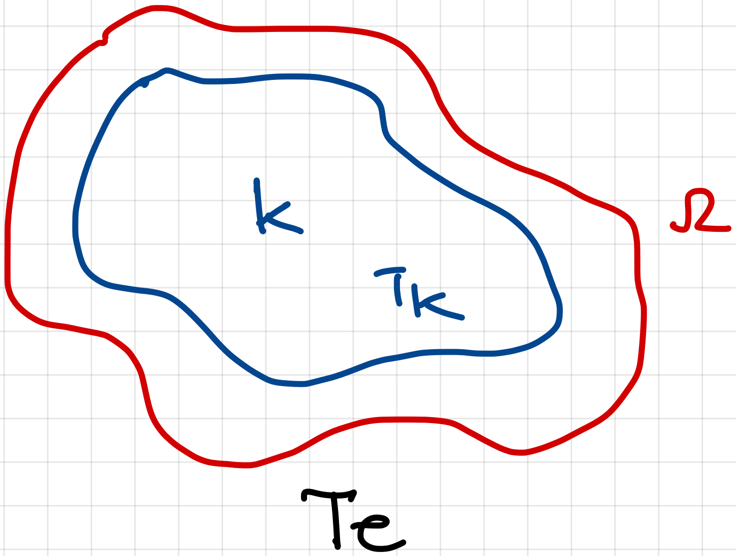
JOINT WORK with D. BUCUR - M. NAHON - C. NITSCH
(CALC. VAR. 2022)



K is a body in $\mathbb{R}^N$ kept at constant Temperature surrounded by an insulator

$\Omega \setminus K$ is the insulator

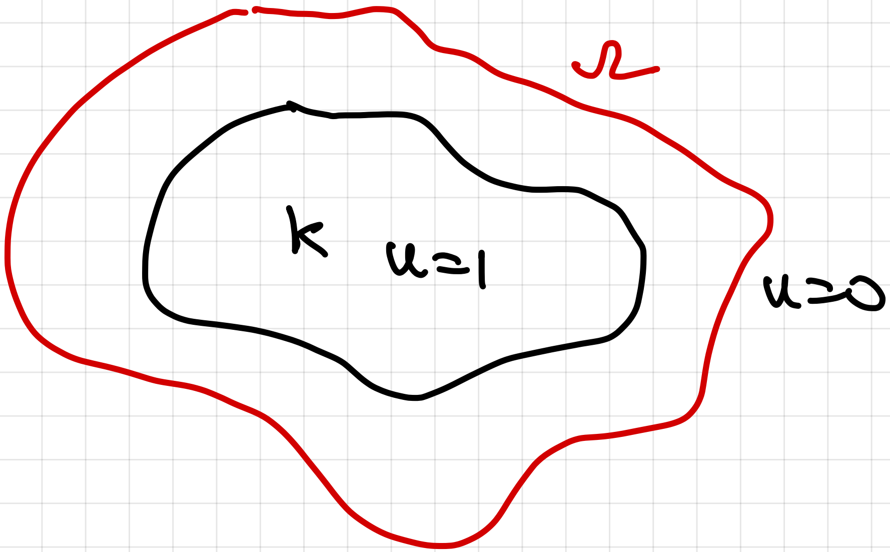
At the equilibrium the Temperature is harmonic in $\Omega \setminus K$

We denote by u the Temperature

$u \in H^1(\Omega)$

$u \equiv 1$ in K

$\Delta u = 0$ in $\Omega \setminus K$



$$\boxed{u = \frac{T - T_e}{T_K - T_e}}$$

$$\frac{\partial u}{\partial \nu} + f(u) = 0$$

$f(u) = \beta u$ CONVECTION

$f(u)$ fourth order polinomial for radiation

CONVECTION $\longrightarrow$ ROBIN B.C

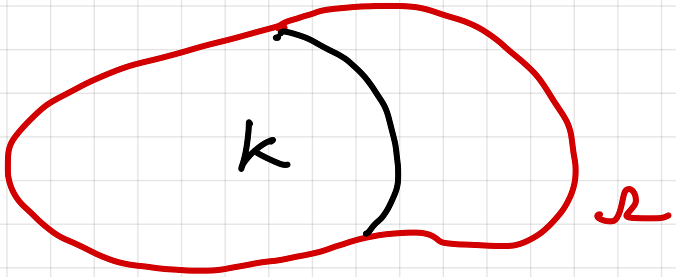
RADIATION $\longrightarrow$ NONLINEAR B.C.

The SIMPLEST way to "DESCRIBE" convection is to say that

The heat loss per unit time and unit surface is proportional to the jump of temperature across the surface.

ROBIN CASE

K compact; $K \subseteq \overline{\Omega}$, Ω open Lipschitz.



Let us consider the following energy functional

$$E(K,\Omega) = \min\left\{ \int_{\Omega} |\nabla v|^2 dx + \beta \int_{\partial\Omega} v^2 d\mathcal{H}^{n-1} : \begin{array}{l} v \in H^1(\Omega) \\ v = 1 \text{ on } K \end{array} \right\}$$

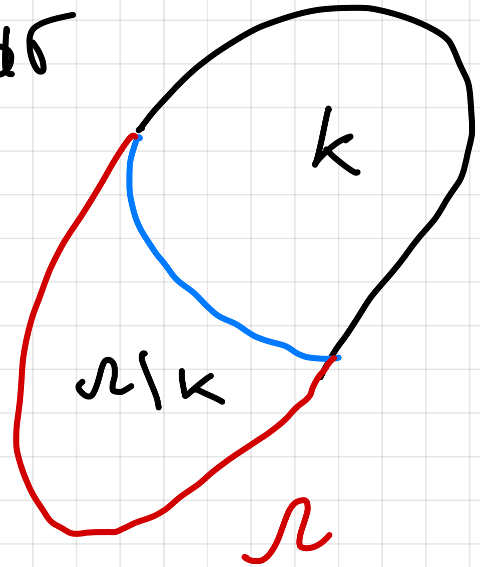
If u is a minimizer then u solves the previous PDE with

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega \setminus \partial K$$

ROBIN B.C.

The energy $E(K,\Omega)$ is the heat rate loss

$$\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\sigma = \int_{\Omega\setminus K} |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\sigma$$



$$= \int_{\partial(\Omega\setminus K)} u \frac{\partial u}{\partial \nu} + \beta \int_{\partial\Omega} u^2 =$$

$$\int_{\partial K \cap \Omega} \frac{\partial u}{\partial \nu} + \int_{\partial\Omega \setminus \partial K} u \cdot \frac{\partial u}{\partial \nu} + \beta \int_{\partial\Omega} u^2 =$$

$$= -\int_{\partial\Omega\setminus\partial K} \overbrace{\frac{\partial u}{\partial \nu}}^{=-\beta u} + \int_{\partial\Omega\setminus\partial K} \overbrace{u \frac{\partial u}{\partial \nu}}^{=-\beta u^2} + \beta \int_{\partial\Omega\setminus\partial K} u^2 + \beta \int_{\partial\Omega\cap\partial K} \overset{u}{\cancel{u^2}} = \beta \int_{\partial\Omega} u$$

The energy $E(k,\Omega)$ is the heat rate loss

Problem 1 For given k and $M > \mathrm{Vol}(k)$

prove that $\inf\left\{ E(k,\Omega) : \begin{array}{l}\Omega \text{ open} \\ \overline{\Omega} \supseteq k\end{array} \quad \mathrm{Vol}(\Omega) \le M \right\}$

is achieved.

CAFFARELLI - KRIVENTSOV.

The RADIAL CASE

Assume $K = B_1$ unit ball and $\Omega = B_R$ $R > 1$

$$u(r) = 1 - \frac{\beta\,(\phi(r) - \phi(1))_+}{\phi'(R) + \beta(\phi(R) - \phi(1))}$$

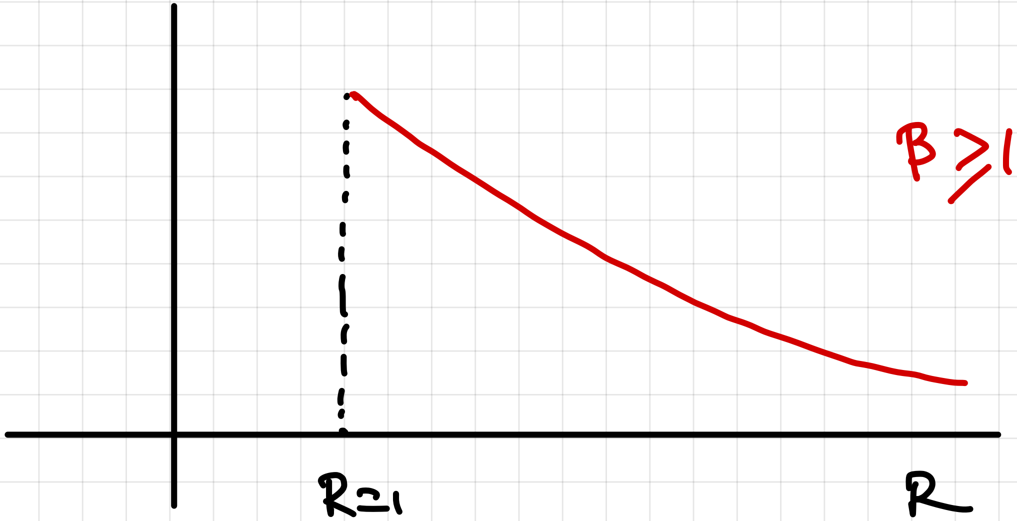
$N = 2$ $\qquad \phi(r) = \log r$

$N \geq 3$ $\qquad \phi(r) = -\frac{r^{(2-N)}}{N-2}$

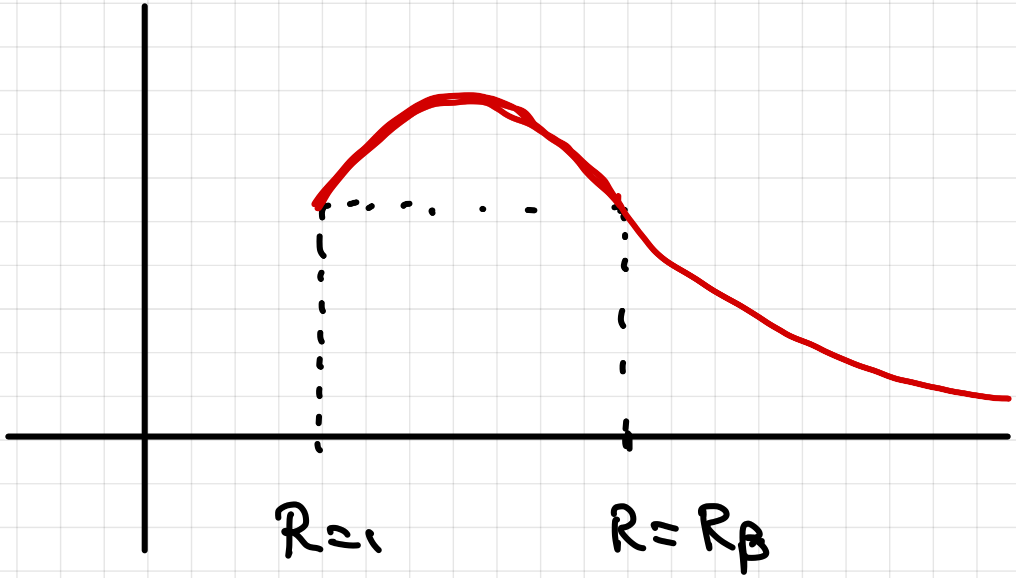
$$E_\beta(B_1, B_R) = \frac{\beta N \omega_N}{\phi'(R) + \beta(\phi(R) - \phi(1))} = E_\beta(R)$$

Let's plot $E_\beta(R)$

$\boxed{N=2}$



$\beta \geq 1$

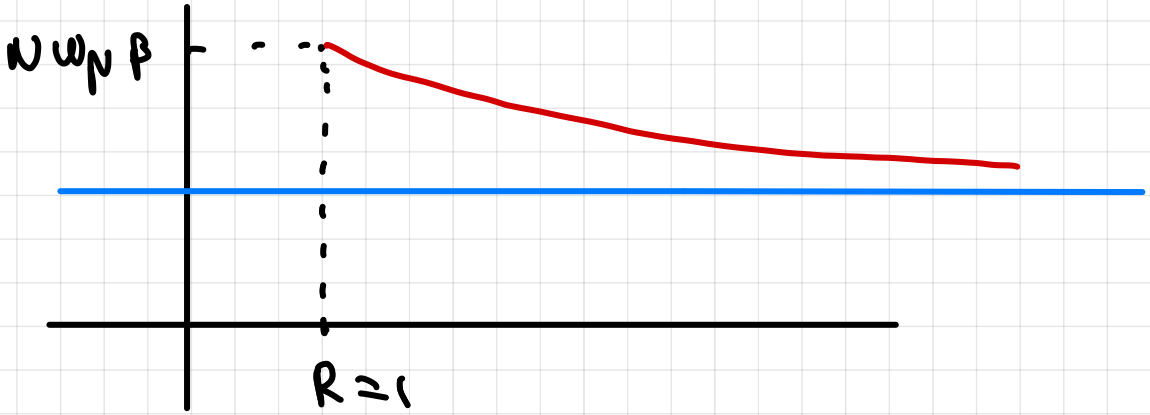


$0 < \beta < 1$

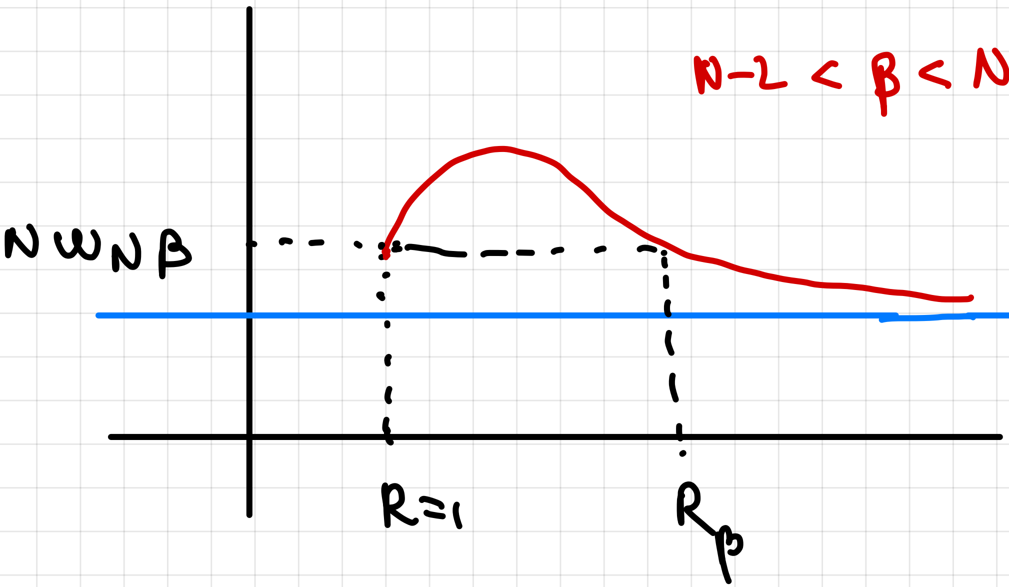
Let's plot $E_\beta(R)$

$\boxed{N \geq 3}$

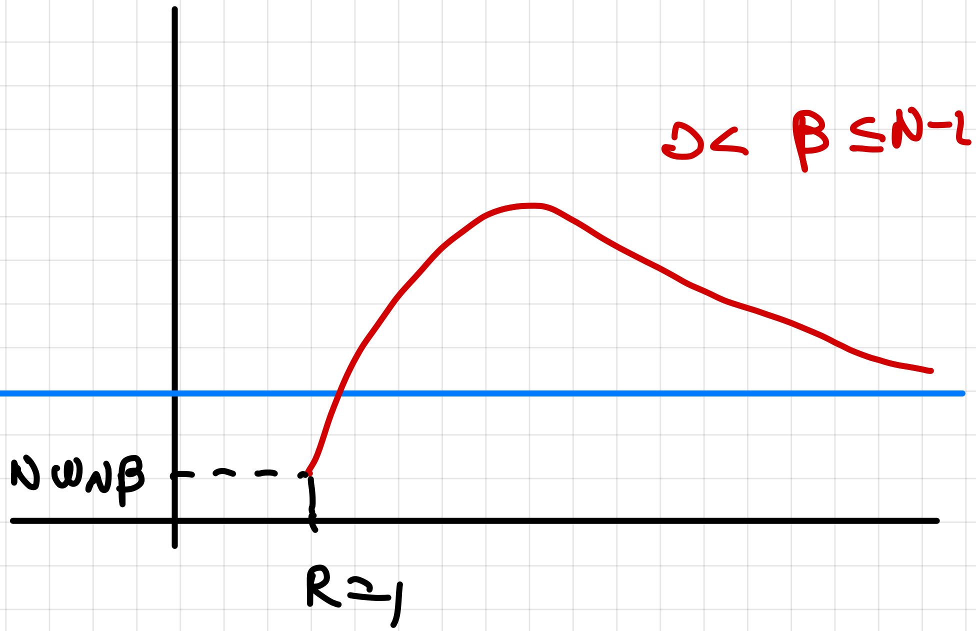
$\beta \geq N-1$



$N-2 < \beta < N-1$



$0 < \beta \leq N-2$

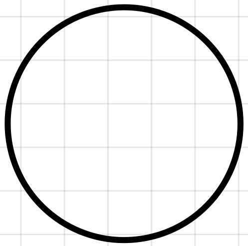


PROBLEM 2 (BUCUR - NAHON - NITSCH - T. CALC VAR)

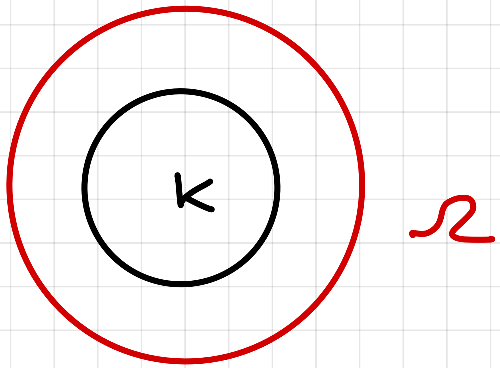
For given $m < M$ find the minimum of

$$\left\{ E(K,\Omega) \; : \; K \subseteq \overline{\Omega}; \quad \mathrm{Vol}(K) = m \quad \mathrm{Vol}(\Omega) \le M \right\}$$

THEOREM: The solution consists of two concentric balls: one of volume m and the other of volume M



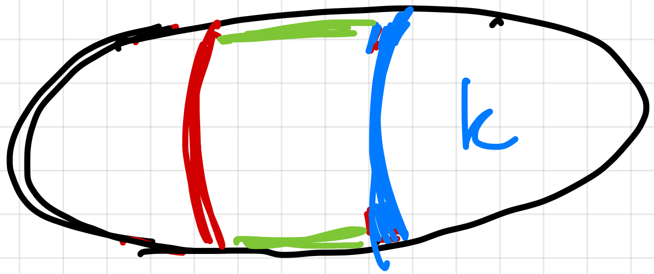
$K \equiv \Omega$



$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K \\ u = 1 & \text{in } K \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \partial\Omega \setminus \partial K \end{cases}$$

Let $\Omega_t = \{ x \in \Omega : u > t \}$

$0 < t < 1$

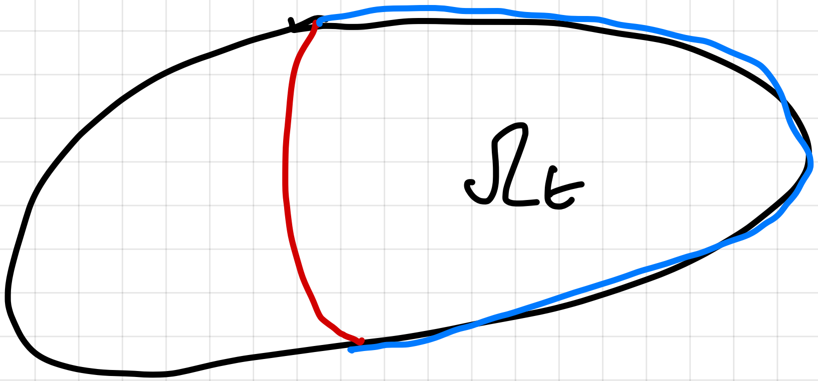


$$0 = \int_{\{t < u < 1\}} \frac{\Delta u}{u} \, dx = \int_{\partial\{t < u < 1\}} \frac{\nabla u}{u} \cdot \nu_{\Omega_t} \, d\sigma + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \, dx =$$

$$= \int_{K \cap \partial\Omega} |\nabla u| \, d\sigma - \int_{\partial\Omega_t} \frac{|\nabla u|}{u} \, d\sigma + \int_{\partial(t<u<1) \cap \partial\Omega} \underbrace{\frac{1}{u} \frac{\partial u}{\partial \nu_\Omega}}_{= -\beta} \, d\sigma + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \, dx$$

$$\int_{K\cap\partial\Omega} |\nabla u|\, d\sigma = \int_{\partial^i\Omega_t} \frac{|\nabla u|}{u} + \beta\, \mathcal{H}^{n-1}\left(\partial(t<u<1)\cap\partial\Omega\right) - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2}$$

$$E(K,\Omega) = \beta\, \mathcal{H}^{n-1}\left(\partial\Omega_t \cap \partial\Omega\right) + \int_{\partial\Omega_t^i} \frac{|\nabla u|}{u}\, d\sigma - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2}\, dx$$



for a.e. $t\in(0,1)$

The H function (Bossel-Daners)

$$H(t,\phi) = \beta \mathcal{H}^{n-1}(\partial\Omega_t \cap \partial\Omega) + \int_{\partial\Omega_t^i} \phi \, d\sigma - \int_{\Omega_t} \phi^2 \, dx$$

- $E(k,\Omega) = H\left(t, \frac{|\nabla u|}{u}\right)$ for a.e. $t \in (0,1)$

  u solution in k, Ω

The H function (Bossel-Daners)

$$H_\Omega(t,\phi) = \beta\, \mathcal{H}^{n-1}(\partial\Omega_t \cap \partial\Omega) + \int_{\partial\Omega_t^i} \phi \, d\sigma - \int_{\Omega_t} \phi^2 \, dx$$

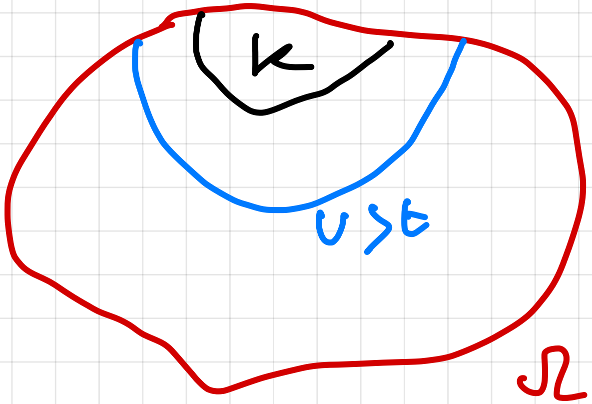
- $$E(k,\Omega) = H_\Omega\left(t, \frac{|\nabla u|}{u}\right) \quad \text{for a.e. } t \in (0,1)$$

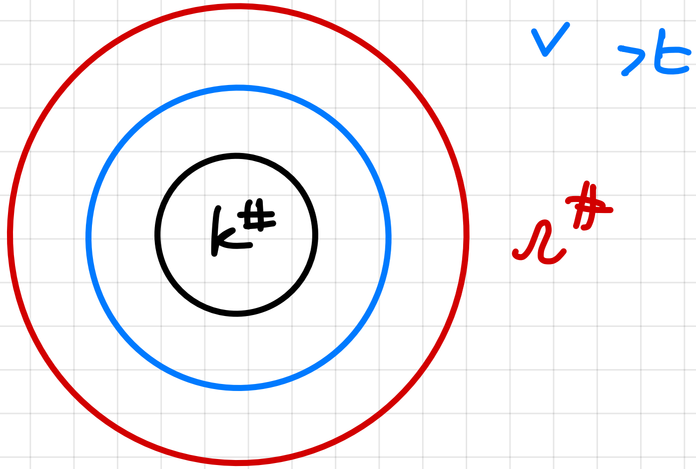
u solution in k, Ω

. Lemma: $\forall \phi \in C^\infty$ there exists $t \in (0,1)$:

$$E(k,\Omega) \geq H_\Omega(t,\phi)$$

De rearrangement (BOSSEL-DANERS)





$$\phi^* = \frac{|\nabla v|}{v}$$

$$\tilde{\phi} : \quad \tilde{\phi}_{|\partial^i\{u>t\}} = \phi^*_{|\{v>g(t)\}}$$

t : : $\mathrm{Vol}(\{u>t\}) = \mathrm{Vol}\{v>g(t)\}$

u : solution in k, Ω

$$E(k,\Omega) = H_{\Omega}\left(t, \frac{|\nabla u|}{u}\right)$$

but for $\phi \in C^{\infty}$ $\exists \bar{t}$:

$$E(k,\Omega) \geq H_{\Omega}(\bar{t}, \phi)$$

v : solution in $k^{\#}, \Omega^{\#}$

$$\boxed{E(k^{\#},\Omega^{\#})} = H_{\Omega^{*}}\left(t, \frac{|\nabla v|}{v}\right)$$

for a.e. t

u : solution in K, Ω

$$E(K,\Omega) = H_{\Omega}\left(t, \frac{|\nabla u|}{u}\right)$$

t : $$\geq H_{\Omega}(t, \tilde{\phi})$$

isoperimetric ineq.

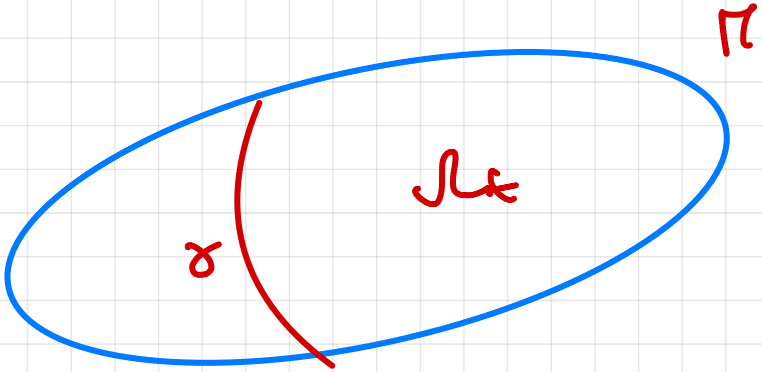
$$\geq H_{\Omega^\#}(t, \phi^*)$$

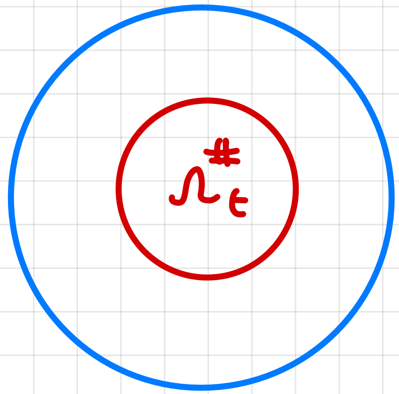
$$= E(K^\#, \Omega^\#)$$

v : solution in $K^\#, \Omega^\#$

$$\boxed{E(K^\#,\Omega^\#)} = H_{\Omega^\#}\Big(t, \underbrace{\frac{|\nabla v|}{v}}_{\phi^*}\Big)$$

$$H(t,\tilde{\phi}) = \beta \mathcal{H}^{n-1}(\partial\Omega_t \cap \partial\Omega) + \int_{\partial\Omega_t^i} \tilde{\phi}\, d\sigma - \int_{\Omega_t} \tilde{\phi}^2\, dx$$





$$\beta H^{n-1}(\Gamma) + H^{n-1}(\gamma)\cdot \phi^*_{\partial\Omega_t^\#} - \int_{\Omega_t} \tilde{\phi}^2$$

$$H^{n-1}(\partial\Omega_t^\#)\cdot \phi^*\big|_{\partial\Omega_t^\#} - \int_{\Omega_t^\#} \phi^{*2}$$

moreover $\quad \phi^* \le \beta$

RELAXATION

$$\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{n-1}$$

$$F_\beta(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \beta \int_{J_u} (u_+^2 + u_-^2) \, d\mathcal{H}^{n-1}$$

$u \in SBV(\mathbb{R}^N)$ $\quad \mathrm{Vol}(u \geq 1) \geq m \quad \mathrm{Vol}(u > 0) \leq k$

$$F_\vartheta(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{J_u} \left(\vartheta(u^+) + \vartheta(u^-) \right) d H^{n-1}$$

Relaxation

$u \in SBV(\mathbb{R}^N)$; $\mathrm{Vol}(u \geq 1) \geq m$; $\cancel{\mathrm{Vol}(\{u>0\}) \leq M}$

$$F_{\vartheta,\Lambda} = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{J_u} \left[\vartheta(u^+) + \vartheta(u^-) \right] d\mathcal{H}^{N-1} +$$

$$+ \Lambda \, \mathrm{Vol}(\{u>0\})$$

$\vartheta : \mathbb{R} \to \mathbb{R}^+$ l.s.c. ; non decreasing ; $\vartheta(0) = 0$

Th. $F_{\vartheta,\Lambda}$ is minimized by a radial function u $\Rightarrow$ K and Ω are concentric balls (possibly $K \equiv \Omega$).

Thank you for your attention!