Poincaré–Sobolev Inequalities in domains

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Shape Optimisation and Geometric Spectral Theory

joint work with Rafael Benguria and Cristóbal Vallejos.

A scale-invariant interpolation inequality

$\Omega \subset \mathbb{R}^d$ bounded, connected.
$\bar{v} = \frac{1}{|\Omega|} \int_\Omega v$.

$$\int_\Omega |v - \bar{v}|^p \leq \frac{1}{C_\Omega} \int_\Omega |\nabla v|^2 \left(\int_\Omega v^2 \right)^q$$

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$C_\Omega = C_{\lambda \Omega}$

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$u = v - \bar{v}, \quad \nabla u = \nabla v, \quad \int u^2 = \int v^2 - |\Omega| \bar{v}^2$

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$$C_\Omega = \inf_{\substack{u \in H^1(\Omega) \\ u \perp 1}} \frac{\int_\Omega |\nabla u|^2 \left(\int_\Omega u^2\right)^{\frac{2}{d}}}{\int_\Omega |u|^{2+\frac{4}{d}}}$$

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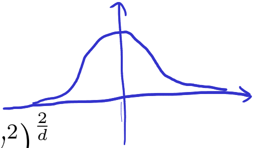

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$C_{GNS}(1) = \frac{\pi^2}{4}$, minimizers exist for any d.

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If $d = 1$, $\Omega = (0,1)$ no minimizers exist and $C_\Omega = C_{\mathrm{GNS}}/4$

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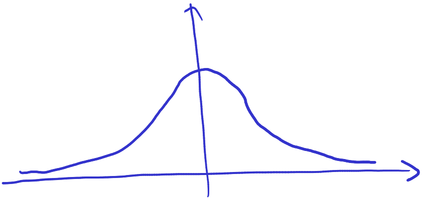

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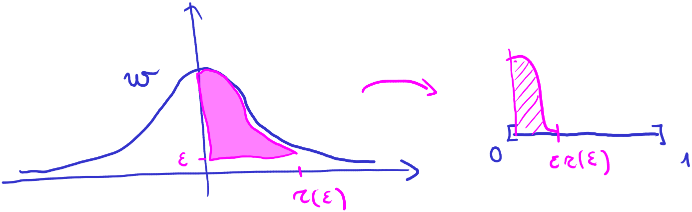

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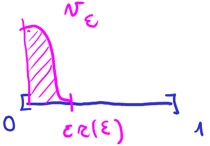
$v_\varepsilon = \left[w\left(\frac{x}{\varepsilon}\right) - \varepsilon \right]_+$



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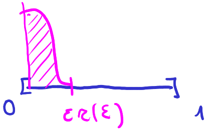

- $\bar{v}_\varepsilon \sim \varepsilon$
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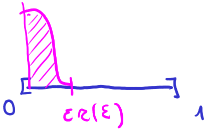

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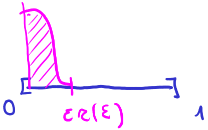

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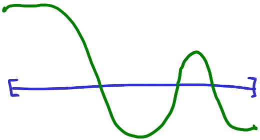

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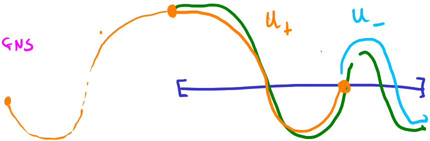
$$\frac{2\int_0^1 u_+'^2 \left(2\int_0^1 u_+^2\right)^2}{2\int_0^1 u_+^6} \ge C_{GNS}$$



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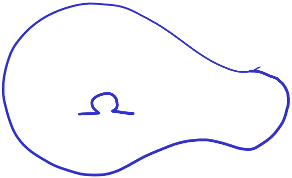

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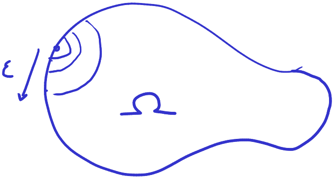

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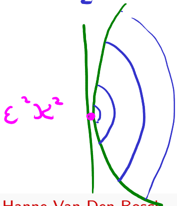

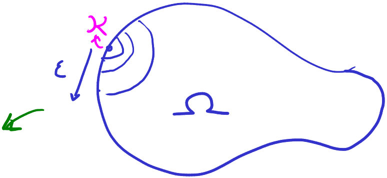

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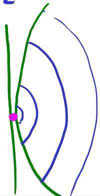
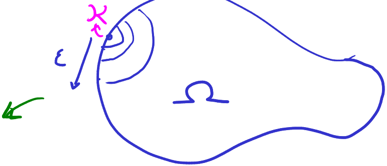
$$\frac{\int |\nabla u_\varepsilon|^2 \left(\int u_\varepsilon^2\right)^{2/d}}{\int |u_\varepsilon - \bar{u}_\varepsilon|^{2+\frac{4}{d}}} = C_{GNS}(d) - \varepsilon^2 \kappa^2 C_d + \dots$$

$\varepsilon^2 \kappa^2$





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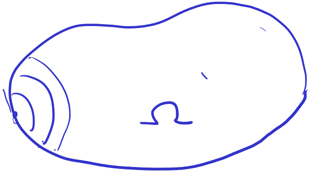
2) u_n minimizing sequence.

Loss of compactness

$\rightsquigarrow$ concentration



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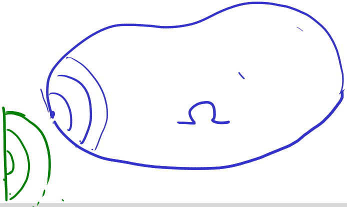

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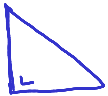
- No minimizer exists for Ω : 

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- Conjecture: nonexistence for $[0,1]^2$.

$\updownarrow$

existence
- for $[0,1] \times [0,L]$ $L \gg 1$.
- for $[0,1]^d$ $d \geq 9$.

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Open questions

- Optimal shape ? (Maximal ?)
- Symmetry of minimizer for ball,...
- On sphere ?
- ...

Thank You !

References: ArXiv:1810.05698 and ArXiv:1802.01740