An inequality for the normal derivative of the Lane–Emden ground state

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joint work with
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Setting up the problem

Consider the variational problem

$$\lambda_q(\Omega) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^q(\Omega)}}\,,$$

with $1 \leq q \leq 2$ and $\Omega \subset \mathbb{R}^d$ is an open set.

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If $\lambda_1(\Omega) > 0$ the embedding $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous and the infimum is unchanged if $C_0^\infty(\Omega)$ is replaced by $\mathcal{D}_0^{1,2}(\Omega)$.

Here we mostly consider Ω for which the embedding is compact. In this case the infimum over $\mathcal{D}_0^{1,2}(\Omega)$ is attained by some $u \in \mathcal{D}_0^{1,2}(\Omega)$. Today we are interested in **properties of minimizers** and how they depend on the set Ω.

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Any minimizer u solves the **Lane–Emden equation**

$$\begin{cases} -\Delta u = \lambda_q(\Omega)\|u\|_{L^q(\Omega)}^{2-q}|u|^{q-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

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Throughout $u_{q,\Omega}$ denotes a non-negative minimizer normalized in $L^q(\Omega)$.

The linear case: For $q = 2$ we recognize $\lambda_2(\Omega)$ as the **lowest** eigenvalue of the Dirichlet Laplacian in Ω and the set of all minimizers is the corresponding eigenspace.

The sub-homogeneous case: For $1 \leq q < 2$ it is common to instead study minimal energy solutions of $-\Delta \tilde{u} = \tilde{u}^{q-1}$ and the energy $\mathbf{F}_q(\Omega) = \|\nabla \tilde{u}\|^2_{L^2}$.

By homogeneity

$$\tilde{u} = \lambda_q(\Omega)^{-1/(2-q)} u_{q,\Omega} \quad \text{and} \quad \mathbf{F}_q(\Omega) = \lambda_q(\Omega)^{-q/(2-q)}.$$

In particular, the quantity $\mathbf{F}_1(\Omega) = 1/\lambda_1(\Omega)$ is the torsional rigidity of Ω and the solution $\tilde{u}$ is the classical torsion function; $-\Delta w = 1$ with $w|_{\partial\Omega} = 0$.

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Remark: There are interesting differences between the cases of $q = 2$ and $1 < q < 2$ (see e.g. Brasco–Franzina '20), for instance:

- If $q = 2$ then the critical values of $u \mapsto \|\nabla u\|_{L^2}^2 / \|u\|_{L^2}^2$ is an infinite discrete set (the spectrum of the Dirichlet Laplacian).
- For $1 < q < 2$ the critical values of $u \mapsto \|\nabla u\|_{L^2}^2 / \|u\|_{L^q}^2$ is a closed infinite set but it is not in general known to be countable. There are examples where the set fails to be discrete.

Basic properties:

i) (*monotonicity*) If $\Omega' \subset \Omega$, then $\lambda_q(\Omega') \geq \lambda_q(\Omega)$.

ii) (*scaling*) Let $\alpha_q = (2 + d(2/q - 1))^{-1}$, then for all $s > 0$

$$\lambda_q(s\Omega) = s^{-1/\alpha_q}\lambda_q(\Omega) \quad \text{and} \quad u_{q,s\Omega}(x) = s^{-d/q}u_{q,\Omega}(x/s)\,.$$

iii) (*disjoint unions*) If $\Omega = \bigcup_{j\geq 1}\Omega_j$ with $\Omega_j \cap \Omega_{j'} = \emptyset$ when $j \neq j'$, then

a) for $1 \leq q < 2$

$$\lambda_q(\Omega) = \Biggl(\sum_{j\geq 1}\lambda_q(\Omega_j)^{-\frac{q}{2-q}}\Biggr)^{-\frac{2-q}{q}} \quad \text{and} \quad u_{q,\Omega} = \sum_{j\geq 1}\Bigl(\frac{\lambda_q(\Omega)}{\lambda_q(\Omega_j)}\Bigr)^{\frac{1}{2-q}}u_{q,\Omega_j}\,.$$

b) for $q = 2$ then $\lambda_2(\Omega) = \min_{j\geq 1}\lambda_2(\Omega_j)$ and the set of minimizers is the linear span of

$$\bigl\{u_{q,\Omega_j} : j \geq 1 \text{ such that } \lambda_2(\Omega_j) = \lambda_2(\Omega)\bigr\}.$$

iv) (*continuity interior exhaustion*) If $\Omega \subset \mathbb{R}^d$ is open and $\{\Omega_j\}_{j\geq 1}$ satisfy $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j\geq 1}\Omega_j = \Omega$ then

$$\lim_{j\to\infty}\lambda_q(\Omega_j) = \lambda_q(\Omega)\,.$$

Main result

Theorem

Fix $1 \leq q \leq 2$, let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary. Then

$$\frac{1}{\lambda_q(\Omega)^{1+\alpha_q}} \int_{\partial\Omega} \left|\frac{\partial u_{q,\Omega}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x) \geq \frac{1}{\lambda_q(B)^{1+\alpha_q}} \int_{\partial B} \left|\frac{\partial u_{q,B}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x),$$

where B is the unit ball and $\alpha_q = (2 + d(2/q - 1))^{-1}$.

Remarks:

- That the normal derivative $\frac{\partial u_{q,\Omega}}{\partial\nu}$ can be made sense of when $\partial\Omega$ is irregular follows from classical work of Dahlberg, Jerison–Kenig, Verchota in the 70's and 80's.
- The theorem combined with Faber–Krahn-type inequalities for λ_q implies

$$|\Omega|^{\frac{1+\alpha_q}{d\alpha_q}} \int_{\partial\Omega} \left|\frac{\partial u_{q,\Omega}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x) \geq |B|^{\frac{1+\alpha_q}{d\alpha_q}} \int_{\partial B} \left|\frac{\partial u_{q,B}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x)\,,$$

and

$$|\partial\Omega|^{\frac{1+\alpha_q}{(d-1)\alpha_q}} \int_{\partial\Omega} \left|\frac{\partial u_{q,\Omega}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x) \geq |\partial B|^{\frac{1+\alpha_q}{(d-1)\alpha_q}} \int_{\partial B} \left|\frac{\partial u_{q,B}}{\partial\nu}\right|^2 d\mathcal{H}^{d-1}(x)\,.$$

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History: For convex sets bounds of this form have appeared earlier, in particular in connection to Minkowski-type problems:

- For $q = 2$ the bound is (implicitly) in Jerison Adv. Math. '96 (for problem of electrostatic capacity an analogue appears in Jerison Acta Math. '96).
- For $q = 1$ the bound is (implicitly) in Colesanti–Fimiani '10.
- For $q \in \{1, 2\}$ the bounds appear in Bucur–Fragala–Lamboley '12.
- Similar results but where the Laplacian is replaced by the p-Laplace operator appear in Colesanti–Nyström–Salani–Xiao–Yang–Zhang '15.

Strategy of proof

Our aim is to mimic a classical argument to pass from the **classical Brunn–Minkowski inequality** to the **classical isoperimetric inequality**.

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(if Ω is regular) $\quad \mathrm{Per}(\Omega) \geq d|B|^{1/d}|\Omega|^{(d-1)/d}. \qquad \square$

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$$\mathcal{SM}_*(\Omega) := \liminf_{t \to 0^+} \frac{|\Omega + tB| - |\Omega|}{t} = \liminf_{t \to 0^+} \frac{|\{x \in \Omega^c : \mathrm{dist}(x, \Omega) < t\}|}{t}.$$

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Today we shall follow this strategy but with the shape functional $\Omega \mapsto |\Omega|$ replaced by $\Omega \mapsto \lambda_q(\Omega)$.

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The essence of the strategy boils down to:

1) a Brunn–Minkowski inequality for λ_q, and
2) computing (one-sided) derivative of $t \mapsto \lambda_q(\Omega + tB)$ at $t = 0$.

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Part 1) is ok.

Theorem

For $1 \le q \le 2$, $0 \le s \le 1$, and $\Omega_0, \Omega_1 \subset \mathbb{R}^d$ open sets

$$\lambda_q((1-s)\Omega_0 + s\Omega_1) \le \Big((1-s)\lambda_q(\Omega_0)^{-\alpha_q} + s\lambda_q(\Omega_1)^{-\alpha_q}\Big)^{-1/\alpha_q}.$$

For $1 \le q < 2$ equality holds for some $s \in (0,1)$ if and only if either

- $\min\{\lambda_q(\Omega_0), \lambda_q(\Omega_1)\} = 0$, *or*
- *both Ω_0 and Ω_1 agree with homothetic copies of a bounded convex set K up to sets of zero capacity.*

This is (essentially) proved in
- Brascamp–Lieb '76 for $q = 2$,
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Part 2) needs more work.

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- Compute derivative of $t \mapsto \lambda_q(\Phi(t, \Omega))$ when $\Phi\colon (-T, T) \times \mathbb{R}^d \to \mathbb{R}^d$ is sufficiently regular. (Hadamard formula)
- Prove that for regular Ω the mapping $t \mapsto \Omega + tB$ can be approximated by regular perturbations of Ω.
- Combining these two results with the BM inequality one proves the main bound for regular Ω.
- Use main inequality for regular sets and approximation argument to obtain the general result.

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In what remains we take a look at the first two points, aiming to prove:

Lemma

Fix $1 \leq q \leq 2$, let $\Omega \subset \mathbb{R}^d$ be open, bounded, connected with C^1 boundary. Then

$$\lim_{t \to 0^+} \frac{\lambda_q(\Omega + tB) - \lambda_q(\Omega)}{t} = -\int_{\partial\Omega} \left| \frac{\partial u_{q,\Omega}}{\partial \nu} \right|^2 d\mathcal{H}^{d-1}(x)\,.$$

Remark: For convex Ω this is simplified by representation of $\lambda_q(\Omega)$ as an integral over $\mathbb{S}^{d-1}$ and properties of the Minkowski sum.

A Hadamard formula for $\lambda_q(\Omega)$

Theorem

Fix $1 \leq q \leq 2$ and $\Omega \subset \mathbb{R}^d$ open set of finite measure such that the normalized minimizer $u_{q,\Omega}$ is unique. Let $\Phi \in C^1((-1,1); W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, be such that $\Phi(t,\,\cdot\,)\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz homeomorphism of a neighbourhood of Ω onto its image, and

$$\Phi(t,x) = x + t\dot{\Phi}(x) + o_{t\to 0}(t) \quad \text{in } W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)\,.$$

Then $t \mapsto \lambda_q(\Phi(t,\Omega))$ is differentiable at $t=0$ and

$$\begin{aligned}\lim_{t\to 0}\frac{\lambda_q(\Phi(t,\Omega))-\lambda_q(\Omega)}{t} &= -2\int_\Omega \nabla u_{q,\Omega}\cdot \big(D\dot{\Phi}\big)\nabla u_{q,\Omega}\,dx \\ &\quad + \int_\Omega \Big(|\nabla u_{q,\Omega}|^2 - \frac{2}{q}\lambda_q(\Omega)u_{q,\Omega}^q\Big)\nabla\cdot\dot{\Phi}\,dx\,.\end{aligned}$$

If Ω has Lipschitz boundary,

$$\lim_{t\to 0}\frac{\lambda_q(\Phi(t,\Omega))-\lambda_q(\Omega)}{t} = -\int_{\partial\Omega}\Big|\frac{\partial u_{q,\Omega}}{\partial\nu}\Big|^2 \nu\cdot\dot{\Phi}\,d\mathcal{H}^{d-1}(x)\,.$$

Remark: For $q=1$ or 2 this is classical.

A Hadamard formula for $\lambda_q(\Omega)$

The result looks standard, but the standard proof runs into problems.

Classically: Differentiability of $t \mapsto (\lambda_q(\Phi(t,\Omega)), u_{q,\Phi(t,\Omega)})$ is established by using the implicit function theorem applied to the mapping

$$H_0^1(\Omega) \times \mathbb{R} \times (-1,1) \to H^{-1}(\Omega) \times \mathbb{R}$$

$$\begin{pmatrix} v \\ \lambda \\ t \end{pmatrix} \mapsto \begin{pmatrix} -(\Delta(v \circ \Phi(t,\cdot)^{-1})) \circ \Phi(t,\cdot) - \lambda v^{q-1} \\ \int_\Omega |v|^q |\det D_x \Phi(t,x)|\, dx - 1 \end{pmatrix}$$

Problem: for $1 < q < 2$ the map $v \mapsto v^{q-1}$ is **not** Fréchet differentiable.

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Problem: for $1 < q < 2$ the map $v \mapsto v^{q-1}$ is **not** Fréchet differentiable.

Our solution: Use a variational proof which avoids differentiating $t \mapsto u_{q,\Phi(t,\Omega)}$.

Approximation of Minkowski sum

Remaining problem: Want to construct regular map Φ so that, for $t > 0$ small, $\Phi(t, \Omega)$ approximates $\Omega + tB$,

$$\Phi(t, x) = x + t\dot{\Phi}(x) + o(t) \qquad \text{and} \qquad \dot{\Phi}|_{\partial\Omega} = \nu_{\partial\Omega}.$$

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Theorem

Let $\Omega \subset \mathbb{R}^d$ be open and bounded with C^1 boundary and fix $\varepsilon, \delta > 0$. There exists a map $\Phi \in C^\infty((-1,1); C^\infty(\mathbb{R}^d;\mathbb{R}^d))$ so that

- $\Phi(t,x) = x + t\dot{\Phi}(x)$ *with* $\dot{\Phi} \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ *supported near* $\partial\Omega$,
- *for* $|t|$ *sufficiently small* $\Phi(t,\cdot)$ *is a diffeomorphism of* $\mathbb{R}^d$ *onto itself,*
- *for sufficiently small* $t > 0$,

$$\Phi(t,\Omega) \subset \Omega + tB \subset \Phi((1+\delta)t, \Omega)$$

- *and* $\|\dot{\Phi} - \nu_{\partial\Omega}\|_{L^\infty(\partial\Omega)} < \varepsilon$.

Remark: The assumptions are essentially sharp: Setting

$$\rho(\Omega) := \inf\{\|X - \nu_{\partial\Omega}\|_{L^\infty(\partial\Omega)} : X \in C^0(\partial\Omega;\mathbb{R}^d), |X| = 1\}$$

then by Hofmann–Mitrea–Taylor '07

$$\rho(\Omega) = 0 \iff \partial\Omega \text{ is } C^1 \qquad \text{and} \qquad \rho(\Omega) < \sqrt{2} \iff \partial\Omega \text{ is Lipschitz.}$$

Approximation of Minkowski sum

Define the **signed distance function**

$$\delta_\Omega(x) = \mathrm{dist}(x, \Omega) - \mathrm{dist}(x, \Omega^c), \quad \text{note that } |\nabla \delta_\Omega| = 1 \text{ a.e.}$$

Then, for $t > 0$,

$$\Omega + tB = \{x \in \mathbb{R}^d : \delta_\Omega(x) < t\}$$

and a natural candidate for Φ is

$$(t, x) \mapsto x + t\nabla\delta_\Omega(x).$$

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But if $\partial\Omega$ is non-regular then the vector field $\nabla\delta_\Omega$ is only defined almost everywhere and is certainly not smooth.

Solution: Replace $\nabla\delta_\Omega$ by a new vector field obtained by localizing $\nabla\delta_\Omega$ close to $\partial\Omega$ and mollifying.

The proof of the theorem is reduced to verifying the stated properties through explicit calculations.

Thank you for your attention!