Preservation of concavity by the Dirichlet heat flow

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Shape Optimisation and Geometric Spectral Theory

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Introduction

Based on joint work with

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and
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Papers

[IST1]*To Logconcavity and Beyond*, Comm. Contemporary Mathematics (2020)

[IST2]*New characterizations of log-concavity via Dirichlet heat flow*, Annali Mat. Pura Appl. (2021)

[IST3]*Characterization of F-concavity preserved by the Dirichlet heat flow*, preprint (2022)

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Logcancave functions

A nonnegative function u *in* $\mathbf{R}^N$ *is said* logconcave *in* $\mathbf{R}^N$ *if*

$$u((1-\mu)x+\mu y) \geq u(x)^{1-\mu}u(y)^{\mu}$$

for $\mu \in [0,1]$ *and* $x, y \in \mathbf{R}^N$ *such that* $u(x)u(y) > 0$.

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This is equivalent to the following:

- the set $S_u := \{x \in \mathbf{R}^N \,:\, u(x) > 0\}$ is convex

and

- $\log u$ is concave in S_u.

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Logconcavity is a very useful *variation of concavity* and plays an important role in various fields such as PDEs, geometry, probability, statics, optimization theory and so on.

Power concavity

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Roughly speaking, for some $p \in [-\infty, +\infty]$ a nonnegative function u is said *p*-concave if the set S_u is convex and:

- Case $p = +\infty$: u is a positive constant in $S_u = \{u > 0\}$;
- Case $p > 0$: u^p is concave in S_u;
- Case $p = 0$: $\log u$ is concave in S_u;
- Case $p < 0$: u^p is convex in S_u;
- Case $p = -\infty$: u is quasiconcave, i.e. all its superlevel sets are convex.

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To give a more precise definition, it is useful to introduce the concept of power mean.

p-means

Let $p \in [-\infty, +\infty]$ and $\mu \in (0, 1)$. Given two real numbers $a > 0$ and $b > 0$, the quantity

$$M_p(a, b; \lambda) = \begin{cases} \max\{a, b\} & p = +\infty \\ [(1-\lambda)a^p + \lambda b^p]^{1/p} & \text{per } p \neq -\infty,\ 0,\ +\infty \\ a^{1-\lambda} b^{\lambda} & p = 0 \\ \min\{a, b\} & p = -\infty \end{cases} \tag{0.1}$$

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For $p = 1$ we have the usual *arithmetic mean*, for $p = 0$ we have the usual *geometric mean*.

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$$u((1-\mu)x+\mu x) \geq M_p(u(x),\, u(y);\, \mu)$$

for $\mu \in [0,1]$ and x, $y \in \mathbf{R}^N$.
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Then we have an order between p-concavity: if u is p-concave for some p, then u is q concave for every $q \leq p$, i.e. $\mathcal{C}[q] \supseteq \mathcal{C}[p]$ for $q \leq p$.

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In particular, quasiconcavity is the weakest conceivable power concavity property and every power concave function is quasiconcave.

The marriage of logconcavity and heat flow

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Much of its relevance, especially for elliptic and parabolic equations, is due to the fact that the Gauss kernel

$$G(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \tag{0.3}$$

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Indeed,

$$\log\, G(x,t) = -\frac{|x|^2}{4t} + \log(4\pi t)^{-\frac{N}{2}} \qquad (0.4)$$

is concave in $\mathbf{R}^N$ for any fixed $t > 0$.

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Preservation of logconcavity by the heat flow

Let u be a bounded nonnegative solution of

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{0.5}$$

where Ω is a convex domain in $\mathbf{R}^N$ and u_0 is a bounded nonnegative function in Ω. Then $u(\cdot, t)$ is logconcave in Ω for any $t > 0$ if u_0 is logconcave in Ω.

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In the case of $\Omega = \mathbf{R}^N$, I will simply write $e^{t\Delta} u_0$ for $e^{t\Delta_{\mathbf{R}^N}} u_0$, that is,

$$[e^{t\Delta} u_0](x) = \int_{\mathbf{R}^N} G(x-y,t) u_0(y)\, dy, \quad x \in \mathbf{R}^N, \ t > 0. \tag{0.6}$$

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where Ω is a convex domain in $\mathbf{R}^N$ and u_0 is a bounded nonnegative function in Ω. Then $u(\cdot, t)$ is logconcave in Ω for any $t > 0$ if u_0 is logconcave in Ω.

Thanks to a limit procedure, i.e.

$$\lim_{t \to +\infty} e^{\lambda_1 t} [e^{t\Delta_\Omega} \chi_\Omega](x) = \phi_1(x)\,,$$

this implies the first eigenfunction of the Laplacian in a convex domain is logconcave.

The marriage of logconcavity and heat flow

Logconcavity is so naturally and deeply linked to heat transfer that $e^{t\Delta}u_0$ spontaneously becomes logconcave in $\mathbf{R}^N$ even without the logconcavity of initial function u_0. Indeed, Lee and Vázquez (2003) proved the following:

Let u_0 be a bounded nonnegative function in $\mathbf{R}^N$ with compact support. Then there exists $T > 0$ such that $e^{t\Delta}u_0$ is logconcave in $\mathbf{R}^N$ for $t \geq T$.

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And this is true, at least among power concavities!

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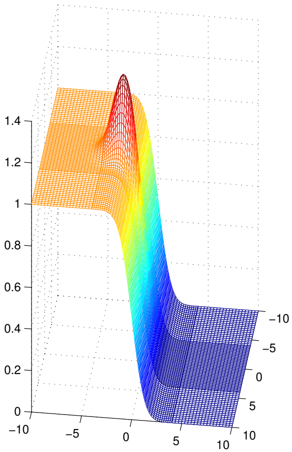
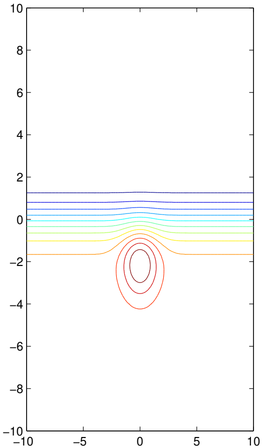
Indeed, in [IST1] we proved that no p-concavity for $p > 0$ is preserved by $e^{t\Delta}$. Furthermore, if we start with a weaker concave initial datum, say p-concave for any $p < 0$, then even quasiconcavity may be immediately lost [IST3] (see also Ishige-S., Arch. Math. (2008) and Int. Free Bound. (2010)).





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Finally, for $N \geq 2$ logconcavity is the weakest and the strongest (then the only) power concavity preserved by DHF.

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For $N = 1$ also quasi-concavity is preserved [Angenent, 1988], but it is a particular case, because in this case quasi-concave means monotone or first increasing up to a certain point, then possibly constant for an interval, finally decreasing.

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(Q) *Is logconcavity the strongest/weakest concavity property preserved by DHF in a convex domain? If not, what is the strongest/weakest concavity property preserved by DHF?*

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To this aim, in [IST1] we introduce a new concavity property – we called it *power log-concavity* – which generalizes logconcavity.

Power logconcavity

For $\alpha > 0$ we consider the strictly increasing function $L_\alpha : [0,1] \to [-\infty, 0]$ defined as follows

$$L_\alpha(s) := -(-\log s)^\alpha \quad \text{for} \quad s \in (0,1], \quad L_\alpha(s) := -\infty \quad \text{for} \quad s = 0$$

and we say that a function $0 \leq u \leq 1$ is α*-logconcave* if $L_\alpha(u)$ is concave.

Power logconcavity

We easily see the following properties:

- Usual logconcavity corresponds to 1-logconcavity;
- Hierarchy: If $0 < \alpha \leq \beta$ and u is α-logconcave in Ω, then u is β-logconcave in Ω (i.e. the smaller α = the stronger α-logconcavity);
- Then α-logconcavity with $\alpha < 1$ is stronger than logconcavity; on the other hand, it is weaker than p-concavity for any $p > 0$;
- The Gauss kernel $G(\cdot, t)$ is $(1/2)$-logconcave in $\mathbf{R}^N$ (for t large enough).
- Let $\alpha \leq 1$. If u is α-logconcave, then κu is also α-logconcave for any $0 < \kappa \leq 1$, while this is in general not true if $\kappa > 1$.

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Why α-logconcavity? Where it comes from?It stems from the observation that the heat kernel is not only logconcave, but also $(1/2)$-logconcave:

$$L_{1/2}(G(x,t)) = -\sqrt{-\log G(x,t)} = -\left[\frac{|x|^2}{4t} + \frac{N}{2}\log(4\pi t)\right]^{\frac{1}{2}}.$$

The divorce of logconcavity and heat transfer

Preservation of α-logconcavity [IST1]

α-logconcavity is *preserved* by heat flow when $\alpha \in [1/2, 1]$
This means: if u_0 is an α-logconcave function in Ω, then $e^{t\Delta_\Omega} u_0$ is α-logconcave in Ω for any $t > 0$.

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The final fate of heat flow [IST1]

Let u_0 be a bounded nonnegative function in $\mathbf{R}^N$ with compact support. Then, for any given $1/2 < \alpha \leq 1$, there exists $T_\alpha > 0$ such that $L_\alpha(e^{t\Delta} u_0)$ is concave in $\mathbf{R}^N$ for any $t \geq T_\alpha$, i.e. $e^{t\Delta} u_0$ is α-logconcave in $\mathbf{R}^N$.

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Conjecture: this theorem holds true even for $\alpha = 1/2$.

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α-logconcavity is *preserved* by heat flow when $\alpha \in [1/2, 1]$
This means: if u_0 is an α-logconcave function in Ω, then $e^{t\Delta_\Omega} u_0$ is α-logconcave in Ω for any $t > 0$.

Now the strongest concavity preserved by the heat flow is $(1/2)$-logconcavity, no longer logconcavity!!! (Which, however, still remains the weakest)
Furthermore, we can see that α-logconcavity spontaneously appears in $\mathbf{R}^N$ when the initial datum has compact support.

The final fate of heat flow [IST1]

Let u_0 be a bounded nonnegative function in $\mathbf{R}^N$ with compact support. Then, for any given $1/2 < \alpha \leq 1$, there exists $T_\alpha > 0$ such that $L_\alpha(e^{t\Delta} u_0)$ is concave in $\mathbf{R}^N$ for any $t \geq T_\alpha$, i.e. $e^{t\Delta} u_0$ is α-logconcave in $\mathbf{R}^N$.

Conjecture: this theorem holds true even for $\alpha = 1/2$.
Conjecture: the first Dirichlet eigenfunction of a convex set is $(1/2)$-logconcave (or at least α-logconcave for some $1/2 \leq \alpha < 1$).

The divorce of logconcavity and heat transfer

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Does $(1/2)$-logconcavity is a sharp property for heat flow?

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Moreover, there are still many other concavity properties that can come into play....

F-concavity

Next we introduce the notion of F-concavity (or *Concavifiability*), which generalizes and embraces all the notions of concavity we have already seen and it is indeed the largest known/possible generalization of concavity.

Definition

Let $a \in (0, \infty]$, $I = [0, a)$ and $\operatorname{int} I = (0, a)$.

(i) A function $F : I \to [-\infty, \infty)$ is said *admissible* on I if $F \in C(\operatorname{int} I)$, F is strictly increasing on I, and $F(0) = -\infty$.

(ii) Let F be admissible on I and Ω a convex set. Set

$$\mathcal{A}_\Omega(I) := \{u : \Omega \to \mathbb{R} \mid u(\Omega) \subset I\}.$$

(iii) Given $u \in \mathcal{A}_\Omega(I)$, we say that u is *F-concave* in Ω if

$$F(u((1-\lambda)x + \lambda y)) \geq (1-\lambda)F(u(x)) + \lambda F(u(y))$$

for all $x, y \in \Omega$ and $\lambda \in (0, 1)$. We denote by $\mathcal{C}_\Omega[F]$ the set of F-concave functions in Ω.

In the universe of F-concavities, it is possible to introduce a hierarchy (which generalizes the one established among power concavities).

Definition

Let a_1, a_2, $a \in (0, \infty]$ and $a \leq \min\{a_1, a_2\}$. Set $I_1 = [0, a_1)$, $I_2 = [0, a_2)$, and $I = [0, a)$. Let F_1 and F_2 be admissible on I_1 and I_2, respectively. We say that *F_1-concavity is weaker* (*resp. strictly weaker*) *than F_2-concavity* in $\mathcal{A}_\Omega(I)$, or equivalently that *F_2-concavity is stronger* (*resp. strictly stronger*) *than F_1-concavity* in $\mathcal{A}_\Omega(I)$, if

$$\mathcal{C}_\Omega[F_2] \cap \mathcal{A}_\Omega(I) \subset \mathcal{C}_\Omega[F_1] \qquad (\text{resp. } \mathcal{C}_\Omega[F_2] \cap \mathcal{A}_\Omega(I) \subsetneq \mathcal{C}_\Omega[F_1]).$$

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Roughly speaking: F_2-concavity is stronger than F_1-concavity is avery F_2-concave function is also F_1-concave.

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Roughly speaking: F_2-concavity is stronger than F_1-concavity is avery F_2-concave function is also F_1-concave.
Criterion: if F_1 and F_2 are admissible on $I = [0, a)$, then F_1-concavity is weaker than F_2-concavity in $\mathcal{A}_\Omega(I)$ if and only if $F_1 \circ F_2^{-1}$ is concave in $F_2(I)$ (*or, equivalently, $F_2 \circ F_1^{-1}$ is convex in $F_1(I)$*).

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Also, a function u is *p-concave* for some $p \in \mathbf{R}$, if u is F-concave with $F = \Phi_p$, where

$$\Phi_p(r) := \int_1^r s^{-p} ds = \begin{cases} \dfrac{r^p - 1}{p} & \text{if} \quad p \neq 0, \\ \log r & \text{if} \quad p = 0, \end{cases}$$

for $r \in (0, \infty)$ and $\Phi_p(0) := -\infty$.

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Clearly, if a function u is F-concave in Ω for some admissible F, then it is quasiconcave in Ω; then quasiconcavity is truly the weakest conceivable concavity properties.
However it does not exist any admissible F which produces quasi-concavity.

Sharp concavities for DHF

Now we can ask the following questions in the framework of F-concavity.

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(A3) when starting with a non-logconcave initial datum, in general the corresponding solution (for $N \geq 2$) is not even quasi-concave, so losing every reminiscence of concavity.

For answering question (Q3), let us introduce a new F-concavity.

Hot Concavity

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Let

$$h(z) := \left(e^{\Delta_{\mathbb{R}}} \mathbf{1}_{[0,\infty)}\right)(z) = (4\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} e^{-\frac{s^2}{4}}\, ds \quad \text{for } z \in \mathbb{R}. \tag{0.8}$$

Then the function h is smooth in $\mathbb{R}$, $\lim_{z\to-\infty} h(z) = 0$, $\lim_{z\to\infty} h(z) = 1$, and $h' > 0$ in $\mathbb{R}$.

Definition

Denote by H the inverse function of h. For any $a \in (0, \infty]$, we define an admissible function H_a on $[0, a)$ by

$$H_a(r) := \begin{cases} H(r/a) & \text{for } r \in (0, a) \text{ if } a > 0, \\ \log r & \text{for } r \in (0, a) \text{ if } a = \infty, \\ -\infty & \text{for } r = 0 \text{ and } a \in (0, \infty]. \end{cases}$$

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Then we deduce the following picture of the hierarchy of F-concavities in $I = [0, 1]$

$$\overbrace{H_1\text{-concavity} \subset \underbrace{1/2\text{-logconcavity} \subset \cdots \subset 1\text{-logconcavity}}_{\alpha\text{-log-concavity } (1/2 \leq \alpha \leq 1)} = \text{logconcavity} = H_\infty\text{-concavity}}^{\text{stronger} \leftarrow \qquad\qquad \rightarrow \text{weaker}}$$

Theorem

Let $I = [0, a)$ with $a \in (0, \infty]$ and Ω a convex domain in $\mathbb{R}^n$ with $n \geq 1$.

(1) *H_a-concavity is preserved by DHF in Ω.*

(2) *Let F be admissible on I. If F-concavity is preserved by DHF in Ω, then F-concavity is weaker than H_a-concavity in $\mathcal{A}_\Omega(I)$ and $\lim_{r \to +0} F(r) = -\infty$.*

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Then we obtain the following answer to question (**Q1**):

(**A1**) H_a-concavity is the strongest F-concavity preserved by DHF in $\mathcal{A}_\Omega(I)$ with $I = [0, a)$.

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Conjecture: ϕ_1 is H_a-concave for a suitable a, say for $a = \|\phi_1\|_\infty$ of for any strictly greater than this...

The revenge of logcocanvity

Notice that the answer depends on the interval I, precisely on a. The dependance on a can be interpreted as dependance on the L^∞ norm of the initial datum. Then if you are happy with initial data u_0 such that $0 \leq u_0 \leq a$, then H_a concavity is the strongest property preserved.

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But if you don't want to have any limitation of the size of the initial datum, then.... Logconcavity is the answer to (Q1).

Corollary

Let F be admissible on $I = [0, \infty)$ and Ω a convex domain in $\mathbb{R}^n$ with $n \geq 2$. Then F-concavity is preserved by DHF in Ω if and only if $\mathcal{C}_\Omega[F] = \mathcal{C}_\Omega[H_\infty]$, i.e. F-concavity coincides with log-concavity.

THANKS!!!