Maximizers beyond the hemisphere for the second Neumann eigenvalue

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Neumann eigenvalues

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \le \mu_3(\Omega) \le \cdots \to \infty$$

Second eigenvalue: μ_2 maximal for ball of same volume

- Szegő (1954), 2-dimensions simply connected
- Weinberger (1956), n-dimensions

Developments. . .

- curved surfaces, simply connected — Bandle
- third eigenvalue — Girouard, Nadirashvili, Polterovich, Bucur, Henrot
- metrics on whole sphere — Hersch, . . .

Second eigenvalue maximal for geodesic disk, up to 50% of sphere

Theorem (Bandle 1972)

$$\left.\begin{array}{r} \textit{simply connected surface } \Omega \\ \textit{curvature} \leq K \in \mathbb{R} \\ \textit{area}(\Omega) K \leq \frac{1}{2} \cdot 4\pi \end{array}\right. \implies \mu_2(\Omega) \leq \mu_2(D_K)$$

where

$$D_K = \textit{geodesic disk, constant curvature } K\textit{, same area as } \Omega.$$

Also, harmonic mean of μ_2 and μ_3 is maximal at D_K.

Corollary

$$\left.\begin{array}{r} \textit{simply connected } \Omega \subset S^2 \\ \textit{area}(\Omega) \leq \frac{1}{2}\textit{area}(S^2) \end{array}\right. \implies \begin{array}{l} \mu_2(\Omega) \leq \mu_2(\textit{spherical cap} \\ \qquad \textit{with same area as } \Omega) \end{array}$$

Open problem for past 50 years:

is "half sphere" obstruction fundamental, or surmountable?

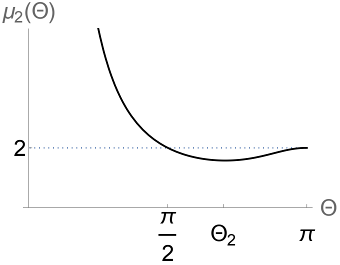
Spherical caps

Non-monotonicity of Neumann eigenvalues w.r.t. cap aperture Θ

Mathematica: `NDEigenvalues`

numerics: Dauge–Pogu (1988)
proofs: Dauge–Helffer (1993)

e.g. $\mu_2(\pi/2) = 2$, eigenfns $u = x, y$



$\Theta_2 \simeq 0.7\pi$
root of $\mu_2(\Theta)\sin^2\Theta = 1$

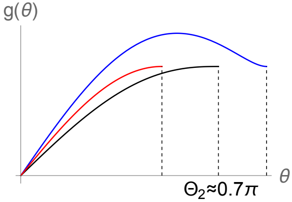
Observations

- $\mu_2(\Theta)$ remains monotonic BEYOND hemisphere 0.5π, until $\Theta_2 \simeq 0.7\pi$
- for $\Theta > \Theta_2$, larger caps have larger eigenvalues

Non-monotonicity of cap eigenfns $u = g(\theta) \begin{cases} \cos\phi \\ \sin\phi \end{cases}$

$\theta =$ angle from z-axis

$\quad =$ geodesic radius from north pole



Observations

- radial part remains monotonic for caps BEYOND hemisphere, until aperture $\simeq 0.7\pi$
- for apertures $\Theta > \Theta_2$, the radial part g is non-monotonic

Second eigenvalue maximal for geodesic disk, up to 94% of sphere

Theorem (Langford–Laugesen, Math. Ann. 2022)

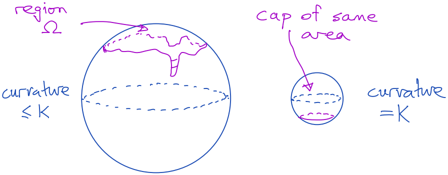
simply connected surface Ω

$$\left.\begin{array}{r} \textit{curvature} \leq K \in \mathbb{R} \\ \textit{area}(\Omega)K \leq \dfrac{16}{17} \cdot 4\pi \end{array}\right. \implies \mu_2(\Omega) \leq \mu_2(D_K)$$

where

$$D_K = \textit{geodesic disk, constant curvature } K\textit{, same area as } \Omega.$$

Also, harmonic mean of μ_2 *and* μ_3 *is maximal at* D_K.



Proof sketch (following Szegő and Bandle in Steps 1 and 2)

Step 1 — domain $\Omega \subset S^2$ with curvature $K = 1$.

Stereographic projection: spherical domain $\Omega \mapsto$ planar domain U
spherical cap $D_K \mapsto$ planar disk D, radius R

Area normalization: U and D have same area w.r.t. density

$$w(r) = 4/(1 + r^2)^2.$$

Weighted eigenvalue problem $\boxed{-\Delta u = \mu w u}$

Eigenfunctions for weighted disk D:

$$f_2 = h(r) \cos \phi, \qquad f_3 = h(r) \sin \phi$$

Transplant to trial functions on U:

$$u_2 = f_2 \circ F^{-1}, \qquad u_3 = f_3 \circ F^{-1}$$

Center of mass: conformal map $F : D \to U$ chosen s.t. $u_2, u_3 \perp 1$.

Dirichlet integral conformal invariance:

$$\int_U |\nabla u_i|^2 \, dA = \int_D |\nabla f_i|^2 \, dA.$$

Proof, cont.

For Rayleigh quotient denominator, **want**:

$$\int_U (h \circ F^{-1})^2 w \, dA \geq \int_D h^2 w \, dA$$
$$\int_D h^2 \big(B' - A'\big) \, dr \geq 0$$

where

$$B(r) = \int_{F(D(r))} w \, dA$$
$$= w\text{-area of } F(D(r))$$
$$A(r) = w\text{-area of } D(r)$$

and area normalization says $A(R) = B(R)$. By parts, we **want**

$$\int_0^R (h^2)' \big(A - B\big) \, dr \geq 0$$

Method: conformal mass transplantation!

Proof, cont.

$A(r) = w$-area of $D(r)$, $B(r) = w$-area of $F(D(r))$, want

$$\int_0^R (h^2)'(A - B)\, dr \geq 0$$

Bol's isoperimetric inequality + curvature bound (Bandle) implies $\boxed{A \geq B}$.

Step 2 — area$/4\pi \leq 0.79 \Longrightarrow$ cap aperture $\leq \Theta_2 \Longrightarrow h' > 0$. QED.
(Bandle: area$/4\pi \leq 0.50$)

Step 3 — $\boxed{0.79 < \text{area}/4\pi < 1} \Longrightarrow$ aperture $> \Theta_2 \Longrightarrow h' \not> 0$. BAD!

NEW IDEA:

factor out A, define $H = \int_0^r (h^2)'A$ (depends only on cap), int. by parts.
Want

$$\int_0^R H\,(B/A)'\, dr \geq 0$$

Area ratio increasing: $\boxed{(B/A)' \geq 0}$ follows from Bol.

Show $H(R) \geq 0$ by **Legendre functions**, when area$/4\pi \leq 0.94 \approx 16/17$.
Since $H' = (h^2)'A$ changes sign from $+$ to $-$, deduce $H(r) \geq 0$. QED.

Corollary

$$\begin{array}{c} \textit{simply connected } \Omega \subset S^2 \\ \textit{area}(\Omega) \leq \frac{16}{17}\textit{area}(S^2) \end{array} \implies \begin{array}{l} \mu_2(\Omega) \leq \mu_2(\textit{spherical cap} \\ \qquad \textit{with same area as } \Omega) \end{array}$$

Note $16/17 \simeq 94\%$. Can one improve to 100% ?

Conjecture

$$\textit{simply connected } \Omega \subset S^2 \implies \begin{array}{l} \mu_2(\Omega) \leq \mu_2(\textit{spherical cap} \\ \qquad \textit{with same area as } \Omega) \end{array}$$

Open problems

• All dimensions, all domains:

Weinberger's method adapts to hyperbolic space (Chavel 1979, Xu 1995). Similarly, domains on sphere S^n by Ashbaugh–Benguria (1995) using "two-pole" Weinberger construction, under "antipodal" restriction and

$$\text{area}(\Omega) \leq \frac{1}{2}\text{area}(S^n).$$

Cannot improve 1/2 to 1, due to multiply connected counterexamples by Bucur, Laugesen, Martinet, Nahon (in progress).

• Third eigenvalue:

is μ_3 maximal for two disjoint equal-sized caps of curvature K??

* Constant curvature: Euclidean 2-dim simply conn. by Girouard–Nadirashvili–Polterovich (2009), higher dim by Bucur–Henrot (2019); hyperbolic domains by Freitas–Laugesen (2022); spherical domains by Bucur–Martinet–Nahon (preprint).

* Variable curvature ≤ 0: simply conn. by Girouard–Polterovich (2010)

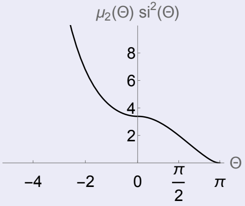
Eigenvalue scaling on hyperbolic space and sphere

Root Θ_2 of $\mu_2(\Theta)\sin^2\Theta = 1$ exists by monotonicity of whole spectrum:

Theorem (Scaling monotonicity; Langford and Laugesen 2022)

For each $k \geq 2$,

$$\Theta \mapsto \begin{cases} \mu_k(\Theta)\,\sinh^2\Theta, & \Theta \in (-\infty, 0), \\ \mu_k(\mathbb{D}), & \Theta = 0, \\ \mu_k(\Theta)\,\sin^2\Theta, & \Theta \in (0,\pi), \end{cases}$$



decreases strictly and continuously from ∞ to 0.
Similarly for Dirichlet eigenvalues when $k \geq 1$.

i.e. monotonicity of "eigenvalue times perimeter squared"
(intrinsic perimeter of disk or cap is $2\pi\sinh|\Theta|$ or $2\pi\sin\Theta$).

Higher dimensions? Scott Harman (PhD student) in progress.