Classical wave methods and modern gauge transforms: Spectral asymptotics in the one dimensional case

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Joint work with J.Galkowski (UCL) and R.Shterenberg (UAB)

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Conjecture (Sommerfeld–Lorentz, 1910)

Let

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\}.$$

Then,

$$N(\lambda) = \frac{\mathrm{vol}_g(M)\mathrm{vol}_{\mathbb{R}^d}(B_1)}{(2\pi)^d}\lambda^d + o(\lambda^d).$$

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Theorem (Weyl, 1911 (slightly modified setting))

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  Proved by Weyl in 1911

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Let (M, g) be a smooth, compact Riemannian manifold of dimension d. Then, there are $\{a_j\}_{j=1}^{\infty}$ such that for all N we have, as $t \to 0^+$:

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Naive Conjecture

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- Let $(M, g) = (\mathbb{S}^2, g_{\text{round}})$.
- Then even the second asymptotic term of $N(\lambda)$ does not exist!

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All based on Levitan's wave method Other people working on this problem: V.Ivrii, R.Melrose, Yu.Safarov, D.Vassiliev, S.Zelditch, and many others.

A second naive conjecture

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- If $V(\mathbf{x}) \to +\infty$ as $\mathbf{x} \to \infty$, the spectrum is discrete, but the conjecture is obviously false (harmonic oscillator). So, take V to be bounded.

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- New problem!: $N(\lambda, g, V)$ does not make sense here (unless V is very structured at infinity).

A replacement for the Weyl law

The local counting function or the *local density of states* (LDS) is given by

$$e(-\Delta_g + V, \lambda)(x) := 1_{(-\infty,\lambda^2]}(-\Delta_g + V)(x, x).$$

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In this case we have

$$N(-\Delta_g + V, \lambda) = \int_M e(-\Delta_g + V, \lambda)(x).$$

High energy asymptotics of the LDS

Theorem (Levitan 1952, Avakumović 1956, Hörmander 1968)

$$e(-\Delta_g + V, \lambda)(x) = (2\pi)^{-d}\mathrm{vol}_{\mathbb{R}^d}(B_1)\lambda^d + O(\lambda^{d-1}).$$

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Theorem (Safarov 1988, Sogge–Zelditch 2002)

If there are few loops from x to itself, then

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If the geodesics through x are all periodic with the same time,

$$\left|e(-\Delta_g + V, \lambda)(x) - (2\pi)^{-d}\mathrm{vol}_{\mathbb{R}^d}(B_1)\lambda^d\right| \neq o(\lambda^{d-1}).$$

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If there are no geodesic loops, then $e(-\Delta_g + V, \lambda)(x)$ has a full asymptotic expansion in powers of λ.

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- Still a problem $V = |x|^2$.

A less naive conjecture

We say $V \in C_b^\infty(\mathbb{R}^d)$ if $V \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$, there are $C_\alpha > 0$ such that

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Conjecture (LP–Shterenberg 2016)

Suppose $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $\{a_j(x)\}_{j=0}^\infty$ such that for any $N > 0$,

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$$\Updownarrow$$

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Suppose $V_1, V_2 \in C_b^\infty(\mathbb{R}^d)$. Then, if $V_1 = V_2$ in a neighborhood of x, for any $N > 0$, we have

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- Absolutely continuous
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CS +P on $\mathbb{R}$	wave method + GT	[Galkowski 2020]

The conjecture is true in 1 dimension

Theorem (Galkowski – LP – Shterenberg 2022)

Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. *Then there are* $\{a_j(x)\}_{j=0}^\infty$ *such that for all* $N > 0$, *there is* $C_N > 0$ *satisfying*

$$\left| e(-\Delta_{\mathbb{R}} + V, \lambda)(x) - \sum_{j=0}^{N-1} a_j(x)\lambda^{1-2j} \right| \leq C_N \lambda^{1-2N}.$$

Moreover, $a_j(x)$ *can be determined from a finite (j-dependent) number of derivatives of* V *at* x.

Corollaries of the theorem: Spectral Gaps

Corollary (Galkowski – LP – Shterenberg 2022)

Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. *Then for all* $N > 0$, *there is* $C_N > 0$ *such that for all* $\lambda \geq 1$ *and* $\epsilon > 0$, *if*

$$\text{spec}(-\Delta_{\mathbb{R}} + V) \cap [\lambda - \epsilon, \lambda + \epsilon] = \emptyset,$$

then

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Corollaries of the theorem: Almost plane waves

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$$(-\Delta_{\mathbb{R}} + V - \lambda^2)u = 0,$$

and any $x_1, x_2 \in \mathbb{R}$ *with* $|x_1 - x_2| < c_N \lambda^N$*, we have:*

$$|u(x_1)|^2 + \lambda^{-2}|u'(x_1)|^2 \leq e^{C\lambda^{-1}}(|u(x_2)|^2 + \lambda^{-2}|u'(x_2)|^2)$$

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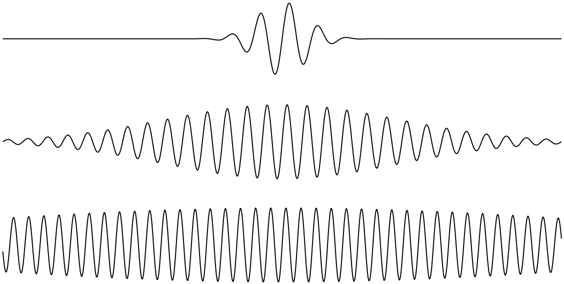

Corollaries of the theorem: Lyapunov exponents

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Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. *If the Lyapunov exponent,* $\Lambda(\lambda)$, *makes sense, then* $\Lambda(\lambda) \leq C_N \lambda^{-N}$.

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Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. If the Lyapunov exponent, $\Lambda(\lambda)$, makes sense, then $\Lambda(\lambda) \leq C_N \lambda^{-N}$.

Heuristic message
The spectrum WANTS to be absolutely continuous for high energies.