The isoperimetric inequality outside a convex set: the case of equality

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Shape Optimisation and Geometric Spectral Theory

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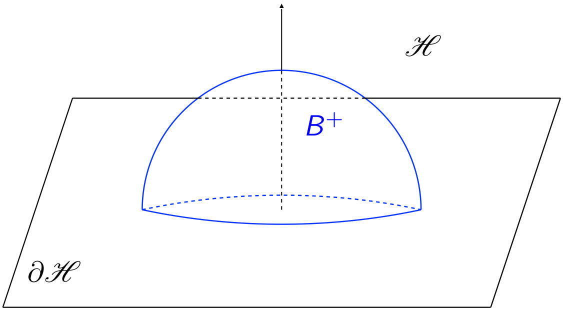
A well-known isoperimetric inequality

Among all sets of finite perimeter $\Omega \subset \mathbb{R}^N$ *of volume* m *contained in a half space* $\mathscr{H}$ *the unique minimizer of* $P(\Omega; \mathscr{H})$ *(the perimeter of* Ω *in* $\mathscr{H}$*) is the half ball* B^+, $|B^+| = m$ *sitting on* $\partial\mathscr{H}$



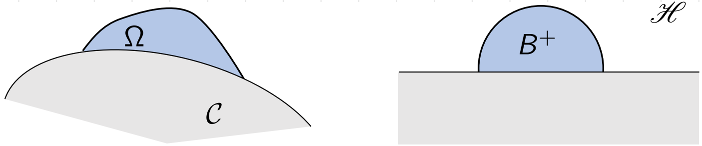
The isoperimetric inequality outside a convex set

Theorem (Choe-Ghomi-Ritoré, 2007)

Let $\mathcal{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. For any set of finite perimeter $\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$ we have

$$(*) \qquad P(\Omega; \mathbb{R}^N \setminus \mathcal{C}) \geq P(B^+; \mathbb{R}^N \setminus \mathscr{H})$$

(the surface measure of the half ball B^+ with the same volume of Ω)



Moreover, if $\mathcal{C}$ is smooth and Ω is a smooth bounded set for which the equality in $(*)$ holds, then Ω is a half ball sitting on a facet of $\mathcal{C}$.

The isoperimetric profile

$$(*) \qquad P(\Omega; \mathbb{R}^N \setminus \mathcal{C}) \geq N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} |\Omega|^{\frac{N-1}{N}}$$

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Introducing the isoperimetric profile of $\mathcal{C}$

$$I_{\mathcal{C}}(m) = \inf\{P(E; \mathbb{R}^N \setminus \mathcal{C}) : E \subset \mathbb{R}^N \setminus \mathcal{C}, |E| = m\}$$

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and the isoperimetric profile of the half space
$\mathscr{H} = \{x \in \mathbb{R}^N : x_N > 0\}$

$$I_{\mathscr{H}}(m) = \min\{P(E; \mathscr{H}) : E \subset \mathscr{H}, |E| = m\}$$

$$= N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} m^{\frac{N-1}{N}}$$

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(*) can be rewritten as

$$I_{\mathcal{C}}(m) \geq I_{\mathscr{H}}(m)$$

Our result

Theorem (The equality case, F.-Morini (2021))

Let $\mathcal{C} \subset \mathbb{R}^N$ be any closed convex set with nonempty interior and let $\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$ be any set of finite perimeter such that

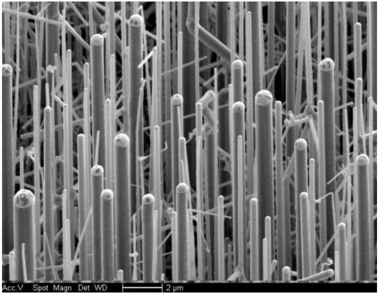
$$P(\Omega; \mathbb{R}^N \setminus \mathcal{C}) = N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} |\Omega|^{\frac{N-1}{N}}$$

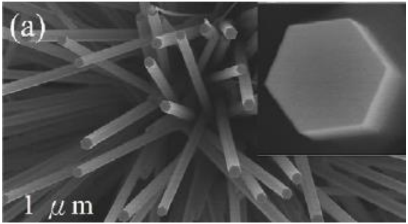
Then Ω is a half ball supported on a facet of $\mathcal{C}$.

Motivation

In models of vapor-liquid-solid grown nanowires

$\mathcal{C}$ = semi-infinite 'cylinder' with sharp edges and (nonsmooth) cross section





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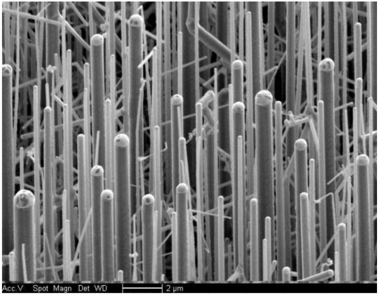
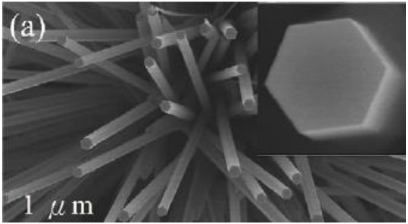
The crystal growth is driven by the capillarity energy functional

$$P(\Omega; \mathbb{R}^N \setminus \mathcal{C}) - \lambda \mathcal{H}^{N-1}(\partial\Omega \cap \partial\mathcal{C}) \qquad -1 < \lambda < 1$$





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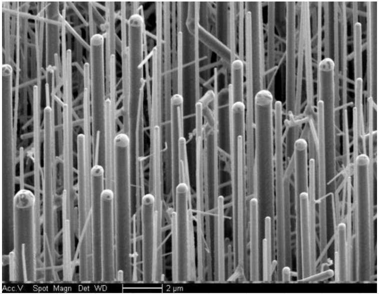
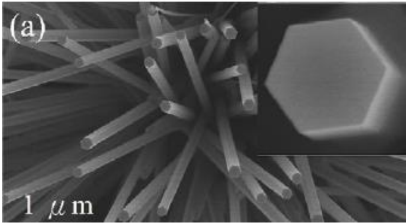
The case $\lambda = 0$ corresponds to the relative perimeter





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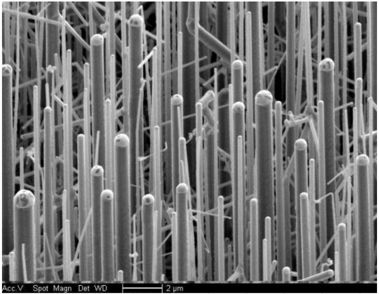
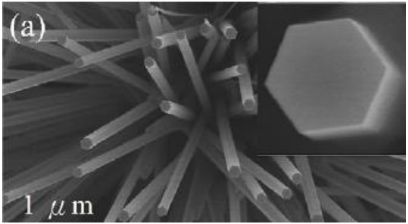
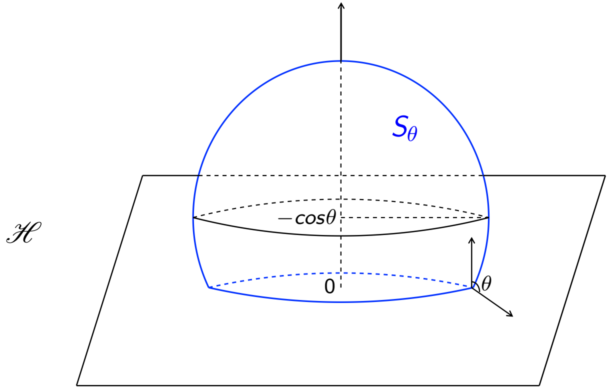
Qualitative properties of local minimizers studied in a recent paper by Fonseca-F.-Leoni-Morini (2021)





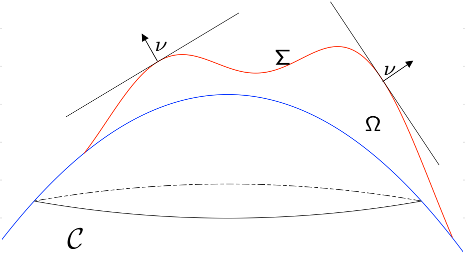
Basic notation

Let $\theta \in (0, \pi)$



S_θ = the unit spherical cap forming an angle θ with $\partial\mathscr{H}$

Basic notation - The total curvature



$\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$ an open set

$\Sigma = \overline{\partial\Omega \setminus \mathcal{C}}$

If $x \in \Sigma$ the normal cone at x to Σ is defined as

$$N_x\Sigma = \{\nu \in \mathbb{S}^{N-1} : \ (y - x) \cdot \nu \leq 0 \ \ \forall y \in \Sigma\}$$

Then we set

$$\Sigma^+ = \left\{x \in \Sigma \setminus \mathcal{C} : \exists \text{ a support hyperplane } \Pi_x, \text{ s.t. } \Pi_x \cap \Sigma = \{x\}\right\}$$

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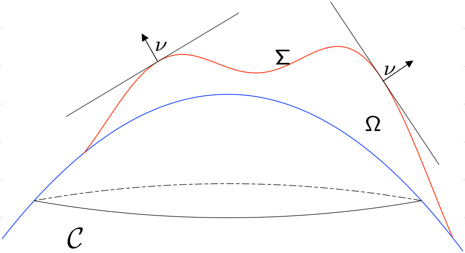

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$$\tau^+(\Sigma) = \mathcal{H}^{N-1}\Big(\bigcup_{x\in\Sigma^+} N_x\Sigma\Big) = \text{the total curvature of } \Sigma$$

An inequality involving the total curvature

Theorem (Choe-Ghomi-Ritoré, 2006)

Let $\mathcal{C}$ be a smooth convex set and let $\Sigma = \overline{\partial\Omega \setminus \mathcal{C}}$, $\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$, be a C^2 hypersurface with boundary. Assume that $\partial\Sigma$ intersects $\partial\mathcal{C}$ orthogonally. Then

$$\tau^+(\Sigma) \geq \frac{N\omega_N}{2}$$

Moreover if the equality holds then $\partial\Sigma$ lies on a support hyperplane to $\mathcal{C}$

In other words:

The measure of the image of Σ^+ through the Gauss map is bigger than the one of a half sphere

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The regularity of $\mathcal{C}$ is crucial to ensure that the minimizer Ω_m of

$$\min\{P(E;\mathbb{R}^N \setminus \mathcal{C}) : E \subset \mathbb{R}^N \setminus \mathcal{C}, |E| = m\}$$

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satisfies the orthogonality condition at $\partial\mathcal{C}$ (this is the Young's law when $\lambda = 0$, i.e. $\theta = \frac{\pi}{2}$)

Theorem (F.-Morini, 2021)

Let $\mathcal{C} \subset \mathbb{R}^N$ be a closed convex set of class C^1, $\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$ a bounded open set and $\Sigma := \overline{\partial\Omega \setminus \mathcal{C}}$. Let $\theta \in (0, \pi)$ such that

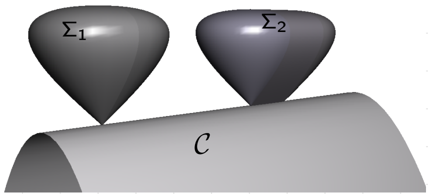
(1) $\nu \cdot \nu_{\mathcal{C}}(x) \leq \cos\theta$ *whenever* $x \in \Sigma \cap \partial\mathcal{C},\ \nu \in N_x\Sigma$

(2) *then* $\tau^+(\Sigma) \geq \mathcal{H}^{N-1}(S_\theta)$

Let $\Sigma \cap \partial\mathcal{C} \subset B_r$. For any $\varepsilon > 0$ there exists δ, depending on ε, θ and r, but not on $\mathcal{C}$ or Σ, such that if

$$\nu \cdot \nu_{\mathcal{C}}(x) \leq \cos\theta + \delta \quad \textit{whenever} \quad x \in \Sigma \cap \partial\mathcal{C},\ \nu \in N_x\Sigma$$
$$\textit{and} \quad \tau^+(\Sigma) \leq \mathcal{H}^{N-1}(S_\theta) + \delta$$

then $\Sigma \cap \partial\mathcal{C}$ is not empty and lies between two parallel ε-distant hyperplanes orthogonal to $\nu_{\mathcal{C}}(x)$ for some $x \in \Sigma \cap \partial\mathcal{C}$.
In particular, if (1) is satisfied and the equality in (2) holds, then $\Sigma \cap \partial\mathcal{C}$ lies on a support hyperplane to $\mathcal{C}$.



Both Σ_1 and Σ_2 satisfy

$$\nu \cdot \nu_{\mathcal{C}}(x) \leq \cos\left(\frac{3\pi}{4}\right) \quad \text{whenever} \quad x \in \Sigma_i \cap \partial\mathcal{C},\ \nu \in N_x\Sigma_i$$

and
$$\tau^+(\Sigma_i) = \mathcal{H}^{N-1}(S_{\frac{3\pi}{4}})$$

and thus $\Sigma_1 \cap \partial\mathcal{C}, \Sigma_2 \cap \partial\mathcal{C}$ lie on a support plane to $\mathcal{C}$

Theorem (A Willmore type inequality)

Let $\mathcal{C} \subset \mathbb{R}^N$ *be a closed convex set with nonempty interior,* $\Omega \subset \mathbb{R}^N \setminus \mathcal{C}$ *a bounded open set,* $\Sigma := \overline{\partial\Omega \setminus \mathcal{C}}$ *and let* $\theta \in (0, \pi)$. *Assume that* $\Sigma \setminus \mathcal{C}$ *is of class* $C^{1,1}$. *Assume also*

$$\nu \cdot \nu' \leq \cos\theta \quad \text{whenever} \quad x \in \Sigma \cap \partial\mathcal{C},\ \nu \in N_x\Sigma \ \text{ and } \ \nu' \in N_x\mathcal{C}$$

Then
$$\int_{\Sigma\setminus\mathcal{C}} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1} \geq (N-1)^{N-1}\mathcal{H}^{N-1}(S_\theta) \qquad (*)$$

Moreover, if equality holds in $(*)$ *and* $H_\Sigma \neq 0$ *a.e., then* $\Sigma \setminus \mathcal{C}$ *coincides, up to a rigid motion, with an omothetic of* S_θ *sitting on a facet of* $\mathcal{C}$.

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When $N = 3$ $(*)$ becomes

$$\int_{\Sigma\setminus\mathcal{C}} H_\Sigma^2\, d\mathcal{H}^2 \geq 4\mathcal{H}^2(S_\theta)$$

Asymptotic behaviour of the isoperimetric profile

$$I_{\mathcal{C}}(m) = \inf\{P(E; \mathbb{R}^N \setminus \mathcal{C}) : \ E \subset \mathbb{R}^N \setminus \mathcal{C}, \ |E| = m\}$$

$$I_{\mathscr{H}}(m) = \min\{P(E; \mathscr{H}) : \ E \subset \mathscr{H}, \ |E| = m\} = N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} m^{\frac{N-1}{N}}$$

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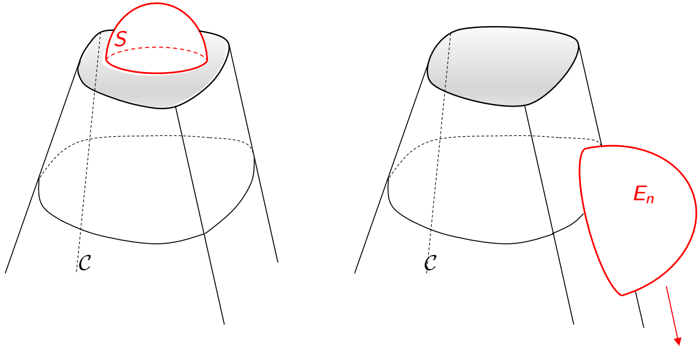

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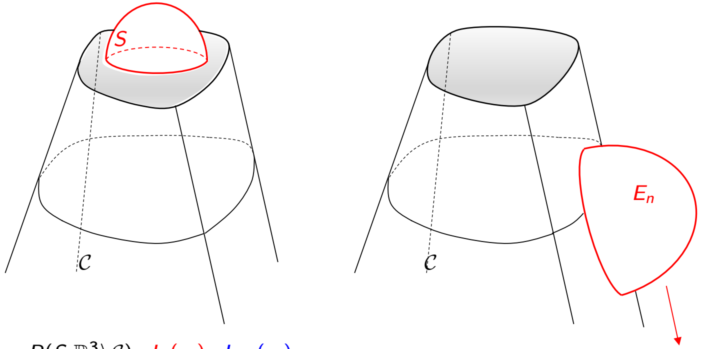

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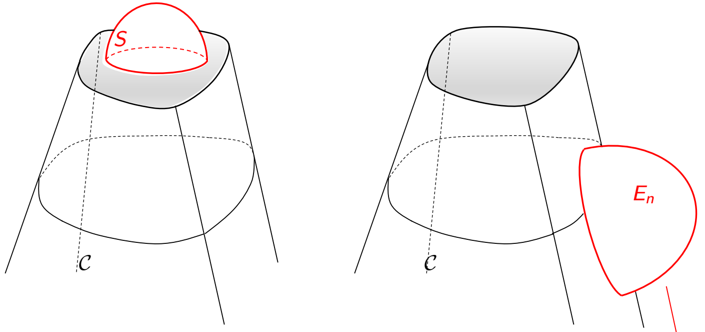
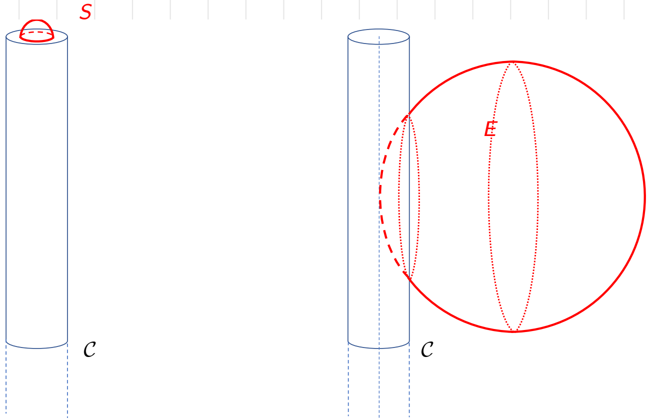


$P(S;\mathbb{R}^3\setminus\mathcal{C}) = I_{\mathcal{C}}(m) = I_{\mathscr{H}}(m)$ $\qquad$ $P(E_n;\mathbb{R}^3\setminus\mathcal{C}) \to I_{\mathcal{C}}(m) = I_{\mathscr{H}}(m)$!

Asymptotic behaviour of the isoperimetric profile



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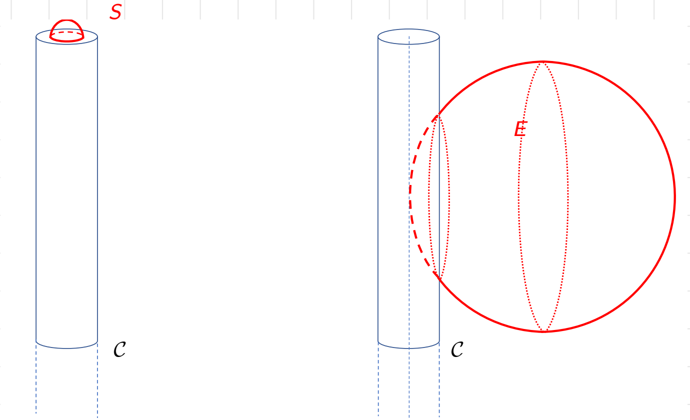

$P(S;\mathbb{R}^3\backslash\mathcal{C})=I_{\mathcal{C}}(m)=I_{\mathscr{H}}(m)$

$P(E;\mathbb{R}^3\backslash\mathcal{C})m^{-\frac{N-1}{N}} \approx P(B)|B|^{-\frac{N-1}{N}}$

If $\mathcal{C} \subset \mathbb{R}^N$ is a convex body, recall

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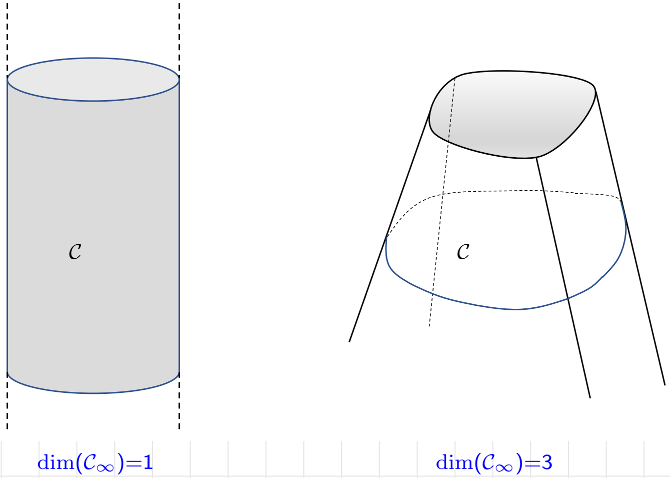

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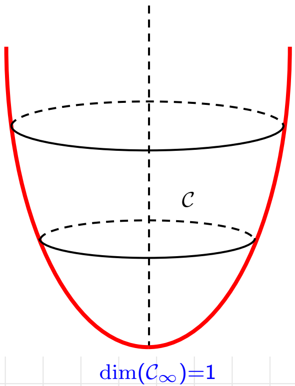

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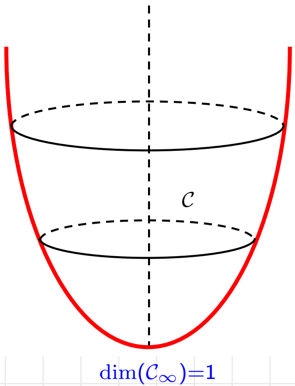

But

$$\lim_{m\to\infty} \frac{l_{\mathcal{C}}(m)}{m^{\frac{2}{3}}} = \frac{P(B)}{|B|^{\frac{2}{3}}}$$

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$$d^*(\mathcal{C}) := \max\Big\{ \dim K : \exists \{x_n\} \subset \mathcal{C}, \lambda_n \to 0, \text{ s.t. } \lambda_n(\mathcal{C} - x_n) \to K \text{ in the Kuratowski sense} \Big\}$$

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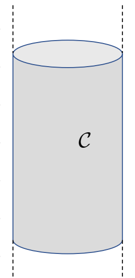
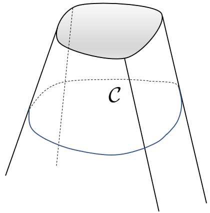
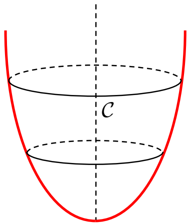
Note: $d^*(\mathcal{C}) \geq \dim \mathcal{C}_\infty$



$d^*(\mathcal{C})=1$



$d^*(\mathcal{C})=3$



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Asymptotic behaviour of the isoperimetric profile

Theorem (F.-Maggi-Morini-Novack, work in progress)

Let $\mathcal{C} \subset \mathbb{R}^N$ *be an unbounded convex body.*

If $d^*(\mathcal{C}) \geq N-1$ *, then* $I_{\mathcal{C}} = I_{\mathscr{H}}$. *Otherwise*

$$\lim_{m\to\infty} \frac{I_{\mathcal{C}}(m)}{m^{\frac{N-1}{N}}} = \frac{P(B)}{|B|^{\frac{N-1}{N}}}$$

Moreover in this case, for m large

$$(*) \qquad I_{\mathcal{C}}(m) \geq P(B)\left(\frac{m}{|B|}\right)^{\frac{N-1}{N}} - C_0 m^{\frac{d^*(\mathcal{C})}{N}}$$

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In the special case $d^*(\mathcal{C}) = 1$ we may improve $(*)$:

$$I_{\mathcal{C}}(m) \geq P(B)\left(\frac{m}{|B|}\right)^{\frac{N-1}{N}} - C_0 m^{\frac{1}{2N}}$$