Harmonic maps and eigenvalue optimisation in higher dimension

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Based on a joint work with Daniel Stern (UChicago)

Laplace-Beltrami operator

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$$\Delta_g f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

where g_{ij} is the Riemannian metric, g^{ij} are the components of the matrix inverse to g_{ij} and $|g| = \det g$.

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Set

$$\bar{\lambda}_i(M,g) = \lambda_i(M,g)\mathrm{Vol}(M,g)^{\frac{2}{n}}.$$

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where $c = [g] = \{e^{\omega}g \,|\, \omega \in C^{\infty}(M)\}$ is a fixed conformal class of metrics.

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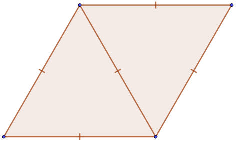

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Equivalently, $\lambda = 1$ is an eigenvalue of the problem with density

$$\Delta_g u = \lambda |d\Phi|_g^2 u.$$

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then

$$0 = \frac{1}{2}\Delta_h(|\Phi|^2) = \langle \Delta_h\Phi, \Phi\rangle - |d\Phi|_h^2 = 1 - |d\Phi|_h^2,$$

i.e. Φ is harmonic.

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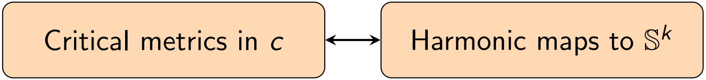

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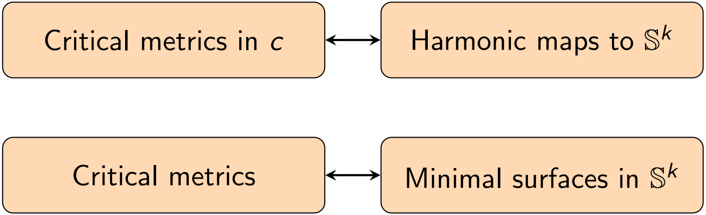

Maximal metrics for λ_1: first examples

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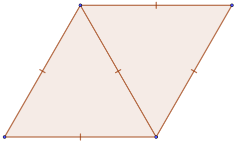

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- $\mathbb{S}^2$: the identity map $\mathbb{S}^2 \to \mathbb{S}^2$ is an isometric minimal immersion.
- $\mathbb{RP}^2$: *Veronese immersion* $v\colon \mathbb{RP}^2 \to \mathbb{S}^4$

$$v(x, y, z) = \left(xy, xz, yz, \frac{\sqrt{3}}{2}(x^2 - y^2), \frac{1}{2}(x^2 + y^2) - z^2\right)$$

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Theorem (K.-Stern, 2020) Any maximal metric has to be "smooth".

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In higher dimensions these quantities are genuinely different.

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K.-Stern (2022): $\mathcal{V}_1(\mathbb{S}^n, g_{st})$ is achieved on constant density.

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Observation: Many results for surfaces in conformal class can be extended to $\mathcal{V}_1(M, g)$.

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Main challenge for dim $M > 2$: there are no multiplicity bounds (Y. Colin de Verdìre)

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- Observe that for $k \gg N$ the densitiy $|d\Phi_k|_g^2$ is maximal.

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- (N, h) is a 3-manifold with $\mathrm{Ric}_h > 0$;
- (N, h) is a k-manifold with positive isotropic curvature, $k \geq 4$.

Open questions

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3. Prove the following identity

$$\sup_{h\in[g]} \mathcal{V}_1(M,h)\mathrm{Vol}(M,h)^{\frac{2-n}{n}} = \Lambda_1(M,[g]).$$

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- conformal automorphisms of $\mathbb{S}^n$ modulo $O(n+1)$ are $\cong \mathbb{B}^{n+1}$
- **Hersch trick:** there exists Φ_b, $b \in \mathbb{B}^{n+1}$ such that

$$\int \Phi_b \rho \, dv_g = 0$$

Hersch's theorem: continued

- Use the components of Φ_b as test-functions for $\lambda_1(\mathbb{S}^n, g, \rho)$

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- Can check that equality iff ρ is constant.

Proof of Hersch's trick

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- Brouwer theorem implies there exists b_0 with $I(b_0) = 0$.

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- **Example:** For any $F_0 \in C^\infty(M, \mathbb{S}^n)$ its *canonical family* F is $F_b = \Phi_b \circ F_0$, where Φ_b are the conformal automorphisms of $\mathbb{S}^n$.

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- Hersch's trick $\implies$ for any $F \in \widetilde{\Gamma}_k$ and any ρ one has

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- Goal: for large n one has $\mathcal{V}_1(M, g) = \widetilde{\mathcal{E}}_n(M, g)$.