Regularity for convex shapes minimizing perimeter

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21/09/22

Shape Optimization and Geometric Spectral Theory

ICMS, Edinburgh

Outline of the talk

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Newton's problem of minimal resistance [1685]

$$\min\left\{\int_{\mathbb{D}}\frac{1}{1+|\nabla u(x)|^2}dx\ \ ,\ \ u:\mathbb{D}\to[0,M],\ \ u\ \textit{concave}\ \right\}$$

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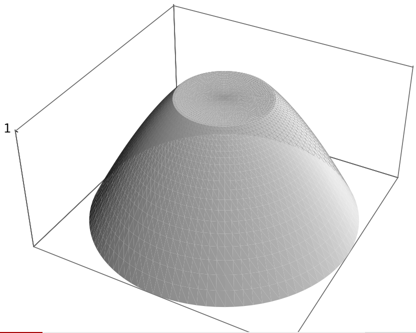
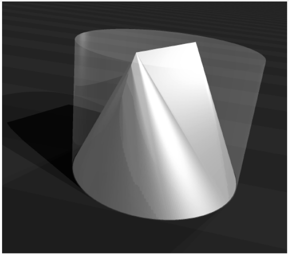
Radial symmetric solution :

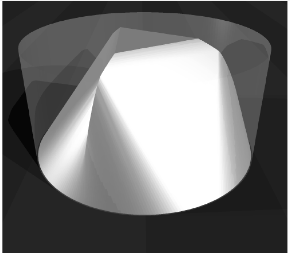


Newton's problem of minimal resistance **[1685]**

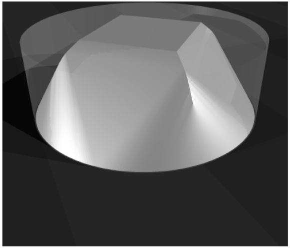
Non radial solutions **[Guasoni 1995], [Buttazzo-Ferone-Kawohl, 1995] - Numerical computations [Lachand-Robert - Oudet, 2004]**



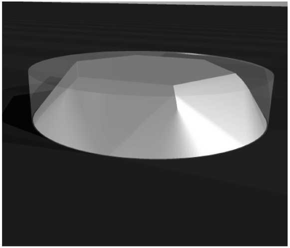
$M = 3/2$



$M = 1$



$M = 7/10$



$M = 4/10$

Shape optimization under convexity constraint

We are interested in problems of the form

$$\min\left\{J(\Omega),\quad \Omega\subset\mathbb{R}^d \ \textit{ is } \ \textit{convex},\quad \Omega\in\mathcal{F}_{ad}\right\}$$

where

- $\Omega\mapsto J(\Omega)$ is a shape functional,
- $\mathcal{F}_{ad}$ is a class of admissible sets (volume constraint, box constraint...)

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Usual questions of optimization :

- Existence of a minimum,
- Properties of optimal shapes (symmetry, regularity...),
- Numerical computations (**[Choné-Le Meur 2001], [Mérigot-Oudet 2014], [Mirebeau 2015], [Antunes-Bogosel 2019], [Ftouhi 2022], [Bogosel 2022]**...)

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Existence is usually "easier",
but the rest is delicate
(the "neighbour" of a convex set is not convex in general)

Some examples

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4. Pólya-Szegö conjecture **[Jerison's talk]**, $\max \frac{T(\Omega)cap(\overline{\Omega})}{|\Omega|^2}$ **[van den Berg's talk]**

An example coming from Blaschke-Santaló diagrams

[Cox-Ross 1995], [Antunes-Freitas 2006], [Ftouhi 2022]

$$\max\left\{\lambda_1(\Omega),\ \ |\Omega|=1,\ \ P(\Omega)=p_0,\ \Omega\ \textit{convex}\ \subset\mathbb{R}^2\right\}$$

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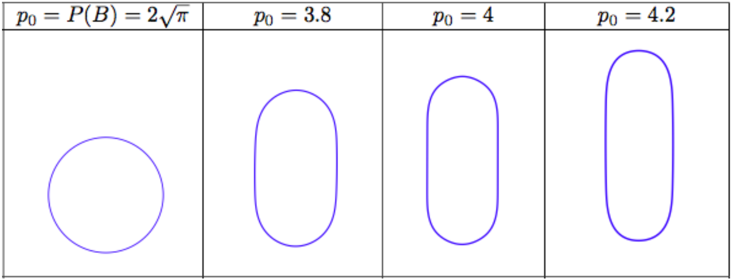

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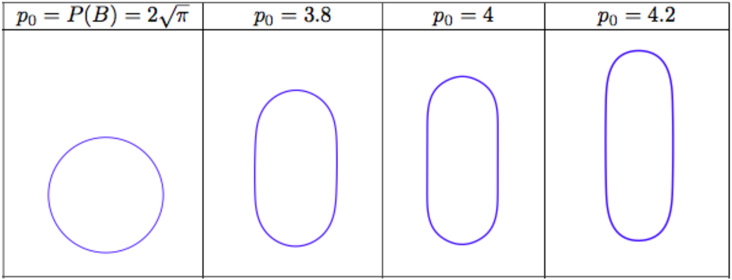

Theorem (**L.-Novruzi-Pierre 2012, Ftouhi-L. 2021**)

For any $p_0 \geq P(B)$, there exists a solution, and it is $C^{1,1}$.

Outline of the talk

Class of isoperimetric problems

Consider

$$\min \left\{ P(\Omega) + R(\Omega), \ \ \Omega \textit{ open convex} \subset \mathbb{R}^n \right\}$$

where $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$, and $\Omega \mapsto R(\Omega)$ is a perturbative term.

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What is the regularity of optimal shapes ?

Classical setting (no convexity constraint) :

Definition

Ω is a quasi-minimizer of the perimeter if for every Ω' such that $\Omega'\Delta\Omega\subset B_r$ for $r\leq r_0$,

$$P(\Omega)\leq P(\Omega')+\Lambda r^{n-1+\alpha}\qquad (\ \alpha\in(0,1]\ )$$

Theorem

A quasi-minimizer of the perimeter is $C^{1,\alpha/2}$ up to a small set of dimension less than $d-8$.

Quasi-minimizer of the perimeter under convexity constraint

Definition

Given $\Lambda, \varepsilon > 0$, Ω is a (Λ, ε)-quasi-minimizer of the perimeter under convexity constraint **(qmpcc)** *if for every Ω' convex in $\mathbb{R}^n$ such that $\Omega' \subset \Omega$ and $|\Omega \setminus \Omega'| \leq \varepsilon$,*

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- The result is sharp [**Alvino-Ferone-Nitsch 2011**] : the solution of

$$\min\left\{\frac{P(\Omega)-P(B)}{P(B)\mathcal{A}(\Omega)^2},\ \ \Omega \text{ convex } \subset \mathbb{R}^2\right\}$$

is a stadium, where $\mathcal{A}$ is the Fraenkel asymmetry.

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Proposition (L.-Prunier 2022)

Given $D' \subset D \subset \mathbb{R}^N$ convex, $n \in \mathbb{N}^*$, then there exists $C = C(n, D', D) > 0$ such that

$$\forall D' \subset \Omega' \subset \Omega \subset D, \;\; |\lambda_n(\Omega) - \lambda_n(\Omega')| \leq C|\Omega \setminus \Omega'|$$

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Corollary

Solutions to

$$\min\left\{P(\Omega) - \gamma\lambda_1(\Omega), \ \ \Omega \text{ convex } \subset B_{R_0}, \ |\Omega| = V_0\right\}$$

are $C^{1,1}$.

Ideas of the proof of the main result (i)

First ingredient : Model problem in the calculus of variations

Given D a convex bounded set, consider :

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- Proof by a slicing procedure.

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$\forall v : D \to \mathbb{R}$ *convex such that* $v \geq u$ *and* $v = u$ *on* ∂D,

$$\int_D \sqrt{1 + |\nabla u|^2} \leq \int_D \sqrt{1 + |\nabla v|^2} + \Lambda \int_D (v - u)$$

Similar strategy as [**Caffarelli-Carlier-Lions**] ?

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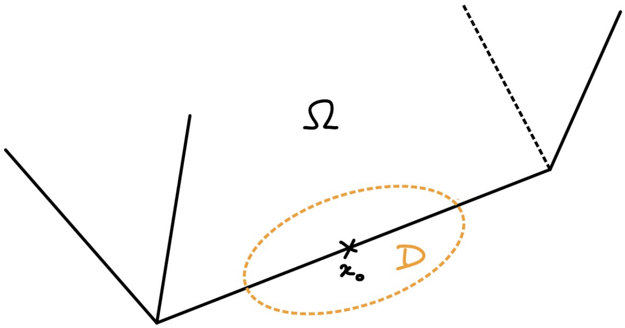

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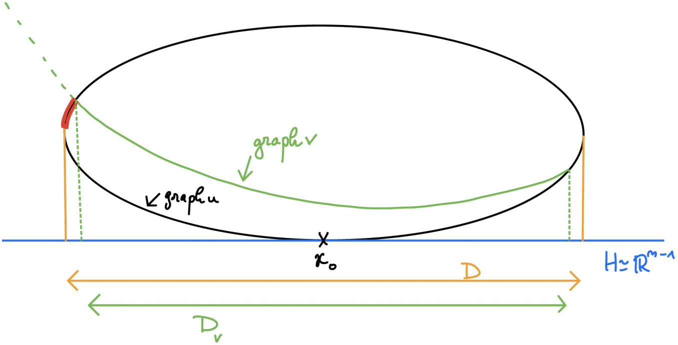

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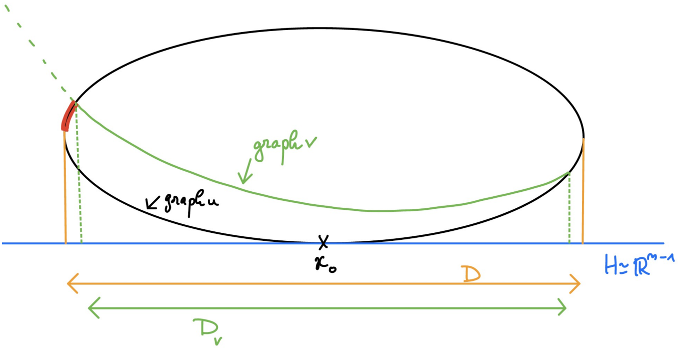

1. $P(\Omega) - P(\Omega_v) \geq \int_D \sqrt{1 + |\nabla u|^2} - \int_{D_v} \sqrt{1 + |\nabla v|^2}$ where $D_v \subset D$.
2. We pick $v = v_r$ as in **[Caffarelli-Carlier-Lions]**
3. We control $|\Omega \setminus \Omega_{v_r}|$ from above.

Outline of the talk

Main question [Fuglede 1989], [Dambrine-Pierre 2000], [Cicalese-Leonardi 2012], [Acerbi-Fusco-Morini 2015], [Allen-Kriventsov-Neumayer 2021]...

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Can we retrieve local minimality among convex sets ?

Work in progress **[Prunier 2022-2023]**

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If $\gamma<\gamma_d$, then the ball is a local minimizer for $P-\gamma\lambda_1$ among convex sets.

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1. Control the third order term in Taylor's expansion for suitable norms.
2. Selection principle among convex sets :

Theorem (Prunier)

Let $(K_j)_{j\in\mathbb{N}}$ a sequence of (Λ,ε)-qmpcc converging in L^1 to the ball and bounded in $C^{1,1}$. Then it converges to B in $C^{1,\alpha}$ for any $\alpha<1$.