The behaviour of the spectrum of the Robin Laplacian with a complex boundary parameter

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The Robin Laplacian

The Laplacian eigenvalue problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, treated as fixed:

$$-\Delta u \equiv -\sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} = \lambda u \qquad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega$$

- Here: ν outer unit normal to $\partial\Omega$; α constant/parameter
- Weak formulation: for all $v \in H^1(\Omega)$,

$$a_\alpha(\psi, v) := \int_\Omega \nabla\psi \cdot \overline{\nabla v}\, dx + \alpha \int_{\partial\Omega} \psi\, \overline{v}\, d\sigma = \lambda \int_\Omega \psi\, \overline{v}\, dx$$

Formally we define the Robin Laplacian $-\Delta_\alpha$ to be the operator on $L^2(\Omega)$ associated with the sesquilinear form a_α on $H^1(\Omega)$.

- If $\alpha \in \mathbb{R}$, then there is an eigenvalue sequence

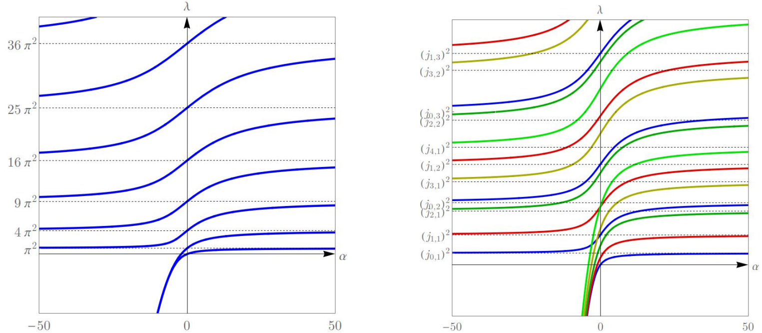
$$\lambda_1(\alpha) < \lambda_2(\alpha) \leq \lambda_3(\alpha) \leq \ldots \to \infty$$

  and the eigenfunctions form an orthonormal basis of $L^2(\Omega)$
- If $\alpha \in \mathbb{R}$, then we have the usual variational characterisations, e.g.:

$$\lambda_1(\alpha) = \inf_{0 \neq u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} |u|^2 \, d\sigma}{\int_\Omega |u|^2 \, \mathrm{d}x}$$

- Kato theory: for fixed Ω the eigenvalues depend analytically on α except at a locally finite number of crossing points
- $\alpha = 0 \rightsquigarrow$ Neumann, $\alpha = \infty \rightsquigarrow$ formally Dirichlet

- As $\alpha \to +\infty$, all eigenvalues converge to their Dirichlet counterparts from below
- As $\alpha \to -\infty$, some eigenvalues converge to their Dirichlet counterparts from above, others diverge to $-\infty$



The first few eigenvalues of the interval of length 1 (left) and the disk of radius 1 (right) as functions of $\alpha \in \mathbb{R}$

- As $\alpha \to \infty$, $\lambda_k(\alpha) \to \lambda_k^D$, kth Dirichlet eigenvalue
  Rate of convergence: $|\lambda_k(\alpha) - \lambda_k^D| \leq C/\sqrt{\alpha}$
  (works of Filinovskiy 2014-17)
- For $\alpha < 0$, $\lambda_1(\alpha) < -\alpha^2$ always (Giorgi–Smits 2007): fix a unit vector $v \in \mathbb{R}^d$ and use $u_v(x) = e^{\alpha x \cdot v}$ as a test function

In 1D, $u(x) = e^{\alpha x}$ solves

$$\begin{aligned} -u'' &= (-\alpha^2)u \qquad \text{in } (0,\infty) \\ -u'(0) + \alpha u(0) &= 0 \end{aligned}$$

- If Ω is C^1, then for each $k \geq 1$

$$\lambda_k(\alpha) = -\alpha^2 + o(\alpha^2) \qquad \text{as } \alpha \to -\infty$$

(Lacey–Ockendon–Sabina 1998, Lou–Zhu 2004, Daners–K. 2010)

- If Ω has corners, then $\lambda_k(\alpha) = -C(\Omega, k)\alpha^2 + o(\alpha^2)$ for some $C(\Omega, k) \geq 1$ (Lacey–Ockendon–Sabina 1998, Levitin–Parnovski 2008, works of Khalile–Pankrashkin 2018+)
- For Ω smooth, more terms in the asymptotic expansion in α are known and involve the maximal mean curvature of $\partial\Omega$ (Exner–Minakov–Parnovski 2014; Freitas–Krejčiřík 2015; Pankrashkin–Popoff 2015; Helffer–Kachmar 2017, ...)

The case of complex α

- $-\Delta_\alpha$ still has discrete spectrum but no longer self-adjoint
- Complex eigenvalues and no variational principles
- The eigenvalues still depend meromorphically on $\alpha \in \mathbb{C}$ (Kato!), but there is no natural "enumeration" of them
- The eigenfunctions no longer form an ONB of $L^2(\Omega)$

Theorem: basic properties (BKL)

Let $\lambda_k(\alpha_0)$, $k \in \mathbb{N}$, be an enumeration of the eigenvalues for some $\alpha_0 \in \mathbb{R}$ (repeated according to their finite multiplicities). Then each $\lambda_k(\alpha_0)$, and its corresponding eigenprojection, may be extended to a meromorphic function of $\alpha \in \mathbb{C}$ (holomorphic outside crossing points), such that for any α, these eigenvalues form the totality of the spectrum of $-\Delta_\alpha$. At crossing points, the weighted eigenvalue mean and the total eigenprojection are holomorphic.

Based on the real case as well as explicit calculations for intervals, balls, hyperrectangles:

Conjecture

$\Omega \subset \mathbb{R}^d$ bounded Lipschitz, $\alpha \to \infty$ in $\mathbb{C}$.

(1) If $\operatorname{Re}\alpha$ remains bounded from below, then each EV converges to a Dirichlet EV
(2) If $\operatorname{Re}\alpha \to -\infty$, then there is a sequence of absolutely divergent EVs. Any limit point of any non-divergent EV curve is a Dirichlet EV
  - Ω smooth (C^1): divergent EVs behave like $-\alpha^2 + o(\alpha^2)$
  - Ω Lipschitz: each divergent EV curve behaves like $-C\alpha^2 + o(\alpha^2)$ for some C depending on that curve (and Ω)

Today: bounds on the *numerical range*, which control the possible rate of divergence of the EVs

Sabine's talk: more on the eigenvalue curves, especially via the *Dirichlet-to-Neumann operator*

Observation ("refined trace inequality")

For any Lipschitz Ω, there exists $C = C_\Omega > 0$ such that

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq C_\Omega \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$.

Lemma

- There exist constants $C_1 \geq 2$ and $C_2 > 0$ depending on Ω such that
$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq C_1 \|\nabla u\|_{L^2(\Omega)} + C_2$$
for all $u \in H^1(\Omega)$ with $L^2(\Omega)$-norm 1.
- If Ω is C^2, then we may choose $C_1 = 2$ and C_2 an explicit constant related to the maximal mean curvature of $\partial\Omega$.

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq C_1 \|\nabla u\|_{L^2(\Omega)} + C_2 \quad (C_1 \geq 2,\ C_2 > 0) \qquad (1)$$

Numerical range of the Robin form a_α is the set of values the Rayleigh quotient can take,

$$\left\{ z \in \mathbb{C} : z = a_\alpha(u) \text{ for some } u \in H^1(\Omega) : \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Every eigenvalue is in the numerical range. For

$$t := \int_\Omega |\nabla u|^2 \, dx, \qquad s := \int_{\partial\Omega} |u|^2 \, d\sigma,$$

(1) means that $s \in [0, C_1\sqrt{t} + C_2]$, for any u. Hence:

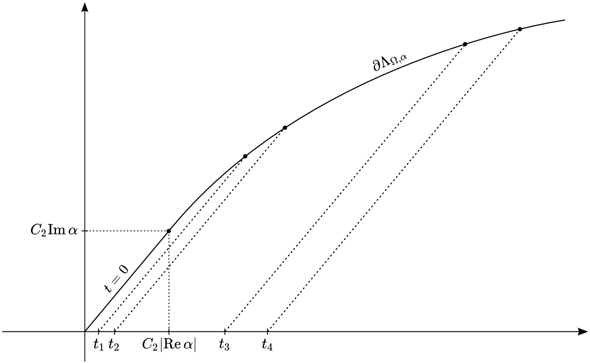
Theorem (BKL)

For fixed $\alpha \in \mathbb{C}$, the numerical range of a_α, in particular every eigenvalue λ of $-\Delta_\alpha$, is contained in

$$\Lambda_{\Omega,\alpha} := \left\{ t + \alpha \cdot s \in \mathbb{C} : t \geq 0,\ s \in [0, C_1\sqrt{t} + C_2] \right\}.$$

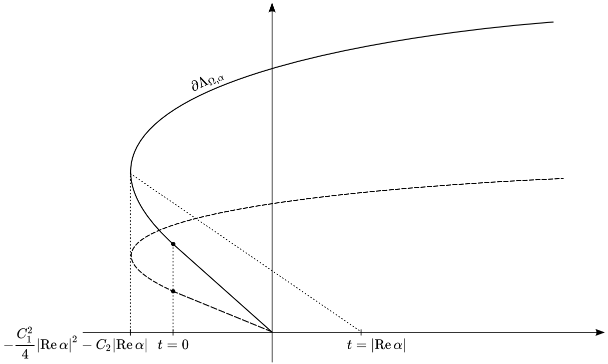
The location of the eigenvalues for complex α

$$\Lambda_{\Omega,\alpha} = \left\{ t + \alpha \cdot s \in \mathbb{C} : t \geq 0,\ s \in [0, C_1\sqrt{t} + C_2] \right\}$$



The region $\Lambda_{\Omega,\alpha}$ for $\operatorname{Re}\alpha > 0$, $\operatorname{Im}\alpha > 0$.

The location of the eigenvalues for complex α



$\Lambda_{\Omega,\alpha}$ for $\mathrm{Re}\,\alpha < 0$ and two different choices of $\mathrm{Im}\,\alpha > 0$.

Claim: $\mathrm{Re}\,\lambda \geq -\frac{C_1^2}{4}(\mathrm{Re}\,\alpha)^2 + C_2\mathrm{Re}\,\alpha$.

(This also means that Δ_α generates a *cosine function*.)

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq C_1 \|\nabla u\|_{L^2(\Omega)} + C_2 \quad (C_1 \geq 2, \; C_2 > 0)$$

Use the elementary inequality

$$2\|\nabla u\|_{L^2(\Omega)} \leq \frac{C_1 |\mathrm{Re}\, \alpha|}{2} + \frac{2}{C_1 |\mathrm{Re}\, \alpha|} \|\nabla u\|_{L^2(\Omega)}^2$$

to get, for all $u \in H^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$,

$$\begin{aligned} \mathrm{Re}\, a_\alpha(u) &= \|\nabla u\|_{L^2(\Omega)}^2 + \mathrm{Re}\, \alpha \int_{\partial\Omega} |u|^2 \, d\sigma \\ &\geq \|\nabla u\|_{L^2(\Omega)}^2 + C_1 \mathrm{Re}\, \alpha \|\nabla u\|_{L^2(\Omega)} + C_2 \mathrm{Re}\, \alpha \\ &\geq -\left(\frac{C_1}{2}\right)^2 (\mathrm{Re}\, \alpha)^2 + C_2 \mathrm{Re}\, \alpha. \end{aligned}$$

Consequences in the real case, $\alpha < 0$

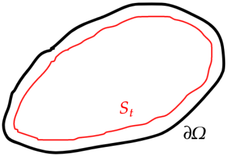
- For general Lipschitz Ω, there exists $C_\Omega \geq 1$ such that
$$\lambda_1(\alpha) \geq -C_\Omega \alpha^2$$
(C_Ω *could* be estimated explicitly based on knowledge of Ω; comes from a covering of $\partial\Omega$ and the behaviour of ν inside each neighbourhood of the covering)
- For $\Omega \in C^2$, there exists $C_2 > 0$ related to the maximal mean curvature such that
$$\lambda_1(\alpha) \geq -\alpha^2 + C_2\alpha;$$
in particular, combined with the test function argument of Giorgi–Smits ($\lambda_1(\alpha) < -\alpha^2$),
$$\lambda_1(\alpha) = -\alpha^2 + O(|\alpha|).$$
With the argument of Daners–K. an alternative proof that $\lambda_k(\alpha) \sim -\alpha^2$ for each fixed k (avoiding the blow-up argument of Lou–Zhu and the Dirichlet-Neumann bracketing/operator decomposition used for the higher terms of the asymptotics)

Not clear if

$$\lambda_1(\alpha) = -\alpha^2 + O(|\alpha|)$$

should hold if $\Omega \in C^1$ only, the constant C_2 is given as follows:

- Choose $\varepsilon > 0$ such that level surface S_t of the distance function to the boundary is a smooth manifold at distance $t \in [0, \varepsilon]$



- $\bar{\kappa}^{S_t}(x) =$ mean curvature of $S_t \in C^2$ at x
- Then we may take

$$C_2 = \varepsilon^{-1} + (d-1) \max_{t \in [0,\varepsilon]} \max_{x \in S_t} |\bar{\kappa}^{S_t}(x)|$$

- For comparison: if $\Omega \in C^3$, then

$$\lambda_1(\alpha) = -\alpha^2 + \left[(d-1) \max_{x \in \partial\Omega} \bar{\kappa}^{\partial\Omega}(x)\right]\alpha + O(|\alpha|^{2/3})$$

Thank you for your attention!

Reference: S. Bögli, J. B. K. and R. Lang, *On the eigenvalues of the Robin Laplacian with a complex parameter*, Anal. Math. Phys. 12 (2022), 39